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A defensive investment strategy for portfolio alpha return and market risk reduction

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Introduction

Financial markets have become increasingly complex and volatile in the past years. During the recent and still on-going financial crisis, many investors, both professional and retail, have experienced great difficulties in managing investments under the violent market changes that have occurred. Most asset classes have been highly correlated one to another, particularly in periods of market downfalls, a fact which has contributed to such difficulties.

This situation has increased the risk aversion of many investors and outlined the necessity of reducing the risk of financial operations. This thesis addresses such widely perceived need and presents an investment strategy which focuses on the containment of market risk in a portfolio while pursuing a medium term positive return.

To this scope, we consider a portfolio comprised of a core-asset and a hedging tool, together with a certain quantity of available cash. The core-asset, typically a mutual fund for which price statistics is available, is selected in function of the correlation to a certain benchmark index and in consideration of the track record pertaining its capacity, if any, of outperforming this same benchmark, both of which we require to be as high as possible. In other words, we are interested in the *relative* performance of the core-asset with respect to the benchmark index rather than the absolute one. If this is positive, the asset achieves a premium return compared to the index, which is often referred to as *alpha*. This quantity depends on the fund manager's ability to beat the market. An accurate fund-picking activity is necessary in order to find assets with these desired characteristics, an activity which can be performed mining data from available databases of funds' performance.

The correlation between core-asset and benchmark index is analyzed in depth in order to decompose the asset return process into two components, the first of which given by the market and the second originated by the alpha dynamics. The latter is characterized by a lower volatility than the market component, and this is reflected in the portfolio once the hedge against the market is implemented. This correlation analysis is performed for both daily returns and returns on longer periods.

Once the core-asset has been selected, its return decomposed and the investment undertaken, we consider two different financial securities which can be traded as hedging tools, in order to eliminate the market risk from the portfolio. To this scope, in the first place we look into a relatively new type of financial security, the Exchange Traded Funds, which are engineered in order to reproduce the performance of an index with a certain proportionality factor. We consider a subclass of these funds, namely the Short ETFs, characterized by a negative factor, so that the security return is inverse to that of the index. There are however a number of difficulties in adopting these funds for hedging purposes, such as the constraint of having to trade them on a daily basis and the significant tracking errors by which they are affected.

As an alternative, we consider short positions on Futures contracts linked to the benchmark index, which perform much better as hedging tools. When adopting these, a minor quantity of cash needs to be kept available and the overall portfolio benefits from an increase in its return. Moreover, these securities may also be traded on a non-daily basis, and to the limit only at the contracts' delivery dates, as long as we are able to evaluate with sufficient accuracy the future correlation between the core-asset and the index.

The result of having implemented a correct hedge is that the market risk from the portfolio is reduced or even cancelled and that the return is directly linked to the alpha return of the core-asset. For this reason, we consider in detail the growth characteristics of such process. Moreover, the low volatility which characterizes the alpha dynamics is also reflected in the portfolio growth path and an enhancement of the risk-return profile is achieved if compared to an investment in the sole core-asset.

Finally, a number of errors arise when comparing the actual return of the portfolio with the one that was expected at the previous stage when the hedge was implemented. These errors are defined for both the daily returns and the compound ones.
The work is organized as follows. In Chapter 1 we set the mathematical framework which will be of reference for the rest of the thesis. In Chapter 2 the core-asset return is analyzed on a daily basis, whereas in Chapter 3 we consider compound returns and growth. In Chapter 4 we then introduce the Short ETFs and the Futures contracts. In Chapter 5 we implement the investment strategy on a daily basis and finally in Chapter 6 we consider non-daily hedging and compare the results of these strategies. The main mathematical formulae are summarized in Appendix A whereas the main implementation data is provided in Appendix B.

This thesis takes inspiration from a study by the same author [1] and it deepens and expands its topics as well as introducing new perspectives to the subject.

Chapter 1

Mathematical framework

In this chapter we set the mathematical framework valid throughout the thesis. All formulas exposed in the following sections and chapters will refer to this framework.

1.1 Method

We approach the study of financial securities and related stochastic processes by presenting an investment strategy from both a theoretical and numerical point of view. We will treat analytically the return processes of financial assets in the attempt to describe their general properties. When possible, we will also introduce approximations and simplifications with the scope of identifying the most important features and quantities. In parallel to the theoretical analysis, we will also perform numerical simulations on real time series.

The work is organized in such a way to emphasize the difference between two fundamental and complementary approaches to the topic. In the first place, we treat the stochastic processes from an *ex-post* point of view and consider the sample data available from past time series. We will perform statistical analysis and introduce sample central moments of the frequency distributions, such as sample means, standard deviations etc. Numerically, we will instead simulate the proposed strategy on real financial data and look into the evolution over past times of the variables introduced.

The second approach is based on the fact that in real life we will be looking into the future rather than into the past, which means that we will have to make probabilistic evaluations on the future evolution of the processes we consider. At any given time, we will rely on the sample past data in order to know what has happened in the past, but then we will consider expectations on the future from an *ex-ante* perspective. Naturally, a link between these two approaches, or in other words between past and future, needs to be identified. This point represents a general and fundamental issue of any application to stochastic processes. Theoretically, we will consider expected central moments on probability distributions, such as expected drift, variance and volatility of adapted stochastic processes defined on a filtered probability space. Numerically, we will instead project these quantities for a given period in the future.

Finally, we will also compare the results estimated from an ex-ante point of view at a given time in the past with the realizations actually occurred and considered from an ex-post perspective, which will lead us to identify the differences between these two approaches and to quantify the errors made.

1.2 Time frame

An effort has been made to treat as far as possible these topics in a homogenous way throughout the thesis, also by trying to adopt a coherent mathematical notation, which we describe as follows, starting from the time frame.
All quantities are measured in function of time $t \in [0, T]$, where T represents the time interval considered. This is discretized in N steps each representing a market trading day, supposing that $t_N = T$ falls at the end of the last market day. Within this time interval, we consider the time t_i with $i \in [1, N-1]$ as the present time, where each t_i is recorded daily at the closing of the market. The index $j \in [0, i-1]$ identifies the time intervals $[t_j, t_{j+1}]$ representing the past for which sample data is known. The index $h \in [i, N-1]$ identifies instead the time intervals $[t_h, t_{h+1}]$ representing the future to be treated in probabilistic terms (Figures 1.1, 1.2 and 1.3).

For simplicity of wording, we will identify the time t_i as the *present* time and indicate times t_j and t_h as *past* and *future* times respectively, although as previously stated it would be more correct to indicate the times *intervals* $[t_j, t_{j+1}]$ and $[t_h, t_{h+1}]$ as representative of past and future times. In fact, when $j = i-1$, $t_{j+1} = t_i$ and when $h = i$, $t_h = t_i$. However, having clarified this point, it appears convenient to adopt such simplified wording.

There will be occasions where we shall indicate the present time t_i as varying in the whole interval $[0, N-1]$, rather than in $[1, N-1]$. This is done particularly when looking into the evolution of the cash and other quantities for which it is desirable to start from time t_0 for homogeneity of treatment. When doing so the notation $j \in [0, i-1]$ is correct for all values of i except when $i = 0$. It is understood that in this particular case there is no past data to refer on, therefore j as an indicator of the past looses its meaning. The index is meaningful starting from $i = 1$.

The methodology will be the following. We will pick a present time t_i and perform statistics for all past times t_j. We will also make probabilistic evaluations of the future evolutions at times t_h and in particular at final time t_N, which will obviously depend on the data available up to time t_i. All this will be performed for each t_i by varying the present time for $i \in [1, N-1]$.

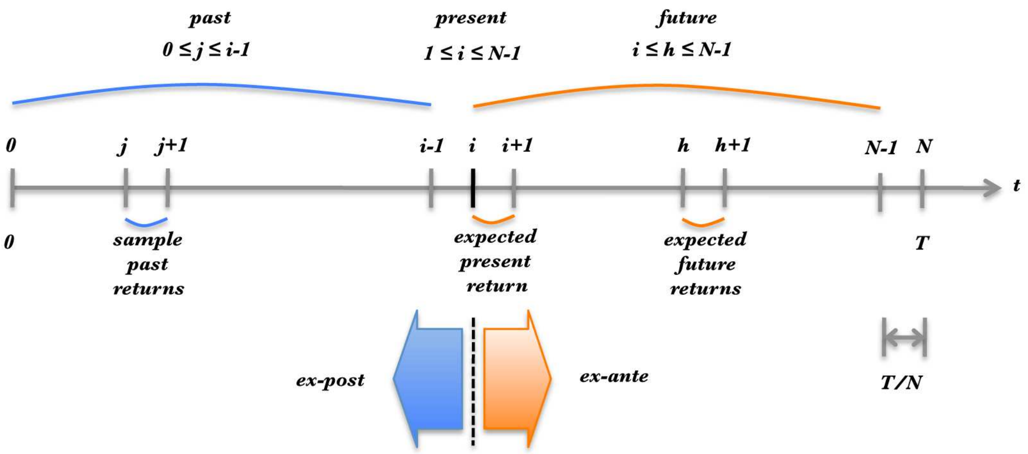


Figure 1.1: Discrete set of times and related indexes. t_i with $i \in [1, N-1]$ represents the present time, t_j with $j \in [0, i-1]$ identifies the time intervals $[t_j, t_{j+1}]$ representing the past with known data and t_h with $h \in [i, N-1]$ identifies instead the time intervals $[t_h, t_{h+1}]$ which represent the future for which probability estimations are made.

