1 Introduction

The underground (or shadow) economy is broadly defined as economic activities which are concealed from public authorities to avoid the payment of taxes and social security contributions, and to avoid compliance with certain legal standards (e.g. labor market regulations, trade licenses). Unreported activities are a universal feature of economic life, and assume considerable proportions even in the industrialized world, where they are estimated to range between 8 and as much as 28 percent of official GDP.[1]

Most of the research on the causes of this particular form of regulation failure has been motivated by the observation that the size of the underground economy (as a fraction of official economic activity) varies considerably across countries. The burden of taxes and social security contributions, excessive market regulation, as well as ineffective law enforcement and corruption, have been suggested to explain these cross-country variations (see Schneider and Enste (2000), Johnson et al. (1998), Friedman et al. (2000), Lemieux et al. (1994)).

This paper proposes a novel rationale for the recent expansion of the underground economy that has been observed in numerous countries: Market deregulation policies that lead to intensified market competition between firms (in the sense that the substitutability of their goods is increased) will have stronger adverse effects on firms in the official economy than on those in the shadow economy, and therefore drive more firms into the underground economy.

The argument runs as follows: A firm which operates in the underground economy can buy its inputs, in particular labor, at a lower price (because it avoids payroll taxes, disregards safety and health standards, etc.), thereby reducing its variable cost relative to a firm in the official economy. The underground firm can pass on its savings to consumers, which will reduce market prices, and as a result its competitors' profits fall. But since the official firm's mark-up falls faster than that of its competitors from shadow economy, such a firm may have to choose between operating underground as well, or going out of business. The fiercer is competition, the higher is the pressure to reduce costs, and the more likely are underground activities to spread in the industry.

This reasoning has some parallels in Shleifer's (2004) argument that competition may promote unethical behavior (e.g. child labor, corruption, etc.). He highlights the trade-off between cost savings and the firm owner's private utility of ethical behavior, and finds that an increase in competition tilts this trade-off in favor of unethical behavior. In my model, firm owners' moral considerations play no role. Instead, I study the interaction between firms and tax enforcement institutions: shadow firms face a risk of being detected and fined by the tax authority, and this threat feeds back more or less strongly into the firm's decision to go underground.

To my knowledge, there is only one paper that builds a model relating the shadow economy to market competition, namely Goldberg and Pavcnik (2003).

[1] See Schneider and Enste (2000) for estimates of the size of the underground economy in numerous countries and a critical discussion of the different measurement methods.

This paper asks whether we should expect the informal sector in developing countries to expand in response to trade liberalization programs (i.e. to an increase in foreign competition on domestic markets). However, in their model all firms behave as price takers, and law enforcement is completely absent. My paper instead highlights the role of strategic interaction among firms and institutions, making it the first one to develop a theory of the underground economy within rigorous models of industrial organization.

As for empirical evidence, Goldberg and Pavcnik (2003) apply the theoretical model they develop to a dataset on two developing countries, namely Brazil and Colombia. While they find no evidence of a relationship between trade policy and the informal sector in Brazil, they do find evidence of such a relationship in Colombia, but only for the period preceding a major labor market reform that increased the flexibility of the Colombian labor market. The role of labor market institutions is also highlighted in Karlinger (2009). This paper analyzes a panel covering 45 countries from 1995 to 2000 and finds that increased competition is indeed correlated with an expansion of the underground economy. The effect is strongest in low-tax, high-corruption countries that do not provide the public services which make it worthwhile for firms to remain official despite growing competitive pressure.

The present paper presents a simple oligopoly model of free entry and free sector choice, where the size of the underground economy is endogenously determined in equilibrium. I model the intensity of competition by the degree of market power, with the source of market power being product differentiation. Product differentiation is a primitive of the model (i.e. I do not consider the possibility of firms choosing their position in the product space, or agreeing to collude, or any other form of endogenous determination of competition), and I find that more intense competition (in the sense that products are closer substitutes, and so market power is lower) translates into a larger underground sector.

Anecdotes support the view that the underground economy may expand in response to keener competition. For example, the head of the Austrian Federal Guild of the Construction Industry (Bundesinnungsmeister des Baugewerbes), Mr. Johannes Lahofer, explains the rise in shadow economic activity in his industry by recent changes in the way public building contracts are assigned, referring in particular to the introduction of compulsory tenders, and the obligation to assign the contract to the lowest-price bid (article in "Kurier" of October 4, 2004)

These new regulations prevent local authorities from discriminating against certain firms (and favoring others) when offering a building contract, and forces them to take all interested construction firms into account. Applying the concept of competition used in this paper, we can say that the new laws rendered the construction industry more competitive by imposing full substitutability of all firms from the point of view of the (institutional) buyer. The ensuing increase in shadow-economic activity in the construction industry is therefore consistent with the line of reasoning laid out above, which suggested precisely that outcome. A forward-looking government should have anticipated this effect, and should have monitored the industry more closely to keep the shadow activities in check.

The paper proceeds as follows: Section 2 presents the model setup. Section 3 solves for the equilibria of the model, and studies their properties. Section 4 discusses modifications and extensions of the benchmark model of Section 2, and Section 5 concludes.

2 The Model Setup

There are two types of agents: firms and the tax authority. Their behavior and decision variables are characterized as follows.

2.1 The firms

There is an industry with a (very large) pool of potential firms. These firms are ex-ante perfectly identical, and play the following two-stage game:

Stage 1: Each firm decides (simultaneously with all other firms) whether to enter the official economy, or to enter the underground economy, or to stay out.[2] One can think of the outside option as a non-entrepreneurial activity. Let us normalize the payoff of the outside option to zero.

The choice between the official and the underground economy is irreversible, and I model it as one between two different "production technologies", which are characterized as follows:

(i) Production costs: If firm i operates in the official economy, its total production cost as function of its output q_i is

$$C_o(q_i) = c_o q_i + C_E \tag{1}$$

while the total production cost of a firm j operating in the underground economy is

$$C_u(q_j) = c_u q_j \tag{2}$$

Denote by $C_E \geq 0$ the entry-regulation cost (red tape) of the official firm, which has to be sunk at stage 1 in order to enter the official economy. Let C_E be smaller than monopoly profits of an official firm, so that the industry is viable. Assume for simplicity that there is no other fixed cost of entry in either sector.

The term $c_o \in (0,1)$ denotes (constant) marginal production cost of the official firm, while $c_u < c_o$ represents marginal cost when operating in the underground economy. The wedge between c_o and c_u can have different sources: If the firm operates in the underground economy, it can avoid payroll taxes for its

[2] Note that I treat the decision to operate in the underground economy as an "all-or-nothing" choice, i.e. I do not allow for a single firm to "split" its operations between the official and the underground sector. This assumption simplifies the analysis considerably. I will come back to this issue in Section 4.

workers, can defy environmental or other regulations which increase the cost of production, and avoid the administrative costs associated with tax compliance itself (like keeping records, registering workers with the social security authority etc.).