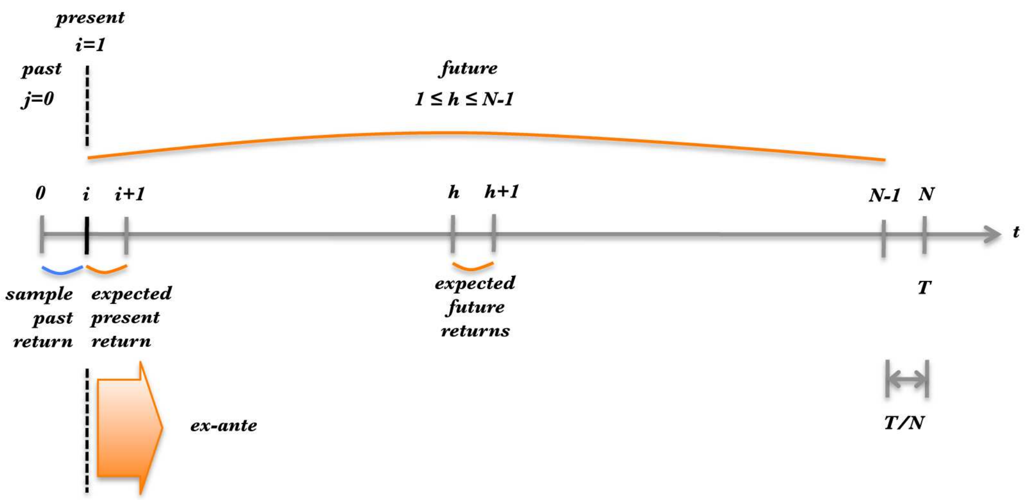


Figure 1.2: Extreme case $i = 1$

1.3 Ex-post evaluation

We now fix a present time t_i with $i \in [1, N-1]$ and consider the sample data relevant to the prices of a financial security (or index) $\{V(t_j)\}$ for all past times $j \in [0, i-1]$. Rather than working on the data set of the prices, we consider its *sample return*, defined as

$$r_V(t_{j+1}) := \frac{V(t_{j+1}) - V(t_j)}{V(t_j)} = \frac{\Delta V(t_{j+1})}{V(t_j)} \tag{1.1}$$

Sample mean. At time t_i, we define the *sample mean* return of $V(t_i)$ as the arithmetic

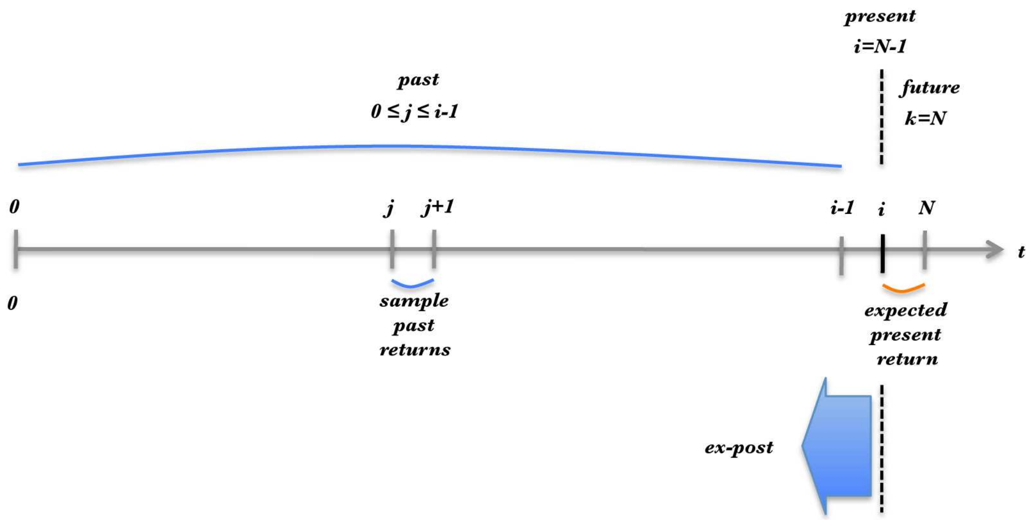


Figure 1.3: Extreme case $i = N - 1$

average of the past returns for $j \in [0, i-1]$

$$\mu_V(t_i) := \frac{1}{i}\sum_{j=0}^{i-1} r_V(t_{j+1}) \tag{1.2}$$

If we have very long time series, we might decide not to consider *all* past data, but only the most recent set of $m \in [2, i]$ data

$$\mu_V(t_i) := \frac{1}{m}\sum_{j=i-m}^{i-1} r_V(t_{j+1}) \tag{1.3}$$

which leads back to equation 1.2 when $m = i$.

Sample variance. Similarly, at time t_i we define the *sample variance* of the past returns for $j \in [0, i-1]$ and its square root the *sample standard deviation* as

$$\sigma_V^2(t_i) := \frac{1}{i-1}\sum_{j=0}^{i-1}\Big(r_V(t_{j+1}) - \mu_V(t_i)\Big)^2 \tag{1.4}$$

When introducing the same memory term $m \in [2, i]$,

$$\sigma_V^2(t_i) := \frac{1}{m-1}\sum_{j=i-m}^{i-1}\Big(r_V(t_{j+1}) - \mu_V(t_i)\Big)^2 \tag{1.5}$$

Sample covariance. We also consider at time t_i the *sample covariance* relevant to the past returns for $j \in [0, i-1]$ of $V(t_i)$ with respect to those of a different random variable $U(t_i)$, which also represents a price process for which the same quantities are similarly defined

$$\sigma^2_{VU}(t_i) := \frac{1}{i-1} \sum_{j=0}^{i-1} \Big[(r_V(t_{j+1}) - \mu_V(t_i))(r_U(t_{j+1}) - \mu_U(t_i)\Big] \tag{1.6}$$

Considering only the past m values

$$\sigma^2_{VU}(t_i) := \frac{1}{m-1} \sum_{j=i-m}^{i-1} \Big[(r_V(t_{j+1}) - \mu_V(t_i))(r_U(t_{j+1}) - \mu_U(t_i)\Big] \tag{1.7}$$

Sample return process. Wrapping up, we can express the sample return data of equation 1.1 as a process written in function of the sample mean and standard deviation

$$r_V(t_{j+1}) := \frac{\Delta V(t_{j+1})}{V(t_j)} = \mu_V(t_i) + \sigma_V(t_i)\Psi_V(t_{j+1}) \tag{1.8}$$

where $\Psi_V(t_{j+1})$ represents the distance of each sample point to the sample mean divided by the sample standard deviation.

$$\Psi_V(t_{j+1}) := \frac{r_V(t_{j+1}) - \mu_V(t_i)}{\sigma_V(t_i)}$$

It is easy to show that this term has zero mean

$$\frac{1}{i}\sum_{j=0}^{i-1} \Psi_V(t_{j+1}) = \frac{1}{\sigma_V(t_i)}\frac{1}{i}\sum_{j=0}^{i-1} r_V(t_{j+1}) - \frac{1}{\sigma_V(t_i)i}\sum_{j=0}^{i-1}\mu_V(t_i) = \frac{\mu_V(t_i)}{\sigma_V(t_i)} - \frac{\mu_V(t_i)i}{\sigma_V(t_i)i} = 0$$

$$\frac{1}{i}\sum_{j=0}^{i-1} \Psi_V(t_{j+1}) = 0 \tag{1.9}$$

and unitary variance.

$$\frac{1}{i-1}\sum_{j=0}^{i-1} \Psi_V(t_{j+1})^2 = \frac{1}{i-1}\sum_{j=0}^{i-1}\left(\frac{r_V(t_{j+1}) - \mu_V(t_i)}{\sigma_V(t_i)}\right)^2 =$$

$$= \frac{1}{\sigma_V(t_i)^2}\frac{1}{i-1}\sum_{j=0}^{i-1}\big(r(t_{j+1}) - \mu_V(t_i)\big)^2 = \frac{\sigma_V(t_i)^2}{\sigma_V(t_i)^2} = 1$$

$$\frac{1}{i-1}\sum_{j=0}^{i-1} \Psi_V(t_{j+1})^2 = 1 \tag{1.10}$$

Sample compound return. We define as *compound growth*, the ratio of the security price at time t_i with respect to the price at time t_0

$$G_V(t_0, t_i) := \frac{V(t_i)}{V(t_0)}$$

and, as *compound return* the difference between these same values divided by the initial value

$$R_V(t_0, t_i) := \frac{V(t_i) - V(t_0)}{V(t_0)} = G_V(t_0, t_i) - 1$$

These quantities may also be written un function of the sample daily returns as follows

$$G_V(t_0, t_i) = \prod_{j=0}^{i-1} \Big[1 + r_V(t_{j+1})\Big] \tag{1.11}$$

$$R_V(t_0, t_i) = \prod_{j=0}^{i-1} \Big[1 + r_V(t_{j+1})\Big] - 1 \tag{1.12}$$

Finally, we introduce the *annualized compound return* defined as

$$R_V^{\text{year}}(t_0, t_i) := \Big(1 + R_V(t_0, t_i)\Big)^{\text{days}/365} - 1 \tag{1.13}$$

1.4 Ex-ante evaluation

Again, we fix the present time t_i with $i \in [1, N-1]$ within our time frame, but now look into the possible future evolution of V and its related quantities. As discussed, we consider the future times t_h with $h \in [i, N-1]$.
The price $\{V(t_h)\}$ is now treated as an $\mathcal{F}_{t_h}$-adapted stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_{t_h}\}, \mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ a sigma-algebra of subsets of Ω, $\mathcal{P}$ a probability measure, and $\{\mathcal{F}_{t_h}\}$ is a filtration generated by $\{V(t_h)\}$ [2]. To our purposes, the sample space Ω is comprised of all the possible prices in $[0, \infty)$, whereas the sigma-algebra $\mathcal{F}$ represents all the possible subsets of prices for which the probability of outcome is given by $\mathcal{P}$.
At this point, we introduce the *expected value*, defined in terms of the *Lebesgue integral* as

$$E[V(t_h)] = \int_{\omega \in \Omega} V(t_h, \omega) d\mathcal{P}(\omega)$$

where ω represents the outcome of a single realization of $V(t_h, \omega)$ and $d\mathcal{P}(\omega)$ the probability density of such outcome. The *conditional expected value* is defined as

$$E\big[V(t_h)|\mathcal{F}_{t_i}\big] = \int_{\omega \in \Omega} V(t_h, \omega) d\big[\mathcal{P}(\omega)|\mathcal{F}_{t_i}\big]$$

The main point we wish to outline is that we establish a link between the *conditional probability* $d\big[\mathcal{P}(\omega)|\mathcal{F}_{t_i}\big]$ and the past frequencies of outcomes deriving from the *ex-post* statistical analysis.
In particular, we consider the evolution of the return $r_V(t_{h+1})$ at future times t_h with $h \in [i, N-1]$ and express it in terms of the following process

$$r_V(t_{h+1}) = E\big[r_V(t_{h+1})|\mathcal{F}_{t_i}\big] + \sqrt{E\Big[\Big(r_V(t_{h+1}) - E\big[r_V(t_{h+1})|\mathcal{F}_{t_i}\big]\Big)^2\Big|\mathcal{F}_{t_i}\Big]}\,\Psi_V(t_{h+1}) \tag{1.14}$$

where $E\big[r_V(t_{h+1})|\mathcal{F}_{t_i}\big]$ represents the *conditional drift* of the process and the *conditional variance* is given by $E\Big[\Big(r_V(t_{h+1}) - E\big[r_V(t_{h+1})|\mathcal{F}_{t_i}\big]\Big)^2\Big|\mathcal{F}_{t_i}\Big]$. The term $\Psi_V(t_{h+1})$ is a stochastic variable assumed to have zero mean and unit variance.

Conditional drift. The assumption we make is that the conditional expected value at time t_i under $\mathcal{P}$ of the future returns for $h \in [i, N-1]$, defined as *conditional drift*, is equal to the sample mean of returns at the same time t_i.

$$E\big[r_V(t_{h+1})|\mathcal{F}_{t_i}\big] = \mu_V(t_i) \tag{1.15}$$

Conditional variance and volatility. Similarly, we assume the *conditional variance* under $\mathcal{P}$ of future returns for $h \in [i, N-1]$, to equal the sample variance of past returns.

$$E\Big[\big(r_V(t_{h+1}) - E\big[r_V(t_{h+1})|\mathcal{F}_{t_i}\big]\big)^2|\mathcal{F}_{t_i}\Big] = E\Big[\big(r_V(t_{h+1}) - \mu_V(t_i)\big)^2|\mathcal{F}_{t_i}\Big] = \sigma_V^2(t_i) \tag{1.16}$$

The square root of the variance we indicate as the return process *volatility.*

Co-variance of returns. We also consider at time t_h the *conditional covariance* under $\mathcal{P}$ relevant to the expected future returns of $V(t_h)$ with respect to those of a different random variable $U(t_h)$, and again we assume this to equal the sample covariance

$$E\Big[\big(r_V(t_{h+1}) - \mu_V(t_i)\big)\big(r_U(t_{h+1}) - \mu_U(t_i)\big)|\mathcal{F}_{t_i}\Big] = \sigma_{VU}^2(t_i) \tag{1.17}$$

Return process. These assumptions, which as we have mentioned represent the fundamental problem of relating past sample data with future expectations, allow us to express from an *ex-ante* point of view at time t_i the return process for future times t_h with $h \in [i, N-1]$

$$r_V(t_{h+1}) = \mu_V(t_i) + \sigma_V(t_i)\Psi_V(t_{h+1}) \tag{1.18}$$

Expected compound return. We also consider the *conditional expected compound growth* from time t_i to the terminal time t_N, defined as

$$E\big[G_V(t_i, t_N)|\mathcal{F}_{t_i}\big] = E\Bigg[\prod_{h=i}^{N-1}\Big[1 + r_V(t_{h+1})\Big]\Big|\mathcal{F}_{t_i}\Bigg]$$

and the related *conditional expected compound return*

$$E\big[R_V(t_i, t_N)|\mathcal{F}_{t_i}\big] = E\Bigg[\prod_{h=i}^{N-1}\Big[1 + r_V(t_{h+1})\Big]\Big|\mathcal{F}_{t_i}\Bigg] - 1$$

1.5 Wiener process

In the following chapters we will consider the return processes of a financial asset $\{S\}$ and of a benchmark index $\{X\}$, with the representation described in the previous sections. Each process will be characterized by a stochastic term, Ψ_S for the asset and Ψ_X for the index, for which the only assumptions will be that they have zero mean and unit variance. The asset return process will then be decomposed into two parts, one given by the index return and one independent to it and characterized by a stochastic term indicated as $\Delta W_\alpha(t_{i+1}) = W_\alpha(t_{i+1}) - W_\alpha(t_i)$. This will be considered in first approximation as a gaussian error and will be described in terms of the difference between two subsequent values of a Wiener process.

To this scope, we now define a generic Wiener process $W(t)$ and recall its main properties [2]. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, where Ω is the sample space, $\mathcal{F}$ a sigma-algebra of subsets of Ω, $\mathcal{P}$ a probability measure. For each $\omega \in \Omega$, we consider a function $W(t)$ of $t \geq 0$ that depends on ω and satisfies the following:

- $W(t)$ is a continuos function of t
- $W(0) = 0$

This function may be defined as brownian motion if for all $0 < t_1 < \cdots < t_m$, the increments

$$W(t_1) = W(t_1) - W(t_0) \qquad W(t_2) - W(t_1) \qquad \cdots \qquad W(t_m) - W(t_{m-1})$$

are independent and each of these is normally distributed with

- $E\big[W(t_{i+1}) - W(t_i)\big] = 0$
- Var $\big[W(t_{i+1}) - W(t_i)\big] = t_{i+1} - t_i$

It also results that for all $0 \leq s < t$

$$E\big[W(t) - W(s)\big] = 0$$

$$\mathrm{Var}\big[W(t) - W(s)\big] = t - s$$

and that it is a martingale, in that

- $E[|W(t)|] < \infty$
- $E\big[W(t)|\mathcal{F}_s\big] = W(s)$

Let's now get back to our analysis of the return process. Consistently with the notation adopted, t_i for $i \in [1, N-1]$ represents the present time up to which data is available. Now we change perspective from an *ex-post* analysis of sample data available from the past times t_j with $j \in [0, i-1]$ to an *ex-ante* perspective of future values at times t_h with $h \in [i, N-1]$ treated in probabilistic terms. Again, the link between past and future is established by adopting a *frequentist* approach to probability, i.e. the statistical data from the past will determine the probability measure of futures values.
Starting from the past data available for $W(t_j)$, we assume now that its future values $W(t_h)$ can be descried by a Wiener process. If we consider for $h \in [i, N-1]$ the future daily increments

$$\Delta W(t_{h+1}) = W(t_{h+1}) - W(t_h)$$

they will be characterized by

$$E\big[\Delta W(t_{h+1})\big] = 0$$

$$\mathrm{Var}\big[\Delta W(t_{h+1})\big] = 1$$

Moreover, for all $m, n \in [i, N-1]$ with $m \neq n$ the daily increments will be independent one to another

$$E\big[\Delta W(t_m)\Delta W(t_n)\big] = E\big[\Delta W(t_m)\big] \cdot E\big[\Delta W(t_n)\big] = 0$$

As mentioned, these results will be exploited when treating the alpha dynamics in the following chapters.