(ii) Auditing: Every firm will be audited by the tax authority with a probability α (where α is common knowledge among all firms). If audited, an agent operating in the underground economy will be detected with certainty and has to pay a fine F; for an agent who operates in the official economy, the audit will remain without consequences, i.e. I assume that the tax authority never makes mistakes (see next section for a discussion of the tax authority and the properties of α and F).

Stage 2: Given that at stage 1, a total number n of firms entered the industry, out of which a share of $1-\mu$ decided to operate in the official economy (while μ operate in the underground economy), at stage 2 the firms will simultaneously choose prices. Then, markets clear, and profits are realized; the tax authority audits a fraction α of all firms, and the underground firms that are caught will be convicted to pay the fine $F > 0$.

Competition among the firms is imperfect in the sense that goods are horizontally differentiated, and each firm produces one variety.

Consumers' valuation for a variety does not depend on how this variety was produced, i.e. whether it was produced in the official or in the underground economy: Consumers may not be able to verify how the good was produced, or if they know, they do not perceive any (vertical) quality difference between goods in the official and the underground sector.[3]

Specifically, consumer demand for variety i, q_i, is characterized by

$$q_i\left(p_i, p_{-i}\right) = \max\left\{\frac{1}{n}\left(1 - p_i\left(1+\gamma\right) + \frac{\gamma}{n}\sum_{j=1}^{n} p_j\right), 0\right\} \tag{3}$$

where p_i is the price chosen by firm i, p_{-i} is the vector of competitors' prices $(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$, and n is the total number of firms operating in the market.

The parameter $\gamma \geq 0$, which will be crucial for the analysis, measures the (symmetric) degree of substitutability (and hence the intensity of competition) between any two varieties i and j; if $\gamma = 0$, the two varieties are completely independent (hence each firm behaves as a monopolist facing demand $q_i\left(p_i\right) = \frac{1}{n}\left(1-p_i\right)$), if γ is large, the two varieties are perceived as close substitutes (and hence competition between the two firms will be very fierce).

This demand function is linearly decreasing in own price, linearly increasing in the average price level (i.e. competitors' prices), and normalized by n, the total number of varieties in the industry. This function has the advantage of

[3] Note that this assumption also implies that consumers do not face any risk of consuming goods produced in the underground economy, i.e. I exclude the possibility of joint legal responsibility of consumer and producer once a firm in the underground economy is caught.

being algebraically convenient, and allows us to capture "competition" (in the sense of sensitivity of own demand to rivals' prices) in a single, exogenous, parameter.

Among the special properties of these demand functions (3), note that the aggregate demand $Q = \sum_{i=1}^{n} q_i = 1 - \frac{1}{n} \sum_{i=1}^{n} p_i$ does not depend on the degree of substitution among the products, γ, and that in the case of price symmetry, i.e. $p_i = p$ for all $i = 1, \ldots, n$, aggregate demand does not change with the number of products n existing in the industry.[4]

2.2 The tax authority

I make the following assumptions:

The tax authority can only intervene at the end of stage 2 of the game (i.e. after firms produced and sold their output), but not at stage 1.[5] At stage 2 of the game, the tax authority cannot directly observe the prices charged (and the quantities sold) by the firms on the final good market.[6]

The tax authority can enforce full payment of the fine, i.e. no partial or total default is possible. This implies that: (i) firms must have sufficient assets to cover the fine[7], and (ii) the tax authority can seize all assets of the underground firms it detects[8].

Both the audit probability α and the fine F are exogenous from the point of view of a single firm. This assumption implies that:

(i) The audit probability α does not vary with a firm's output; in particular, an underground firm is not more likely to attract the tax authority's attention because it produces more.

(ii) F is independent of the incriminated firm's output and profits, i.e. the fine is the same for all firms, no matter what their scale of operation is.[9]

However, the expected fine, αF, is allowed to vary with the aggregate share of underground firms in the industry, μ. In particular, assume that $\alpha F(\mu)$ is some continuous function of μ. I do not make any assumptions about the exact shape of $\alpha F(\mu)$ (it may be constant, increasing, decreasing, or non-monotonic in

[4] For a discussion of the derivation and properties of this demand function, see Shubik and Levitan (1980) and Motta (2003).

[5] Recall that firms entering the official sector pay entry regulation cost C_E. Thus, their number and identity becomes immediately observable to the tax authority. Underground firms, however, cannot be distinguished from non-entering firms until they become active, i.e. produce (and sell) a strictly positive quantity at stage 2.

[6] As we will see later, the prices charged by underground firms will differ systematically from those of official firms; thus, if the tax authority could observe these prices, it could easily identify the underground firms, and the detection probability would have to be 1.

[7] This will be the case if firms have revenues from activities outside of the industry considered in this model, or if the fine is (partially) non-pecuniary (e.g. prison sentences, reputational penalties).

[8] This assumption may not always be satisfied in practice, where underground firms may just shut down their premises and "disappear" when they are caught.

[9] In practice, tax authorities set fines based on rule-of-thumb estimates of turnover (since actual turnover is not verifiable) which comes very close to an exogenous fine.

μ) and about the determinants of the tax authority's behavior (such as resource or informational constraints, revenue targets, etc.) that could give rise to such a function.[10]

3 Equilibria and Their Properties

I will now identify the subgame-perfect pure-strategy equilibria of the game described above.

3.1 Equilibrium in the Product Market (stage 2)

Moving backwards, let us first solve for the equilibrium of the price-choice stage. Given that at stage 1, a total number n of firms entered the market, out of which $n(1-\mu)$ firms decided to operate in the official economy (while $n\mu$ operate in the underground economy), a firm i which decided to operate in the official economy will maximize its gross profits as follows:

$$\max_{p_i} \left\{ (p_i - c_o)\, q_i(p_i, p_{-i}) \right\} \tag{4}$$

while a firm j that opted for the underground economy has to solve

$$\max_{p_j} \left\{ (p_j - c_u)\, q_j(p_j, p_{-j}) \right\} \tag{5}$$

where $q_i(p_i, p_{-i})$ and $q_j(p_j, p_{-j})$ are defined as in equation (3)[11].

The first-order conditions read

$$\begin{aligned} (p_i - c_o)\frac{\partial q_i(p_i, p_{-i})}{\partial p_i} + q_i(p_i, p_{-i}) &= 0 \\ (p_j - c_u)\frac{\partial q_j(p_j, p_{-j})}{\partial p_j} + q_j(p_j, p_{-j}) &= 0 \end{aligned}$$

where $\frac{\partial q_i(p_i, p_{-i})}{\partial p_i} = -\frac{1}{n}\left(1 + \gamma - \frac{\gamma}{n}\right)$.

Let us impose symmetry among the $n(1-\mu)$ firms which operate in the official economy (i.e. all firms in this sector charge the same price, p_o) and among the $n\mu$ firms that operate in the underground economy (which will all charge p_u). Thus, $\sum_{l=1}^{n} p_l = n(1-\mu)\, p_o + n\mu p_u$.