In Appendix A we summarize the main formulae presented throughout the work.

Chapter 2

Core-asset daily returns

2.1 Investment strategy

2.1.1 Outline

Topic of this thesis is an investment strategy characterized by low volatility and liable to be executed not only by professional investors but also at the retail level. The portfolio through which implement such strategy is based on a core financial security, such as a mutual fund, accurately selected in function of its correlation to some index and its historic capacity of outperforming such benchmark. In addition to this security, the portfolio comprises also cash and another security used as hedging tool, traded in order to eliminate from the portfolio performance the risk and return given by the benchmark index. By doing so, the portfolio volatility results significantly lower than the core-asset one, and this will also lead to an increase of the portfolio risk-return profile with respect to the asset one. Both these results address the need for risk reduction of a wide class of investors, particularly felt in periods of violent volatility in financial markets.
We will investigate and compare in detail three different hedging strategies:

1. Adopt Short ETFs inversely replicating the benchmark index and trade them on a daily basis;

2. Enter into short positions on Futures contracts linked to the benchmark index and trade them on a daily basis.

3. Enter into the same short positions on Futures contracts but trade them on longer time frames.

2.1.2 Portfolio elements

We refer to the mathematical framework set out in Chapter 1 and consider the three different portfolios at times t_i with $i \in [1, N-1]$.

1. The first portfolio is composed of the core asset, cash and a Short ETF replicating inversely the return of the benchmark index to which the core asset is mostly correlated. The Short ETFs are traded daily and the prices and values of the elements considered are indicated as follows:

- $\{S(t_i)\}$ price of the core financial security;
- $\{X(t_i)\}$ value of the benchmark index to which $\{S(t_i)\}$ is most highly correlated;
- $\{H(t_i)\}$ price of a Short ETF on the benchmark index;
- $\{C_H(t_i)\}$ value of cash held in the portfolio;
- $\{\Pi_H(t_i)\}$ value of the portfolio.

2. The second portfolio is instead composed of the core asset, cash and a short position into a Futures contract on the benchmark index. The Futures are traded with daily frequency and for these we indicate:
   - $\{S(t_i)\}$ as above;
   - $\{X(t_i)\}$ as above;
   - $\{F(t_i)\}$ price of a Futures contract on the same index;
   - $\{\Pi_F(t_i)\}$ value of the portfolio.
3. The third portfolio has the same elements of the second but the Futures are traded on a non-daily base.

2.1.3 Sample cases

Numerical simulations on three different sample cases are presented, as shown in Table 2.1.

1. The first sample case considers as core asset a mutual fund invested on European equities and as benchmark index the Euro Stoxx 50;
2. The second refers to a US equity mutual fund with the S&P 500 benchmark index;
3. Finally, the third adopts a German equity mutual fund benchmarking against the DAX index.

The specific mutual funds considered as core assets have been selected because they are characterized, in the time frame for which the sample data is available, by a high correlation to their benchmark index and a positive premium return (*alpha*) with respect to such index, and independently from their return in absolute terms.

Both hedging strategies are presented and compared for each of the three sample cases. Starting from the price and value time series mentioned above, we will compute the corresponding daily and compound returns, for which the sample mean and standard deviation will be considered. We will also look into future estimation of these quantities by introducing the expected drift and volatility of the return processes.
Figure 2.1 shows for Sample Case 1 the growth in value of the core asset and the benchmark index together with an ideal Short ETF and a Futures contract on the same index.

Table 2.1: **Sample Cases**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
Isin code	IT0004255219	FR0010612770	LU0325630076
Management	8a Investimenti	PIM Gestion France	BNP Paribas
Fund type	Large Cap European Equity	Large Cap U.S. Equity	Large Cap German Equity
From date	14.Mar.2008	06.Jun.2008	01.Jul.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Benchmark	**Euro Stoxx 50**	**S&P 500**	**DAX**
Short ETF	Amundi ETF Short Dow Jones Euro Stoxx 50	ProShares UltraProShort S&P 500	db x-trackers Short DAX
Isin code	FR0010757781	US74347X8561	LU0292106241
Management	Amundi Investment	ProShares	DB Platinum Advisor
From date	15.Jul.2009	29.Jun.2009	01.Jul.2009
To date	18.Oct.2011	19.Oct.2011	15.Jun.2011
Futures	Euro Stoxx 50 Index Futures	S&P 500 Futures	DAX Futures
Ticker	FESX	FSP	FDAX
Isin code	DE0009652388	-	DE0008469594
Management	Eurex	CME Group	Eurex
From date	02.May.2001	07.01.2005	02.May.2001
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011

2.2 Core-asset and benchmark index returns

2.2.1 Core-asset returns

The core-asset present in the portfolio of each of the three strategies is treated as non-tradable, in the sense that it is purchased when the portfolio is being set up and no trading activity is done on the asset. The rationale for assuming so, is that, transaction costs and execution time can be significant, particularly when dealing with mutual funds. This implies that if the goal is to hedge a portfolio against market risk, then it is better to do so by trading the specific hedging tools which may be transacted on the market at low costs and in short time, rather than acting on the core asset.

For $i \in [1, N-1]$, the core asset return process may be written as

$$r_S(t_{i+1}) := \frac{S(t_{i+1}) - S(t_i)}{S(t_i)} = \frac{\Delta S(t_{i+1})}{S(t_i)} \tag{2.1}$$

and recalling equations 1.8 and 1.18 also as

$$r_S(t_{i+1}) = \mu_S(t_i) + \sigma_S(t_i)\Psi_S(t_{i+1}) \tag{2.2}$$

where Ψ_S represents a non-gaussian stochastic term with zero mean and unit variance.
As seen in Chapter 1 such representation requires the introduction of the sample mean return $\mu_S(t_i)$ and standard deviation $\sigma_S(t_i)$ as given in equations 1.2 and 1.4 (where the respective sums consider all the available sample points):

$$\mu_S(t_i) := \frac{1}{i}\sum_{j=0}^{i-1} r_S(t_{j+1}) \qquad \sigma_S^2(t_i) := \frac{1}{i-1}\sum_{j=0}^{i-1}\Big(r_S(t_{j+1}) - \mu_S(t_i)\Big)^2$$

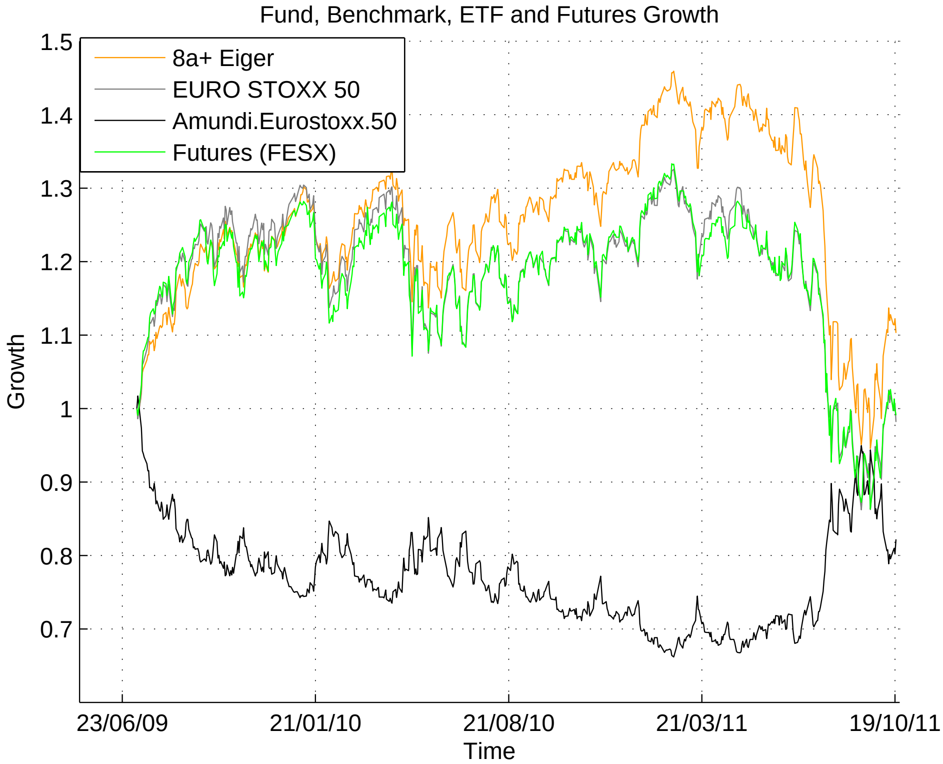


Figure 2.1: Portfolio elements for Sample Case 1: growth in value of the asset, the benchmark index, an ideal Short ETF and a Futures contract.

As an alternative, these quantities may be computed limiting the past data to a number $m \in [2, 1]$ of elements as in equations 1.3 and 1.5

$$\mu_S(t_i) := \frac{1}{m} \sum_{j=i-m}^{i-1} r_S(t_{j+1}) \qquad \sigma_S^2(t_i) := \frac{1}{m-1} \sum_{j=i-m}^{i-1} \Big(r_S(t_{j+1}) - \mu_S(t_i) \Big)^2$$

Figure 2.2 compares for the mutual fund of Sample Case 1 the daily returns $r_S(t_{i+1})$ with the corresponding sample mean $\mu_S(t_i)$ and standard deviation $\sigma_S(t_i)$. The red vertical line identifies the period after which we may consider both parameters stable, and thus determines the *memory* term m for the mentioned alternative computation method. Figure 2.3 shows instead the resulting frequency distribution for the daily returns.

In Table 2.3 we summarize the main data relevant to the performance of the mutual funds considered in the three sample cases over the implementation period of the strategy. Since we perform a comparison between core-asset, Short ETF and Futures, the implementation period considers only the time frame for which data is available for each of these securities, and allows for an initial phase of stabilization for the statistical parameters, as shown in Table 2.2. We outline that on a daily basis, mean returns μ_S are of an order of magnitude of 10^{-4} whereas the volatility is two orders of magnitude higher $\sigma_S = o(10^{-2})$.

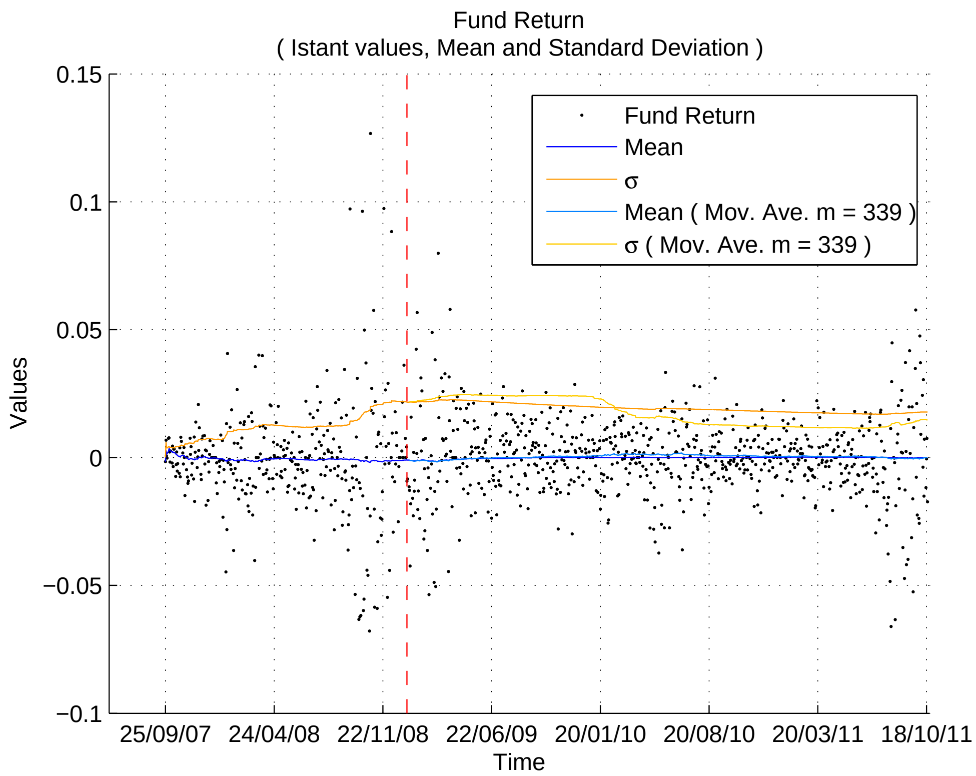


Figure 2.2: Evolution over time of the daily returns $r_S(t_{i+1})$ for Sample Case 1 fund, together with the sample mean return $\mu_S(t_i)$ and the sample standard deviation $\sigma_S(t_i)$, both computed with *full* and *partial* memory statistics. The red vertical line identifies the memory term m.

Table 2.2: **Implementation period**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
Benchmark	**Euro Stoxx 50**	**S&P 500**	**DAX**
From date	22.07.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631

Sample values also of the compound return

$$R_S(t_0, t_i) := \frac{S(t_i)}{S(t_0)} - 1 = \prod_{j=0}^{i-1} \Big(1 + r_S(t_{j+1})\Big) - 1$$

and the annual return are shown.

$$R_S^{\text{year}}(t_0, t_i) := \Big(1 + R_S(t_0, t_i)\Big)^{\text{days}/365} - 1$$

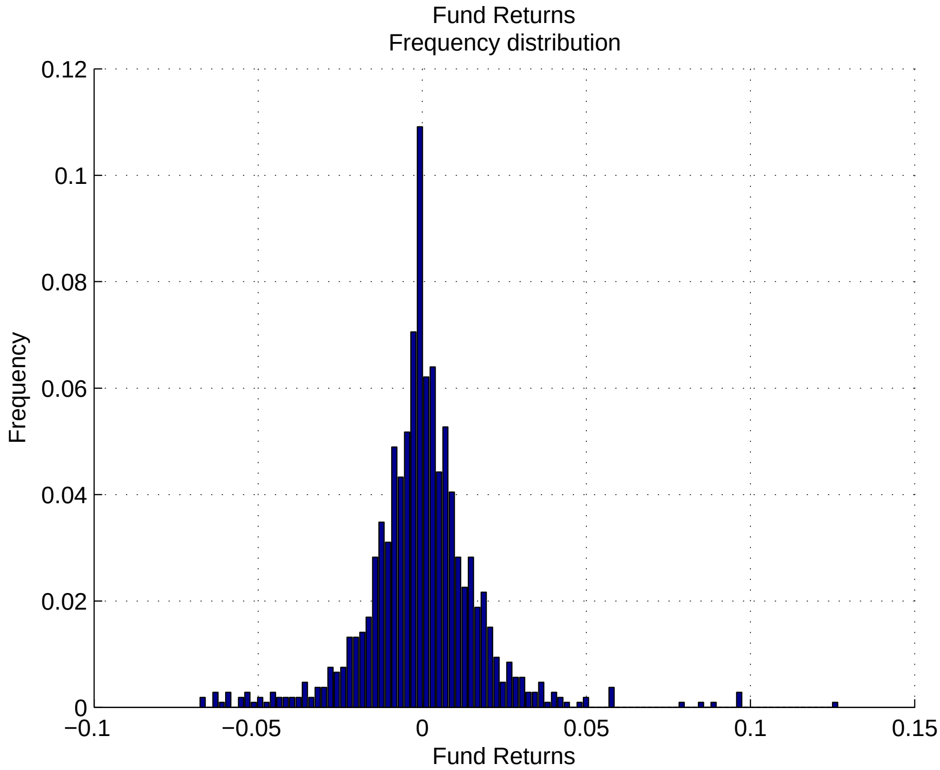


Figure 2.3: Frequency distribution of daily returns $r_S(t_{i+1})$ for Sample Case 1 fund.

2.2.2 Benchmark index returns

In a similar way, for $i \in [1, N-1]$ we define the benchmark index return process as

$$r_X(t_{i+1}) := \frac{X(t_{i+1}) - X(t_i)}{X(t_i)} = \frac{\Delta X(t_{i+1})}{X(t_i)} = \mu_X(t_i) + \sigma_X(t_i)\Psi_X(t_{i+1}) \tag{2.3}$$

where Ψ_X represents a non-gaussian stochastic term with zero mean and unit variance and for which the sample mean return $\mu_X(t_i)$ and standard deviation $\sigma_X(t_i)$ are, considering all available sample points

$$\mu_X(t_i) := \frac{1}{i}\sum_{j=0}^{i-1} r_X(t_{j+1}) \qquad \sigma_X^2(t_i) := \frac{1}{i-1}\sum_{j=0}^{i-1}\Big[r_X(t_{j+1}) - \mu_X(t_i)\Big]^2$$

and the compound growth and return are

$$G_X(t_0, t_i) := \frac{X(t_i)}{X(t_0)} = \prod_{j=0}^{i-1}\Big[1 + r_X(t_{j+1})\Big]$$

$$R_X(t_0, t_i) := \frac{X(t_i)}{X(t_0)} - 1 = \prod_{j=0}^{i-1}\Big[1 + r_X(t_{j+1})\Big] - 1 = G_X(t_0, t_i) - 1$$

Table 2.3: **Core-asset performance**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
From date	22.07.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
μ_S	1.486E-04	8.142E-04	6.061E-04
σ_S	1.422E-02	1.326E-02	9.767E-03
R_S	4.575E-02	5.158E-01	2.838E-01
Annualized R_S	2.014E-02	1.999E-01	1.552E-01
Sharpe ratio	8.956E-03	5.981E-02	6.015E-02
Sortino ratio	5.247E-04	3.551E-03	4.184E-0
RAP (benchmark $r\alpha$*)*	-1.503E-04	2.185E-05	-6.685E-06
RAP (benchmark r_X*)*	2.213E-04	1.217E-04	1.405E-04