After inserting this term into the demand function (3), I solve the first-order conditions for p_o^* and p_u^* (the equilibrium prices charged by the typical firm in

[10] We will see below that the properties of $\alpha F(\mu)$ are decisive for the type of equilibria that can arise in this game.

[11] Note that at this stage, i.e. conditional on having opted for the underground sector, the threat of detection has no influence on the firm's behavior anymore. This is due to the assumption that αF is independent of $q_j(p_j, p_{-j})$, implying that second-stage (price) choices will be unaffected by the expected fine.

the official and underground economy, respectively) to obtain:

$$p_o^*(n,\mu) = \frac{\left(2+2\gamma-\frac{\gamma}{n}\right)\left(1+c_o\left(1+\gamma-\frac{\gamma}{n}\right)\right)+\left(c_u-c_o\right)\gamma\left(1+\gamma-\frac{\gamma}{n}\right)\mu}{\left(2+2\gamma-\frac{\gamma}{n}\right)\left(2+\gamma-\frac{\gamma}{n}\right)} \tag{6}$$

and

$$p_u^*(n,\mu) = \frac{\left(2+2\gamma-\frac{\gamma}{n}\right)\left(1+c_u\left(1+\gamma-\frac{\gamma}{n}\right)\right)+\left(c_o-c_u\right)\gamma\left(1+\gamma-\frac{\gamma}{n}\right)(1-\mu)}{\left(2+2\gamma-\frac{\gamma}{n}\right)\left(2+\gamma-\frac{\gamma}{n}\right)} \tag{7}$$

The first-order conditions also imply that

$$q_i(p_i,p_{-i}) = -(p_i-c_o)\frac{\partial q_i(p_i,p_{-i})}{\partial p_i} \text{ and } q_j(p_j,p_{-j}) = -(p_j-c_u)\frac{\partial q_j(p_j,p_{-j})}{\partial p_j}$$

so that the equilibrium quantities sold by each firm are

$$q_o^*(n,\mu) = \max\left\{\frac{1}{n}(p_o^*-c_o)\left(1+\gamma-\frac{\gamma}{n}\right),0\right\} \tag{8}$$

and

$$q_u^*(n,\mu) = \max\left\{\frac{1}{n}(p_u^*-c_u)\left(1+\gamma-\frac{\gamma}{n}\right),0\right\} \tag{9}$$

Finally, gross profits in the price-choice equilibrium are

$$\Pi_o(n,\mu) = \left\{\begin{array}{cc} \frac{1}{n}(p_o^*-c_o)^2\left(1+\gamma-\frac{\gamma}{n}\right) & \text{if } q_o^*>0 \\ 0 & \text{if } q_o^*=0 \end{array}\right. \tag{10}$$

and

$$\Pi_u(n,\mu) = \left\{\begin{array}{cc} \frac{1}{n}(p_u^*-c_u)^2\left(1+\gamma-\frac{\gamma}{n}\right) & \text{if } q_u^*>0 \\ 0 & \text{if } q_u^*=0 \end{array}\right. \tag{11}$$

The following Lemma highlights some of the properties of the product market equilibrium:

Lemma 1: *In equilibrium, firms operating in the underground economy:*
(i) charge a lower price than firms in the official economy, i.e. $p_u^* < p_o^*$
(ii) have higher mark-ups than firms in the official economy, i.e. $p_u^* - c_u > p_o^* - c_o$
(iii) make larger gross profits than official firms, i.e. $\Pi_u > \Pi_o$
Proof: see Appendix A

Note that the price and profit relations described in Lemma 1 are entirely driven by the fact that underground firms produce at a lower marginal cost than official firms. The resulting cost advantage is partly passed on to consumers (through lower prices), partly retained by the underground firms (through higher markups).

3.2 Equilibrium at the Entry-Stage of the Game (stage 1)

Recall that equilibrium prices, $p_o^*(n,\mu)$ and $p_u^*(n,\mu)$, as well as gross profits, $\Pi_o(n,\mu)$ and $\Pi_u(n,\mu)$, are all functions of n and μ, which will be determined simultaneously at stage 1 of the game.

Any equilibrium of the first stage will have to satisfy the following conditions:

(1) (free entry) None of the inactive firms could make strictly positive net profits by entering the industry;

(2) (breaking even) None of the active firms makes losses (i.e. none of them would strictly prefer to remain inactive);

(3) (free sector choice) None of the firms active in one sector could make higher net profits by switching to the other sector.

More formally, we can define a subgame-perfect, pure-strategy equilibrium of the first stage of the game as a pair (μ^*, n^*) such that[12]:

(i) "Coexistence Equilibria": If firms are active in both the official and the underground sector of the industry, i.e. if $\mu^* \in (0,1)$, then (μ^*, n^*) must solve

$$\Pi_o(\mu, n; \cdot) - C_E = \Pi_u(\mu, n; \cdot) - \alpha F(\mu) = 0$$

(ii) "Pure Official Equilibria": If all active firms operate in the official sector, and no firm is active in the underground economy, i.e. $\mu^* = 0$, then $(\mu^* = 0, n^*)$ must solve

$$\Pi_u(\mu, n; \cdot) - \alpha F(\mu) \le \Pi_o(\mu, n; \cdot) - C_E = 0$$

(iii) "Pure Underground Equilibria": If all active firms operate in the underground sector, and no firm is active in the official economy, i.e. $\mu^* = 1$, then $(\mu^* = 1, n^*)$ must solve

$$\Pi_o(\mu, n; \cdot) - C_E \le \Pi_u(\mu, n; \cdot) - \alpha F(\mu) = 0$$

Which of these equilibria will actually arise depends on how the threat of detection plays out against the higher marginal cost and entry cost of operating in the official economy.

Proposition 2: *The game described above*

(i) has at least one subgame-perfect pure-strategy equilibrium (this may be a coexistence equilibrium, or a pure official or pure underground equilibrium);

(ii) may have multiple equilibria (both pure equilibria, or multiple coexistence equilibria, or any combination of pure and coexistence equilibria).

Proof: see Appendix A

Intuitively, if the expected fine is very high (e.g. so high that even an underground monopolist's gross profits do not cover the expected fine), firms

[12] To simplify the analysis, I will treat both n and $n\mu$ as real numbers, even though they are of course constrained to be positive integers. Thus, the equilibria described and analyzed in the following are in fact just quasi-equilibria.

will be fully deterred from entering the underground economy, and we will only see official firms operating. Conversely, if enforcement is close to inexistent, then all firms will operate underground. If, instead, the expected fine is somewhere in-between, so that not all firms will want to be in the same sector, we obtain coexistence equilibria.