2.2.3 Correlation between core-asset and benchmark index

Exclusively core assets that present a high correlation to some benchmark index for which hedging tools are available, are considered under this strategy. Securities which are poorly linked to the market are not suitable for this intent, since the higher is the correlation to the market and the more effective is the hedge against it. Usually this correlation is approached in terms of *linear regression* [4, 5].
Given the present time t_i for $i \in [1, N-1]$ up to which data is available, the sample returns $r_S(t_{j+1})$ and $r_X(t_{j+1})$ for past times t_j with $j \in [0, i-1]$ (Figure 2.4) are considered. The correlation between them is measured by the sample covariance, defined as

$$\sigma^2_{SX}(t_i) := \frac{1}{i-1} \sum_{j=0}^{i-1} \left[(r_S(t_{j+1}) - \mu_S(t_i))(r_X(t_{j+1}) - \mu_X(t_i) \right]$$

or in terms of the *correlation coefficient* $\rho(t_i)$ when such covariance is normalized with respect to non-null standard deviations $\sigma_S(t_i) \neq 0$ and $\sigma_X(t_i) \neq 0$

$$\rho(t_i) := \frac{\sigma^2_{SX}(t_i)}{\sigma_S(t_i)\sigma_X(t_i)}$$

Moreover, the sample returns are plotted one against the other in a cartesian graph (Figure 2.6) and each $r_S(t_{j+1})$ is represented as a linear function of the index return $r_X(t_{j+1})$ in the parameters $\alpha(t_i)$ and $\beta(t_i)$, with an accuracy given by the same $\rho(t_i) \in [0, 1]$

$$r_S(t_{j+1}) = \alpha(t_i) + \beta(t_i) r_X(t_{j+1}) + \sigma(t_i)\phi(t_{j+1}) \tag{2.4}$$

where $\phi(t_{j+1})$ is a random variable with zero mean and unit variance. The distance between each sample point $r_S(t_{j+1})$ and the mean line $\alpha(t_i) + \beta(t_i) r_X(t_{j+1})$ represents the error of approximation we incur into when assuming such linear relation. These errors are characterized by the standard deviation $\sigma(t_i)$.

Note that all statistical parameters $(\alpha, \beta, \rho, \sigma)$ are computed at present time t_i and represent

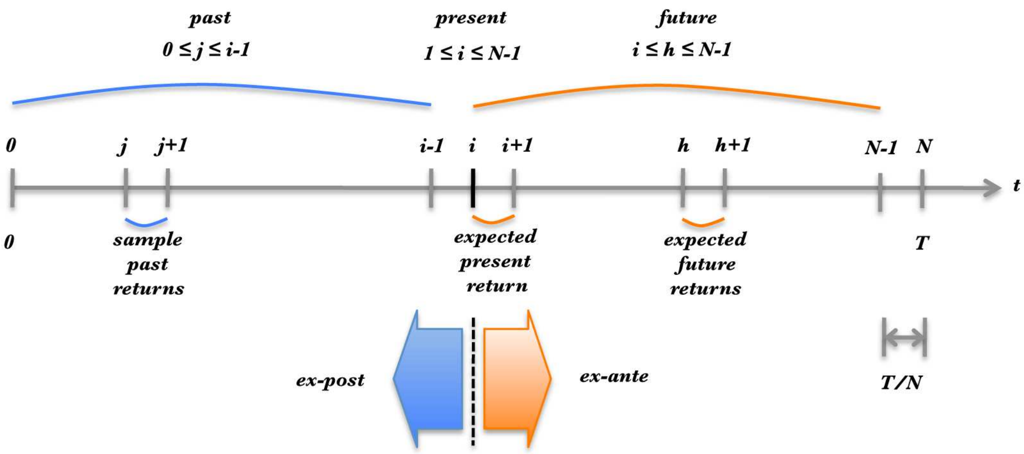


Figure 2.4: Discrete set of times and related indexes.

global values valid for each single point $[r_X, r_S]$ at past times t_{j+1} with $j \in [0, i-1]$.

2.2.4 Mean square distance minimization

Parameters $\alpha(t_i)$ and $\beta(t_i)$ are determined by minimizing the mean square distance of the sample points $r_S(t_{j+1})$ from the line $\alpha(t_i) + \beta(t_i) r_X(t_{j+1})$

$$MSD(t_i) := \frac{1}{i} \sum_{j=0}^{i-1} \Big[r_S(t_{j+1}) - \alpha(t_i) - \beta(t_i) r_X(t_{j+1}) \Big]^2$$

For clarity, we now simplify the notation substituting $[](t_i)$ with $[]_i$ and introduce:

$$x_{j+1} := r_X(t_{j+1}) \qquad y_{j+1} := r_S(t_{j+1})$$

$$\alpha_i := \alpha(t_i) \qquad \beta_i := \beta(t_i)$$

$$MSD_i := \frac{1}{i} \sum_{j=0}^{i-1} \big(y_{j+1} - \alpha_i - \beta_i x_{j+1} \big)^2$$

Before proceeding with the minimization, it is useful to express the mean, variance and covariance of the asset and benchmark returns:

$$\bar{x}_i := \frac{1}{i} \sum_{j=0}^{i-1} x_{j+1} = \frac{1}{i} \sum_{j=0}^{i-1} r_X(t_{j+1}) = \mu_X(t_i)$$

$$\bar{y}_i := \frac{1}{i} \sum_{j=0}^{i-1} y_{j+1} = \frac{1}{i} \sum_{j=0}^{i-1} r_S(t_{j+1}) = \mu_S(t_i)$$

$$\sigma_{x_i}^2 := \frac{1}{i-1} \sum_{j=0}^{i-1} \big(x_{j+1} - \bar{x}_i \big)^2 = \frac{1}{i-1} \sum_{j=0}^{i-1} \big(r_X(t_{j+1}) - \mu_X(t_i) \big)^2 = \sigma_X(t_i)^2$$

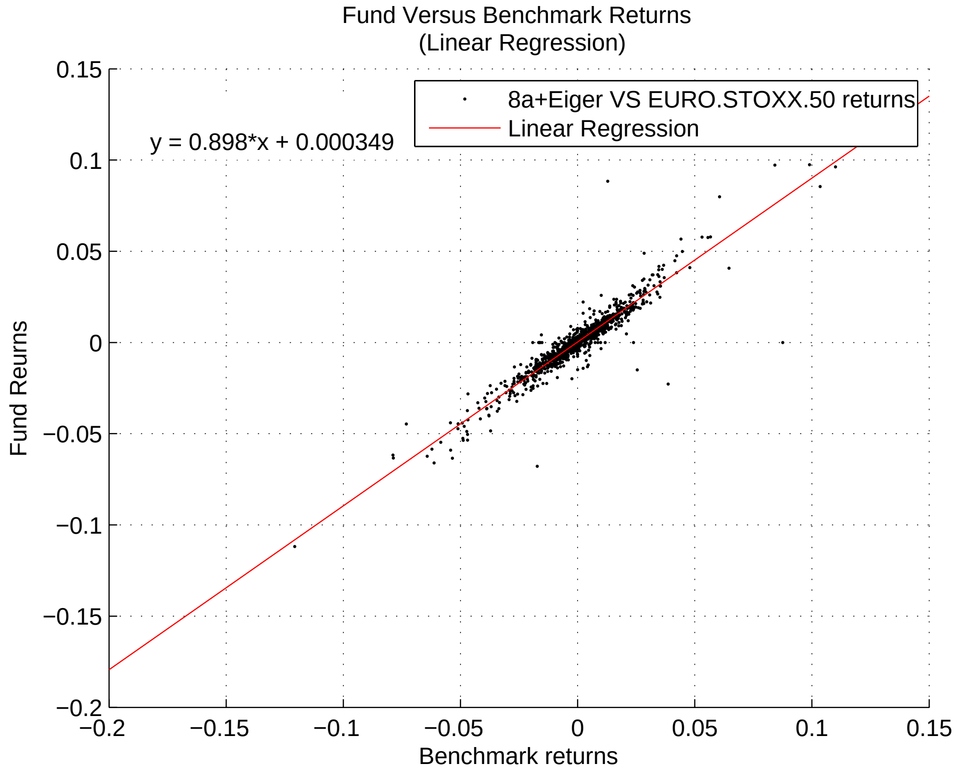


Figure 2.5: Linear regression plot between r_S and r_X for Sample Case 1.

$$\sigma^2_{y_i} := \frac{1}{i-1}\sum_{j=0}^{i-1}\left(y_{j+1}-\bar{y}_i\right)^2 = \frac{1}{i-1}\sum_{j=0}^{i-1}\left(r_S(t_{j+1})-\mu_S(t_i)\right)^2 = \sigma_S(t_i)^2$$

$$\sigma^2_{x_iy_i} := \frac{1}{i-1}\sum_{j=0}^{i-1}(x_{j+1}-\bar{x}_i)(y_{j+1}-\bar{y}_i) = \sigma_{SX}(t_i)^2$$

We also introduce the correlation coefficient between the asset and the benchmark return, defined as

$$\rho_i := \frac{\sigma^2_{x_iy_i}}{\sigma_{x_i}\sigma_{y_i}} = \frac{\sigma^2_{SX}(t_i)}{\sigma_S(t_i)\sigma_X(t_i)} = \rho(t_i)$$

Let's impose the first order necessary conditions to obtain a minimum for MSD_i in the parameters α_i and β_i.

$$\frac{\partial MSD_i}{\partial \alpha_i} = 0$$

$$\frac{\partial MSD_i}{\partial \beta_i} = 0$$

Starting with the first,

$$\frac{\partial MSD_i}{\partial \alpha_i} = -\frac{2}{i}\sum_{j=0}^{i-1}\left(y_{j+1}-\alpha_i-\beta_i x_{j+1}\right) = -2\left(\frac{1}{i}\sum_{j=0}^{i-1}y_{j+1}-\frac{i}{i}\alpha_i-\frac{\beta_i}{i}\sum_{j=0}^{i-1}x_{j+1}\right) = 0$$

$$\bar{y}_i - \alpha_i - \beta_i\bar{x}_i = 0$$

$$\alpha_i = \bar{y}_i - \beta_i\bar{x}_i \tag{2.5}$$

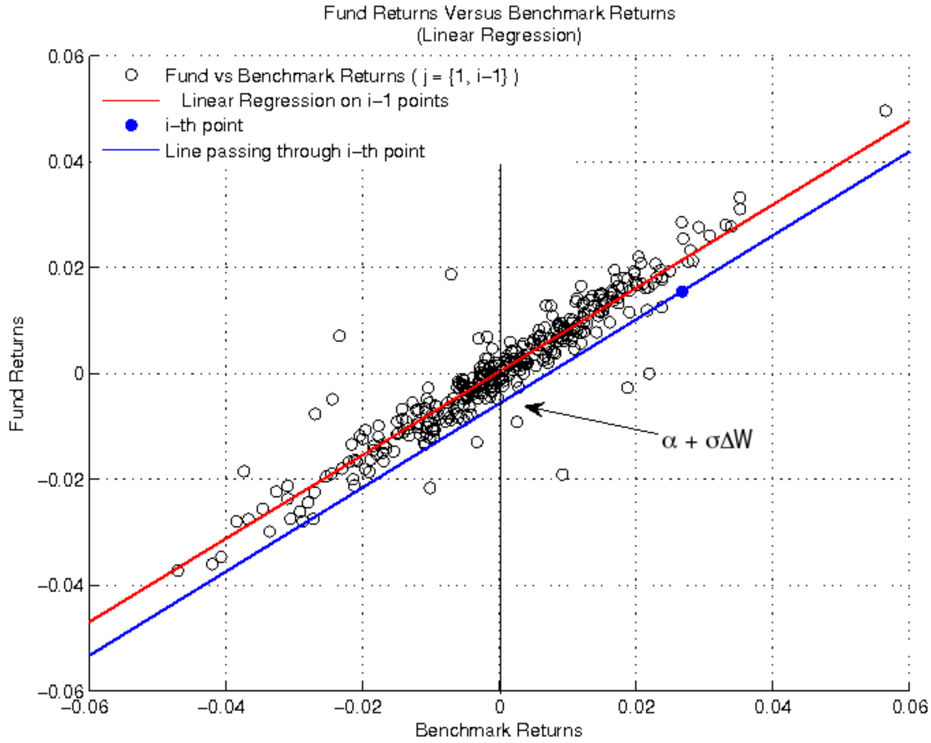


Figure 2.6: Linear Regression. Black dots represent the sample data $[r_X, r_S]$ for all past times t_{j+1}, and the red line the linear regression $r_S(t_{j+1}) = \alpha(t_i) + \beta(t_i)r_X(t_{j+1})$. A single sample point is identified by the blue circle, whereas the blue line represents $r_S(t_{j+1}) = \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1}) + \beta(t_i)r_X(t_{j+1})$.

Considering now the second,

$$\frac{\partial MSD_i}{\partial \beta_i} = -\frac{2}{i}\sum_{j=0}^{i-1}\big(y_{j+1} - \alpha_i - \beta_i x_{j+1}\big)x_{j+1} = -2\Big(\frac{1}{i}\sum_{j=0}^{i-1} x_{j+1}y_{j+1} - \frac{\alpha_i}{i}\sum_{j=0}^{i-1} x_{j+1} - \frac{\beta_i}{i}\sum_{j=0}^{i-1} x_{j+1}^2\Big) = 0 \tag{2.6}$$

To express in equation 2.6 the term $\frac{1}{i}\sum_{j=0}^{i-1} x_{j+1}^2$, we look into the variance

$$\sigma_{x_i}^2 := \frac{1}{i-1}\sum_{j=0}^{i-1}\big(x_{j+1} - \bar{x}_i\big)^2 = \frac{1}{i-1}\sum_{j=0}^{i-1}\big(x_{j+1}^2 - 2x_{j+1}\bar{x}_i + \bar{x}_i^2\big) =$$

$$= \frac{1}{i-1}\sum_{j=0}^{i-1} x_{j+1}^2 - 2\bar{x}_i\frac{1}{i-1}\sum_{j=0}^{i-1} x_{j+1} + \frac{i}{i-1}\bar{x}_i^2$$

$$\frac{i-1}{i}\sigma_{x_i}^2 = \frac{1}{i}\sum_{j=0}^{i-1} x_{j+1}^2 - 2\bar{x}_i\frac{1}{i}\sum_{j=0}^{i-1} x_{j+1} + \bar{x}_i^2 = \frac{1}{i}\sum_{j=0}^{i-1} x_{j+1}^2 - 2\bar{x}_i^2 + \bar{x}_i^2$$

$$\frac{1}{i}\sum_{j=0}^{i-1} x_{j+1}^2 = \bar{x}_i^2 + \frac{i-1}{i}\sigma_{x_i}^2$$

Now, to express in equation 2.6 the term $\frac{1}{i}\sum_{j=0}^{i-1} x_{j+1}y_{j+1}$, we look into the covariance

$$\sigma^2_{x_i y_i} := \frac{1}{i-1}\sum_{j=0}^{i-1}(x_{j+1}-\bar{x}_i)(y_{j+1}-\bar{y}_i) =$$

$$= \frac{1}{i-1}\sum_{j=0}^{i-1} x_{j+1}y_{j+1} - \frac{\bar{x}_i}{i-1}\sum_{j=0}^{i-1} y_{j+1} - \frac{\bar{y}_i}{i-1}\sum_{j=0}^{i-1} x_{j+1} + \frac{i}{i-1}\bar{x}_i\bar{y}_i$$

$$\frac{i-1}{i}\sigma^2_{x_i y_i} = \frac{1}{i}\sum_{j=0}^{i-1} x_{j+1}y_{j+1} - \frac{\bar{x}_i}{i}\sum_{j=0}^{i-1} y_{j+1} - \frac{\bar{y}_i}{i}\sum_{j=0}^{i-1} x_{j+1} + \bar{x}_i\bar{y}_i$$

$$\frac{i-1}{i}\sigma^2_{x_i y_i} = \frac{1}{i}\sum_{j=0}^{i-1} x_{j+1}y_{j+1} - 2\bar{x}_i\bar{y}_i + \bar{x}_i\bar{y}_i$$

$$\frac{1}{i}\sum_{j=0}^{i-1} x_{j+1}y_{j+1} = \frac{i-1}{i}\sigma^2_{x_i y_i} + \bar{x}_i\bar{y}_i$$

Substituting in equation 2.6 these two results,

$$\frac{\partial MSD_i}{\partial \beta_i} = -2\big(\frac{i-1}{i}\sigma^2_{x_i y_i} + \bar{x}_i\bar{y}_i - \alpha_i\bar{x}_i - \beta_i\bar{x}_i^2 - \beta_i\frac{i-1}{i}\sigma^2_{x_i}\big) = 0$$

$$\frac{i-1}{i}\sigma^2_{x_i y_i} + \bar{x}_i\bar{y}_i - \alpha_i\bar{x}_i - \beta_i\bar{x}_i^2 - \beta_i\frac{i-1}{i}\sigma^2_{x_i} = 0$$

and now using 2.5

$$\frac{i-1}{i}\sigma^2_{x_i y_i} + \bar{x}_i\bar{y}_i - \bar{x}_i\bar{y}_i + \beta_i\bar{x}_i^2 - \beta_i\bar{x}_i^2 - \beta_i\frac{i-1}{i}\sigma^2_{x_i} = 0$$

$$\frac{i-1}{i}\big(\sigma^2_{x_i y_i} - \beta_i\sigma^2_{x_i}\big) = 0$$

Finally, we obtain

$$\beta_i = \frac{\sigma^2_{x_i y_i}}{\sigma^2_{x_i}}$$

or, equivalently,

$$\rho_i := \frac{\sigma^2_{x_i y_i}}{\sigma_{x_i}\sigma_{y_i}} \qquad \sigma^2_{x_i y_i} = \rho_i\sigma_{x_i}\sigma_{y_i}$$

$$\beta_i = \rho_i\frac{\sigma_{y_i}}{\sigma_{x_i}}$$

2.2.5 Asset return decomposition

We have thus determined the linear regression parameters $\alpha(t_i)$ and $\beta(t_i)$ appearing in equation

$$r_S(t_{j+1}) = \alpha(t_i) + \beta(t_i)r_X(t_{j+1}) + \sigma(t_i)\phi(t_{j+1}) \tag{2.7}$$