Figure 1 illustrates such a coexistence equilibrium. First, consider all pairs $(\hat{\mu}, \hat{n})$ that set net profits in the official sector equal to zero.[13] By Lemma 1, the underground firm's gross profits, evaluated at any such $(\hat{\mu}, \hat{n})$ must be strictly positive (the dotted line in Figure 1). Now, a pair $(\hat{\mu}, \hat{n})$ constitutes an entry-stage equilibrium whenever it sets the underground firm's net profits $\Pi_u(\mu, n; \cdot) - \alpha F(\mu)$ equal to zero as well. The solid line in Figure 1 shows one of many possible shapes that the expected fine can take. For low values of μ, the expected fine is low (below gross underground profits), and so more firms will enter the underground sector, until $\Pi_u(\mu, n; \cdot)$ intersects with $\alpha F(\mu)$. At this point, the entry-stage game reaches an equilibrium. Any further entry into the underground sector would lead to all firms in this sector making ever higher expected losses, so that there cannot be another equilibrium.

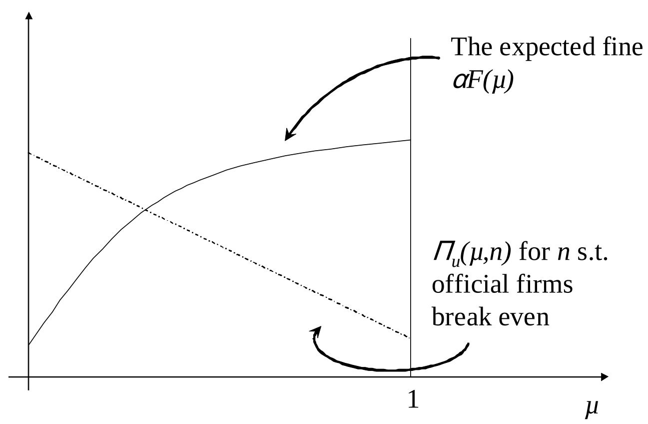

Figure 1: Unique Stable Coexistence Equilibrium

The number of equilibria that our game has will depend on the shape of the expected fine $\alpha F(\mu)$. Since we have not restricted the shape of $\alpha F(\mu)$, there may be multiple values of $\hat{\mu}$ at which $\Pi_u(\hat{\mu}, \hat{n}; \cdot)$ intersects with $\alpha F(\hat{\mu})$,

[13] Since $\Pi_o(\mu, n; \cdot) - C_E$ is continuously decreasing in both μ and n, higher values of $\hat{\mu}$ must be associated with lower values of $\hat{n}$.

including $\mu = 0$ and $\mu = 1$. Any such intersection represents an equilibrium, and hence we can have any arbitrary number of equilibria, and any combination of pure and coexistence equilibria. Figure 2 shows an example where several coexistence equilibria exist, while Figure 3 illustrates the case where both pure and coexistence equilibria exist.

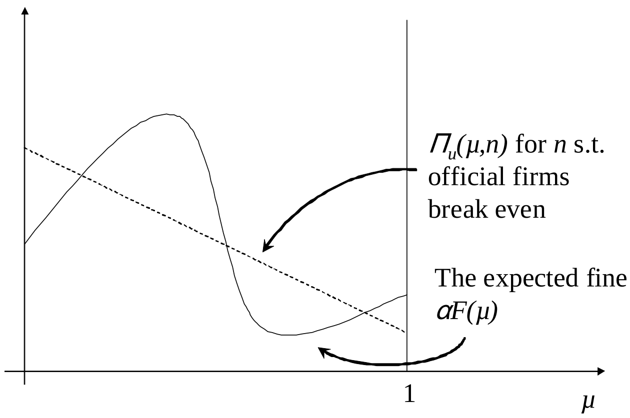

Figure 2: Multiple (Stable and Unstable) Coexistence Equilibria

This multiplicity of equilibria recalls a widely held view that countries with a large underground economy are simply trapped in a bad equilibrium where government cannot raise enough revenue to provide the kind of public services (in particular state-guided contract enforcement mechanisms) that could induce firms to move to the official sector and pay the taxes that are needed to fund such services (Posner (1996)). Note, however, that in my model, the tax revenues raised from the official sector do not flow back to the industry in any way; there is no benefit from paying taxes, other than avoiding the threat of punishment by the tax authority. Thus, Proposition 2 shows that the "trap" may even arise in a setup where there is no link between tax revenues and the quality of public services.

3.3 Comparative Statics

Recall that our objective was to evaluate the impact of intensity of competition, represented by parameter γ, on the size of the underground economy. For this purpose, let us restrict attention to the coexistence equilibria, that is equilibria

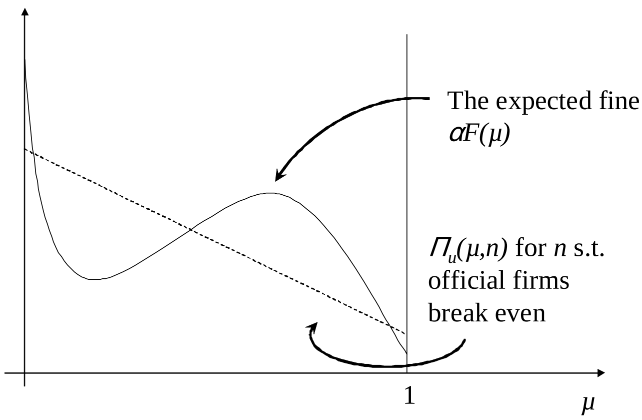

Figure 3: Pure Official, Pure Underground, and Several Coexistence Equilibria

where firms are active in both the official and the underground sector of the industry.

More formally, let $\alpha F(\mu)$ be such that there exists at least one pair (μ^*, n^*), where $\mu^* \in (0,1)$ and $n^* > 1$, solving the equilibrium conditions

$$\Pi_o(\mu, n; \gamma, \cdot) - C_E = \Pi_u(\mu, n; \gamma, \cdot) - \alpha F(\mu) = 0$$

Starting from such an equilibrium, let the competition parameter γ vary slightly. Then, this change will affect the firms' gross profits in both sectors (gross profits will decrease if γ increases), and so μ and n will have to adapt accordingly to allow the industry to settle at a new equilibrium. Proposition 3 tells us in which direction this change in μ will go.

Proposition 3: *If the coexistence equilibrium is stable, then the equilibrium share of firms operating in the underground economy, μ^*, is increasing in the intensity of competition γ. In other words, as the industry becomes more competitive, firms will be more likely to operate in the underground economy.*
Proof: see Appendix A

Thus, we find that an increase in competition raises the share of underground firms in the total number of firms. However, the standard measure for the size of the underground sector (see Introduction) refers to the output of the underground sector as a share of official GDP. Now, in my model, both the number of underground firms and their output are endogenous variables that vary with γ, so that our result with respect to μ^* does not automatically imply that the relative output of the underground sector is increasing in γ as well.

To verify that our model replicates this stylized fact as well, denote by s^* the total output of all underground firms as share of the total output of official firms[14]:

$$s^*(\mu^*, n^*) = \frac{\mu^* n^* q_u^*(\mu^*, n^*)}{(1-\mu^*) n^* q_o^*(\mu^*, n^*)}$$

Corollary 4: *If the coexistence equilibrium is stable, then the total output of the underground sector relative to the official sector, s^*, is increasing in the intensity of competition γ.*

Proof: see Appendix A

To summarize: When deciding which sector to enter, firms face a trade-off: In the official sector, they make lower gross profits than underground firms, and have to pay the entry-regulation cost; in the underground economy, however, they face the risk of detection and punishment. If, in equilibrium, firms are active in both the official and the underground economy, then the share of firms in either sector will exactly balance this trade-off. Now, as competition becomes more intense, markups in both sectors of the industry will drop, but markups in the official sector will drop faster, thus shifting the balance in favor of the underground economy.

4 Discussion and Extensions

The result derived above relies on two key features of the model:

(i) Operating in the underground economy allows firms to produce at lower marginal cost than firms in the official sector;

(ii) Firms' product-market (i.e. price) choices can be separated from the entry and sector choices (sequential decision making) and from all considerations regarding the risk of detection (the expected fine is independent of an underground firm's price or profits).

The results obtained in the previous section are robust to several modifications of the setup:

- allowing for product market competition in quantities instead of prices
- introducing (flat) taxation of official firms' profits; this creates additional incentives for firms to go underground
- introducing additional fixed cost (physical setup costs) in both sectors on top of the entry-regulation cost that firms have to pay to enter the official economy

[14] This variable will help us link our model to the empirical analysis in the following section.

- allowing the detection probability α to depend on these physical setup costs (to incorporate the idea that the larger the facilities required for production, the more "visible" a firm will be, and the more difficult it will find it to hide its operations from the tax authority)

- parameterizing market size (where market size is captured by the intercept of the demand function, and was set to 1 in the analysis above)

- allowing for different functional forms of the demand function (note that both the existence of equilibria and the comparative statics rely on the continuity of gross profits in all parameters, and the signs of the corresponding partial derivatives, not on the specific functional form assumed for demand)

Some of the assumptions in the model may seem strong and deserve a more thorough discussion:

(i) Recall that in this model, the term competition refers to a firms' ability to price above marginal cost. This is not the only sense in which this term can be used; "competition" may refer to both market structure and market outcome. As for market structure, we may think of competition as being restricted by the presence of entry barriers (in particular administrative barriers like trade licenses), which reduce the number of firms that can enter the (official) industry, and which may entice entrepreneurs into "bypassing" them by offering their goods or services without the required permits. Thus, if we equate low competition with high entry barriers, we should expect low competition to be associated with a high incidence of shadow-economic activity.

However, it is important to distinguish between the entry aspect and firms' behavior in the market after entry. Once entry decisions have been made, firms may compete fiercely in the sense that they charge prices close to marginal cost, or they may enjoy market power, that is they may be able to raise prices well above marginal cost without losing all their buyers to their competitors

(ii) The assumption that underground firms operate at a lower marginal cost than their competitors in the official sector is of course more likely to be met in an industry where the efficient minimum scale is rather small. Otherwise, official firms have a clear advantage over underground firms, because they have easier access to external financing that allows them to make the necessary investments. In such an industry, underground firms are stuck at an inefficiently low scale, producing at a higher (rather than lower) marginal cost than their competitors in the official sector.

(iii) The assumption that the fine F is independent of the incriminated firm's output and profits may seem unrealistic, because, in practice, enforcement authorities tend to tailor the punishment "to fit the crime". For instance, tax authorities may set fines according to an estimate of the amount of taxes evaded.

Yet, the scope for variable fines may be limited for several reasons. First, to make an estimate of taxes evaded, the firms's profits would have to be verifiable, which may not always be the case.[15] Second, apart from evading taxes, avoiding

[15] Recall the assumption that prices and quantities are not directly observable by the tax authority; it could be the case that even after investigating an underground firm and finding the evidence necessary for conviction, the firm's output and profits remain non-verifiable (though potentially observable).

compliance with labor and environmental laws may be an important motivation for operating underground. Yet, this damage is more difficult to quantify and to translate into monetary terms, and so fixed-fee punishments are more likely applied to these types of infringements.

(iv) Another feature of the model that may raise concerns is the "all-or-nothing" nature of the sector choice. In practice, there are many firms that split their operations between the two sectors, a decision that cannot arise in the model considered so far.

However, a simple illustration will show that my model prediction holds good even in a very different setting, where I allow for both types of operations within the same firm: Consider a perfectly competitive industry, where each firm behaves as a price taker. Each firm chooses the total output q it wants to sell at the going market price. Each unit of output can be produced in one of two ways: either "officially", i.e. using declared inputs, in which case marginal cost is some convex function $c_o(q)$; or "underground", i.e. using undeclared inputs, which is associated with convex marginal cost $c_u(q)$.

Interpret $c'_o(q) > 0$ as an inherent property of the production technology, which may be due to short-run capacity constraints, while $c'_u(q) > c'_o(q)$ reflects the combined effect of the technological constraints and the threat of detection and punishment, which I assume to be increasing in the underground output. Let $c_u(q = 0) < c_o(q = 0)$, so that, for very low levels of output, producing "underground" is unambiguously more profitable for the firm.

Suppose that the two marginal cost curves intersect at some output level, call it $q^* > 0$, so that for all $q > q^*$, the benefits of underground production (payroll tax evasion etc.) are outweighed by the increasing risk of detection. Then, the firm's short-run supply curve is the lower envelope of these two marginal-cost functions, that is, the firm will produce part of its output (up to q^*) using undeclared inputs, and any q exceeding q^* using declared inputs.

In equilibrium, our firm will produce the q that solves

$$p = C'(q) \text{ where } C'(q) \equiv \begin{cases} c_u(q) & \text{for } q \leq q^* \\ c_o(q) & \text{for } q > q^* \end{cases}$$

Suppose that initially the equilibrium price was high enough to induce the firm to produce more than q^*, i.e. to have some positive official output. Next, assume that there is a negative shock to the equilibrium price, i.e. the price falls to $p' < p$. This could be the result of a drastic cut in tariffs which allowed more efficient foreign firms access to the domestic market, or some other exogenous event that makes the environment for domestic firms more "competitive".

Then, the firm will reduce its output to the level which solves $p' = C'(q)$. Note that the first units of output that will be "crowded out" are the officially produced ones; only when output falls even below q^* will the firm start reducing its underground operations as well. In either case, the ratio of underground to official output will increase, and if all firms are symmetric, then the industry-wide underground economy will have grown in size.

5 Conclusion

This paper develops a simple oligopoly model of price competition with differentiated goods to analyze equilibrium outcomes of the decision to operate underground. In this model, the individual firm can freely choose whether to enter the official or the underground sector, and the intensity of competition in the industry is captured by a single parameter that represents the homogeneity of product varieties. A tax authority monitors the industry and imposes fines on those underground firms that it detects.

When deciding which sector to enter, firms face a trade-off: In the official sector, they incur higher marginal costs of production, and have to pay the entry-regulation cost; in the underground economy, however, they face the risk of detection and punishment. I first show that in equilibrium, firms operating in the underground economy charge a lower price than firms in the official economy, but earn higher mark-ups than firms in the official economy, which implies that they make larger gross profits than official firms.

Next, I show that the entry game has at least one subgame-perfect pure-strategy equilibrium, and that it may have multiple equilibria (where the size of the underground economy can be anything from 0 to 100 percent). My main result is that as the industry becomes more competitive (in the sense that the firms' product varieties become closer substitutes), a larger share of firms will operate in the underground economy: Competition reduces profit margins in the official sector faster than in the underground sector, thus increasing the temptation to go underground. This result also carries over to the standard measure of the size of the underground economy, i.e. the output of the underground sector as a share of official GDP.

Several issues are raised by this paper that deserve further investigation: One key element of the model is the tax authority's behavior, which is taken as given in my model without looking into its determinants. Another issue to investigate are the welfare effects of underground activity. On the one hand, underground firms evade taxes and fail to comply with labor and environmental regulations, thus generating considerable social costs. On the other hand, their presence exerts downward pressure on the prices charged by official firms, which benefits consumers. I would need to make precise assumptions on the weights of these effects in the social welfare function to draw firm conclusions.

References

[1] Friedman, Eric, Simon Johnson, Daniel Kaufmann and Pablo Zoido-Lobaton. 2000. "Dodging the Grabbing Hand: The Determinants of Unofficial Activity in 69 Countries". Journal of Public Economics, 76(3): 459-493.

[2] Goldberg, Pinelopi K. and Nina Pavcnik. 2003. "The Response of the Informal Sector to Trade Liberalization". Journal of Development Economics, 72(2): 463-96.

[3] Gupta, Satya Dev and Nanda K. Choudhry. 1997. "Dynamics of Globalization and Development: An Introduction", in Gupta, Satya Dev (ed.), Dynamics of Globalization and Development. Boston/Dordrecht/ London: Kluwer Academic Publishers.

[4] Johnson, Simon, Daniel Kaufmann and Pablo Zoido-Lobaton. 1998. "Regulatory Discretion and the Unofficial Economy". American Economic Review, 88(2): 387-392.

[5] Karlinger, Liliane. 2009. "The Underground Economy in the Late 1990s: Evading Taxes, or Evading Competition?". World Development, 37(10): 1600-1611.

[6] Kaufmann, Daniel and Aleksander Kaliberda. 1996. "Integrating the Unofficial Economy into the Dynamics of Post Socialist Economies: A Framework of Analyses and Evidence". World Bank Policy Research Working Paper 1691.

[7] Lemieux, Thomas, Bernard Fortin and Pierre Frechette. 1994. "The Effect of Taxes on Labor Supply in the Underground Economy". American Economic Review, 84(1): 231-254.

[8] Motta, Massimo. 2004. "Competition Policy: Theory and Practice". Cambridge, UK: Cambridge University Press.

[9] OECD. 2002. "Measuring the Non-Observed Economy - A Handbook". Paris: OECD Publications.

[10] Posner, Eric A. 1996. "The Regulation of Groups: The Influence of Legal and Nonlegal Sanctions on Collective Action". University of Chicago Law Review, 63: 133-197.

[11] Schneider, Friedrich and Dominik H. Enste. 2000. "Shadow Economies: Size, Causes, and Consequences". Journal of Economic Literature, 38(1): 77-114.

[12] Schneider, Friedrich. 2005. "Shadow Economies Around the World: What Do We Really Know?". European Journal of Political Economy, 21(3): 598-642.

[13] Shleifer, Andrei. 2004. "Does Competition Destroy Ethical Behavior?". American Economic Review, 94 (2): 414-418.

[14] Shubik, Martin and Richard E. Levitan. 1980. "Market Structure and Behavior". Cambridge, MA: Harvard University Press.

6 Appendix - Proofs

Proof of Lemma 1: (i) Subtracting $p_u^*(n,\mu)$ from $p_o^*(n,\mu)$ we obtain: $p_o^* - p_u^* = \frac{(c_o - c_u)(\gamma n - \gamma + n)}{(2n + 2\gamma n - \gamma)} > 0$.

(ii) Rearrange $p_u^* - c_u > p_o^* - c_o$ to have $p_o^* - p_u^* < c_o - c_u$. Then, recall from (i) that $p_o^* - p_u^* = (c_o - c_u)\frac{\gamma n - \gamma + n}{2n + 2\gamma n - \gamma}$. Given that $\gamma > 0$ and $n > 0$, we must have $\frac{\gamma n - \gamma + n}{2n + 2\gamma n - \gamma} < 1$, from which we can conclude that $p_o^* - p_u^* < c_o - c_u$.

(iii) This follows immediately from (ii): Gross profits are determined by squared markups, and $p_u^* - c_u > p_o^* - c_o$ for all parameter values.□

Proof of Proposition 2: (i) For a "pure official equilibrium" to exist, we must have:

$\Pi_o(\mu = 0, n; \cdot) = C_E$. Call the n that solves this equation n_o. (Such an n_o will always exist, as I assumed viability of the industry, i.e. $\Pi_o(\mu = 0, n = 1; \cdot) > C_E$.) Then, evaluate the underground firm's net profit at $(\mu = 0, n_o)$; if $\Pi_u(\mu = 0, n_o; \cdot) - \alpha F(\mu = 0) \leq 0$, there exists a "pure official equilibrium", with $(\mu^*, n^*) = (0, n_o)$.

For a "pure underground equilibrium" to exist, we must have:

$\Pi_u(\mu = 1, n; \cdot) - \alpha F(\mu = 1) = 0$. Call the n that solves this equation n_u. Then, evaluate the official firm's net profit at $(\mu = 1, n_u)$; if $\Pi_o(\mu = 1, n_u) - C_E \leq 0$, there exists a "pure underground equilibrium", with $(\mu^*, n^*) = (1, n_u)$.

Now, suppose that neither a "pure official equilibrium" nor a "pure underground equilibrium" exist, i.e. $\Pi_u(\mu = 0, n_o; \cdot) - \alpha F(\mu = 0) > 0$ and $\Pi_o(\mu = 1, n_u) - C_E > 0$. (If no $n_u \geq 1$ exists that solves $\Pi_u(\mu = 1, n; \cdot) - \alpha F(\mu = 1) = 0$, assume that $\Pi_o(\mu = 1, n) - C_E > 0$ holds for $n = 1$, which implies strengthening the viability assumption.) Then, note that Π_o is strictly decreasing in n; thus, there must be an $n > n_u$, call it n_1, that solves $\Pi_o(\mu = 1, n) - C_E = 0$. Since Π_u is decreasing in n as well, and $n_1 > n_u$, an underground firm's profits evaluated at $(\mu = 1, n_1)$ must be negative, i.e. $\Pi_u(\mu = 1, n_1; \cdot) - \alpha F(\mu = 1) < 0$.

Next, consider all pairs (μ, n) that set official firms' net profits equal to zero. Since Π_o is strictly and continuously decreasing in both μ and n, there will be a unique n for each value of $\mu \in [0, 1]$ such that official firms' profits are exactly equal to zero. In other words, $\Pi_o(\mu, n) - C_E = 0$ defines an implicit function $n(\mu)$ that is continuously decreasing in μ.

Likewise, Π_u is continuous in $(\mu, n(\mu))$ as defined above, and will therefore take on any value between $\Pi_u(\mu = 0, n_o; \cdot)$ and $\Pi_u(\mu = 1, n_1; \cdot)$ as we let μ run from 0 to 1. Recall that at $(\mu = 0, n_o)$, the underground firm's gross profits are strictly larger than $\alpha F(\mu = 0)$, while at $(\mu = 1, n_1)$, the underground firm's net profits are strictly smaller than $\alpha F(\mu = 1)$. By continuity of $\alpha F(\mu)$ in μ, $\alpha F(\mu)$ will take on all values between $\alpha F(\mu = 0)$ and $\alpha F(\mu = 1)$ as we let μ run from 0 to 1. Hence, there must be at least one $\mu \in (0, 1)$ such that $\Pi_u(\mu, n(\mu))$ and $\alpha F(\mu)$ intersect. Denote this value by μ^*, and denote $n(\mu^*)$ by n^*.

Then, at (μ^*, n^*), we have $\Pi_u(\mu^*, n^*) - \alpha F(\mu^*) = 0$, and by construction of $n(\mu)$, we also have $\Pi_o(\mu^*, n^*) - C_E = 0$. Therefore, we found a pair (μ^*, n^*)

that satisfies the equilibrium conditions for a coexistence equilibrium, which proves that if neither of the two pure equilibria exists, there must be at least one coexistence equilibrium.

(ii) Note first that the conditions for existence of a "pure official equilibrium" and of a "pure underground equilibrium" may be satisfied simultaneously, i.e. $\alpha F(\mu)$ and C_E may be such that both $\Pi_u(\mu = 0, n_o; \cdot) - \alpha F(\mu = 0) \leq 0$ and $\Pi_o(\mu = 1, n_u) - C_E \leq 0$ hold.

Moreover, no matter if the two pure equilibria exist or not (or only one of them exists), there is nothing that prevents $\Pi_u(\mu, n(\mu))$ as defined above from intersecting more than once with $\alpha F(\mu)$ as μ runs from 0 to 1. To see this, recall that I have not imposed any restrictions on the shape of $\alpha F(\mu)$ (other than continuity in μ); now, while $\Pi_u(\mu, n(\mu))$ can be shown to be monotonically increasing in μ, $\alpha F(\mu)$ need not be monotonic in μ, thus allowing for more than one intersection with $\Pi_u(\mu, n(\mu))$. In fact, the number of coexistence equilibria can be arbitrarily large: Define $\alpha F(\mu)$ to be exactly equal to $\Pi_u(\mu, n(\mu))$ for some or all μ on the interval $[0, 1]$ to obtain infinitely many coexistence equilibria.□

Proof of Proposition 3: Let (μ^*, n^*) be a coexistence equilibrium, so that $\Pi_o(\mu^*, n^*; \gamma) - C_E = 0$ and $\Pi_u(\mu^*, n^*; \gamma) - \alpha F(\mu^*) = 0$ both hold. Then, we can take the total differential of both equations at solution (μ^*, n^*) to have:

$$\begin{aligned} d\{\Pi_o(\cdot) - C_E\} &= \frac{\partial \Pi_o}{\partial \gamma} d\gamma + \frac{\partial \Pi_o}{\partial \mu} d\mu + \frac{\partial \Pi_o}{\partial n} dn \\ d\{\Pi_u(\cdot) - \alpha F(\mu^*)\} &= \frac{\partial \Pi_u}{\partial \gamma} d\gamma + \left(\frac{\partial \Pi_u}{\partial \mu} - \frac{\partial \alpha F}{\partial \mu} \right) d\mu + \frac{\partial \Pi_u}{\partial n} dn \end{aligned}$$

Note that if $d\mu$ and dn represent adjustments of μ and n to a new equilibrium, following a change in γ, we must have

$$\begin{aligned} d\{\Pi_o(\cdot) - C_E\} &= 0 \text{ and} \\ d\{\Pi_u(\cdot) - \alpha F(\mu^*)\} &= 0 \end{aligned}$$

These two equations allow us to solve for $\frac{d\mu^*}{d\gamma}$, i.e. the change in the equilibrium share of firms in the underground economy relative to the change in the competition parameter γ, which yields:

$$\frac{\partial \mu^*(\cdot)}{\partial \gamma} = \frac{\frac{\partial \Pi_u}{\partial n}\frac{\partial \Pi_o}{\partial \gamma} - \frac{\partial \Pi_o}{\partial n}\frac{\partial \Pi_u}{\partial \gamma}}{\left(\frac{\partial \Pi_u}{\partial \mu} - \frac{\partial \alpha F}{\partial \mu} \right)\frac{\partial \Pi_o}{\partial n} - \frac{\partial \Pi_u}{\partial n}\frac{\partial \Pi_o}{\partial \mu}}$$

Now, to evaluate the sign of this expression, first note that the following inequalities apply: Both the official and the underground firm's gross profits are decreasing in μ, n and γ; the underground firm's profits drop faster than the official firm's profits when μ or n increases, while the opposite is true for an increase in γ:

$$\frac{\partial \Pi_o(\cdot)}{\partial \mu} < 0, \frac{\partial \Pi_o(\cdot)}{\partial n} < 0, \frac{\partial \Pi_o(\cdot)}{\partial \gamma} < 0$$

$$\frac{\partial \Pi_o(\cdot)}{\partial \mu} - \frac{\partial \Pi_u(\cdot)}{\partial \mu} = 2\frac{1}{n}\left(1+\gamma-\frac{\gamma}{n}\right)\frac{\partial p_u^*}{\partial \mu}\left[(p_o^*-c_o)-(p_u^*-c_u)\right] > 0$$

because $2\frac{1}{n}\left(1+\gamma-\frac{\gamma}{n}\right) > 0$, $\frac{\partial p_u^*}{\partial \mu} = \frac{\partial p_o^*}{\partial \mu} < 0$, and $(p_o^*-c_o)-(p_u^*-c_u) < 0$ by Lemma 1.

$$\frac{\partial \Pi_o(\cdot)}{\partial n} - \frac{\partial \Pi_u(\cdot)}{\partial n} = 2\frac{1}{n}\left(1+\gamma-\frac{\gamma}{n}\right)\left[(p_o^*-c_o)\frac{\partial p_o^*}{\partial n}-(p_u^*-c_u)\frac{\partial p_u^*}{\partial n}\right] +$$
$$+\left(-\frac{1}{n^2}\right)\left(1+\gamma-\frac{2\gamma}{n}\right)\left[(p_o^*-c_o)^2-(p_u^*-c_u)^2\right] > 0$$

because $2\frac{1}{n}\left(1+\gamma-\frac{\gamma}{n}\right) > 0$, $(p_o^*-c_o)\frac{\partial p_o^*}{\partial n}-(p_u^*-c_u)\frac{\partial p_u^*}{\partial n} > 0$ by Lemma 1 and $\frac{\partial p_o^*-\partial p_u^*}{\partial n} > 0$, $\left(-\frac{1}{n^2}\right)\left(1+\gamma-\frac{2\gamma}{n}\right) < 0$, and $(p_o^*-c_o)^2-(p_u^*-c_u)^2 < 0$ by Lemma 1.

$$\frac{\partial \Pi_o(\cdot)}{\partial \gamma} - \frac{\partial \Pi_u(\cdot)}{\partial \gamma} = 2\frac{1}{n}\left(1+\gamma-\frac{\gamma}{n}\right)\left[(p_o^*-c_o)\frac{\partial p_o^*}{\partial \gamma}-(p_u^*-c_u)\frac{\partial p_u^*}{\partial \gamma}\right] +$$
$$+\frac{1}{n}\left(1-\frac{1}{n}\right)\left[(p_o^*-c_o)^2-(p_u^*-c_u)^2\right] < 0$$

because $2\frac{1}{n}\left(1+\gamma-\frac{\gamma}{n}\right)\left[(p_o^*-c_o)\frac{\partial p_o^*}{\partial \gamma}-(p_u^*-c_u)\frac{\partial p_u^*}{\partial \gamma}\right] < \frac{1}{n}\left(1-\frac{1}{n}\right)\left[(p_u^*-c_u)^2-(p_o^*-c_o)^2\right]$ since $\frac{\partial \Pi_o(\cdot)}{\partial \gamma} < 0$ and $\frac{\partial \Pi_u(\cdot)}{\partial \gamma} < 0$, $\frac{1}{n}\left(1-\frac{1}{n}\right) > 0$, and $(p_o^*-c_o)^2-(p_u^*-c_u)^2 < 0$ by Lemma 1.

Given these inequalities, we can conclude that the numerator of $\frac{\partial \mu^*(\cdot)}{\partial \gamma}$ will be strictly positive. Thus, the sign of $\frac{\partial \mu^*(\cdot)}{\partial \gamma}$ is determined by the sign of its denominator. Now, if the denominator is positive, this is equivalent to having:

$$-\frac{(\partial \Pi_u/\partial \mu - \partial \alpha F/\partial \mu)}{\partial \Pi_u/\partial n} < -\frac{\partial \Pi_o/\partial \mu}{\partial \Pi_o/\partial n}$$

The left-hand side of this inequality is equivalent to the $[dn/d\mu]_u < 0$ that solves $d\{\Pi_u(\cdot) - \alpha F(\mu^*)\} = 0$ when γ is kept constant, i.e., $[dn/d\mu]_u$ identifies the locus in the (μ, n) space along which the underground firm's profits are unchanged. The right-hand side of the inequality is the corresponding $[dn/d\mu]_o < 0$ that solves $d\{\Pi_o(\cdot) - C_E\} = 0$ when γ is kept constant.

Note that the equilibrium pair (μ^*, n^*) is the intersection of the two loci $[dn/d\mu]_u$ and $[dn/d\mu]_o$. Now, if $[dn/d\mu]_u < [dn/d\mu]_o$, then this implies that anywhere on the $[dn/d\mu]_o$ locus to the right of μ^*, i.e. where $\mu > \mu^*$, the underground firms would make negative profits, thus inducing them to leave the underground sector until μ is back to its equilibrium value. (If the underground firms were instead to make positive profits, further entry into the underground sector would occur, until the industry settles at a new, pure underground, equilibrium).

Likewise, anywhere on the $[dn/d\mu]_o$ locus to the left of μ^*, i.e. where $\mu < \mu^*$, the underground firms would make positive profits, thus inducing more firms to enter the underground sector until μ is back to its equilibrium value.

In other words, if $[dn/d\mu]_u < [dn/d\mu]_o$, this means that the coexistence equilibrium (μ^*, n^*) is stable (the industry will revert to this equilibrium after a small perturbation, rather than moving to an entirely different equilibrium); then, the denominator of $\frac{\partial \mu^*(\cdot)}{\partial \gamma}$ will be strictly positive as well, and this implies that $\frac{\partial \mu^*(\cdot)}{\partial \gamma} > 0$, as stated in the Proposition.□

Proof of Corollary 4: Inserting for $q_u^*(\mu^*, n^*)$ and $q_o^*(\mu^*, n^*)$ from equations (9) and (8), we can simplify s^* to read:

$$s^*(\mu^*, n^*) = \frac{\mu^*(p_u^*(\mu^*, n^*) - c_u)}{(1-\mu^*)(p_o^*(\mu^*, n^*) - c_o)}$$

Then, the first derivative of s^* with respect to γ is:

$$\frac{\partial s^*(\cdot)}{\partial \gamma} = \frac{\frac{\partial \mu^*(\cdot)}{\partial \gamma}(p_u^*(\cdot) - c_u)(p_o^*(\cdot) - c_o) + \mu^*(1-\mu^*)\left((p_o^* - c_o)\frac{\partial p_u^*}{\partial \gamma} - (p_u^* - c_u)\frac{\partial p_o^*}{\partial \gamma}\right)}{\left[(1-\mu^*)(p_o^*(\cdot) - c_o)\right]^2}$$

The sign of $\frac{\partial s^*(\cdot)}{\partial \gamma}$ will be determined by the sign of its numerator. By Proposition 3, we know that if the coexistence equilibrium (μ^*, n^*) is stable, then $\frac{\partial \mu^*(\cdot)}{\partial \gamma} > 0$. The markups of underground and official firms are strictly positive as well, and so is their product: $(p_u^*(\cdot) - c_u)(p_o^*(\cdot) - c_o) > 0$. In any coexistence equilibrium, $\mu^*(1-\mu^*) > 0$. Finally, $0 < (p_o^* - c_o) < (p_u^* - c_u)$ by Lemma 1, and $0 > \frac{\partial p_u^*}{\partial \gamma} > \frac{\partial p_o^*}{\partial \gamma}$, which implies that

$$(p_o^* - c_o)\frac{\partial p_u^*}{\partial \gamma} - (p_u^* - c_u)\frac{\partial p_o^*}{\partial \gamma} = (p_u^* - c_u)\left|\frac{\partial p_o^*}{\partial \gamma}\right| - (p_o^* - c_o)\left|\frac{\partial p_u^*}{\partial \gamma}\right| > 0$$

Thus, both numerator and denominator of $\frac{\partial s^*(\cdot)}{\partial \gamma}$ are strictly positive, implying that $\frac{\partial s^*(\cdot)}{\partial \gamma} > 0$, as stated in the Corollary.□