Getting back to full notation, equation 2.5 determines *alpha* as equal to

$$\alpha(t_i) = \mu_S(t_i) - \beta(t_i)\mu_X(t_i) \tag{2.8}$$

which represents the average return premium achieved by the asset with respect to the index. We may also write this expression in the form

$$\beta(t_i) = \frac{\mu_S(t_i) - \alpha(t_i)}{\mu_X(t_i)}$$

to express *beta* as a proportionality factor between the average return of the fund net of its premium and the average index return. In fact, Getting back to Figure **??**, *alpha* is given by the intercept point between the vertical axis r_S and the mean line $\alpha + \beta r_X$, whereas *beta* is the angular coefficient of such mean line.

Beta is equal to

$$\beta(t_i) = \frac{\sigma^2_{SX}(t_i)}{\sigma^2_X(t_i)} = \rho(t_i)\frac{\sigma_S(t_i)}{\sigma_X(t_i)} \tag{2.9}$$

which means that it is proportional, by means of the correlation coefficient, to the ratio between the asset returns standard deviation and the index one. With high beta, large changes in the index returns will be reflected in large variations of asset returns, whereas with low values of beta, asset returns will be less affected by variations on the index returns. In this way, beta represents the sensitivity of the asset returns with respect to the index ones.

Getting back to equation 2.7, each asset return point $r_S(t_{j+1})$ is composed of three terms. The first is the average return premium $\alpha(t_i)$, the second is a term proportional to the related index return $r_X(t_{j+1})$ by means of $\beta(t_i)$ and the latter $\sigma(t_i)\phi(t_{j+1})$ is a stochastic term proper of each point, which indicates the distance between the point itself and the mean line $\alpha + \beta r_X$.

$$\sigma(t_i)\phi(t_{j+1}) = r_S(t_{j+1}) - \Big[\alpha(t_i) + \beta(t_i) r_X(t_{j+1})\Big]$$

Since this quantity is a vertical distance on the cartesian graph $[r_X, r_S]$ it can be interpreted as an additional return premium achieved by each single point, to be added to the mean premium given by alpha. In other words, we suggest to interpret such term by adding it to the alpha component of the return, and in this way to express it as a stochastic term proper of the same alpha.
If we define

$$\sigma_\alpha(t_i) := \sigma(t_i)$$

$$\Delta W_\alpha(t_{j+1}) = W_\alpha(t_{j+1}) - W_\alpha(t_j) := \phi(t_{j+1})$$

where at first approximation W_α is assumed to be a Wiener process as defined in Chapter 1, then we can express the asset return as

$$r_S(t_{j+1}) = \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1}) + \beta(t_i) r_X(t_{j+1}) \tag{2.10}$$

Under this light, each sample point $\big[r_X(t_{j+1}), r_S(t_{j+1})\big]$ lies on a different line with respect to the mean one, specifically on that line that intercepts the r_S axis in the point $\alpha(t_i) + \sigma(t_i)\Delta W_\alpha(t_{j+1})$ rather than in $\alpha(t_i)$, but which has the same inclination $\beta(t_i)$ as the mean one, as shown in Figure 2.6.

Continuing with this approach, we now introduce a new return process (*alpha dynamics*) defined as follows:

$$r_\alpha(t_{j+1}) := \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1}) \tag{2.11}$$

Recalling that the index return process is represented as

$$r_X(t_{j+1}) := \mu_X(t_i) + \sigma_X(t_i)\Psi_X(t_{j+1}) \tag{2.12}$$

this allows us to write the asset return process as decomposed into two components.

$$r_S(t_{j+1}) = r_\alpha(t_{j+1}) + \beta(t_i) r_X(t_{j+1}) \tag{2.13}$$

The first is the alpha dynamics component r_α, which is characterized by a mean return of α and a stochastic term $\sigma_\alpha \Delta W_\alpha$ which is independent to the market return under this representation. All the dependence to the market return is given by the second term βr_X, which has mean value μ_X and stochasticity given by $\sigma_X \Psi_X$. Target of the investment strategy will be to eliminate this latter term. Expanding all terms,

$$r_S(t_{j+1}) = \Big[\alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1})\Big] + \beta(t_i)\Big[\mu_X(t_i) + \sigma_X(t_i)\Psi_X(t_{j+1})\Big]$$

or

$$r_S(t_{j+1}) = \Big[\alpha(t_i) + \beta(t_i)\mu_X(t_i)\Big] + \Big[\sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1}) + \sigma_X(t_i)\Psi_X(t_{j+1})\Big]$$

where the first term indicates the drift and the second one the stochastic components. This representation leads to the definition of the asset return process in terms of its decomposition as hereby introduced.

The return sample mean is

$$\mu_S(t_i) := \frac{1}{i}\sum_{j=0}^{i-1} r_S(t_i) = \alpha(t_i) + \beta(t_i)\mu_X(t_i)$$

since, recalling equation 1.9,

$$\frac{1}{i}\sum_{j=0}^{i-1}\Psi_X(t_{j+1}) = \frac{1}{i}\sum_{j=0}^{i-1}\Delta W_\alpha(t_{j+1}) = 0$$

The process sample variance is

$$\sigma_S^2(t_i) := \frac{1}{i-1}\sum_{j=0}^{i-1}\Big[r_S(t_i) - \mu_S(t_i)\Big]^2$$

dropping for a moment the full notation

$$\sigma_S^2 = \frac{1}{i-1}\sum_{j=0}^{i-1}\Big[\sigma_\alpha \Delta W_\alpha + \beta\sigma_X\Psi_X\Big]^2 =$$

$$\frac{\sigma_\alpha^2}{i-1}\sum_{j=0}^{i-1}\Delta W_\alpha^2 + \frac{\beta^2\sigma_X^2}{i-1}\sum_{j=0}^{i-1}\Psi_X^2 + 2\frac{\beta\sigma_X\sigma_\alpha}{i-1}\sum_{j=0}^{i-1}\Delta W_\alpha\Psi_X = \sigma_\alpha^2 + \beta^2\sigma_X^2$$

having exploited equation 1.10 for which

$$\frac{1}{i-1}\sum_{j=0}^{i-1}\Delta W_\alpha^2 = \frac{1}{i-1}\sum_{j=0}^{i-1}\Psi_X^2 = 1$$

and the fact that ΔW_α and Ψ_X are independent, so that

$$\frac{1}{i-1}\sum_{j=0}^{i-1}\Delta W_\alpha\Psi_X = 0$$

Wrapping up, ad re-introducing the explicit dependance on time, the sample mean and variance of the asset return process are

$$\mu_S(t_i) = \alpha(t_i) + \beta(t_i)\mu_X(t_i) \tag{2.14}$$

$$\sigma_S^2(t_i) = \sigma_\alpha^2(t_i) + \beta^2(t_i)\sigma_X^2(t_i) \tag{2.15}$$

2.3 Alpha dynamics

2.3.1 Sample data

We have introduced the *alpha dynamics* component of the asset return defined, for $j \in [0, i-1]$ as

$$r_\alpha(t_{j+1}) := \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1})$$

Figure 2.7 shows for the mutual fund of Sample Case 1 the evolution over time of $r_\alpha(t_{j+1})$ compared to the mean $\alpha(t_i)$ and $\sigma_\alpha(t_i)$. Again, the red vertical line denotes the memory term adopted for the computation of these parameters using only the last $m \in [2, i]$ values of $r_\alpha(t_{j+1})$. By zooming in, as shown in figure 2.8, we observe that the sample mean, both with *full* and *partial* memory statistics, has a stable and positive value, as also represented in the frequency distribution graph of figure 2.9. In the asset and return processes, we have adopted non-gaussian stochastic terms with zero mean and unitary standard deviation (Ψ_S and Ψ_X). It is well known in fact that return processes are far from being gaussian and that a number of so-called *stylized facts* have been discovered relatively to their statistical characteristics []. It is not however scope of this work to present or investigate in further detail this point, also considered that once the hedge is adopted, the influence of these processes on the portfolio will be negligible.

On the other hand instead, we have defined at first approximation the alpha return process as being characterized by a gaussian stochastic term ΔW_α of unitary variance and null mean expressed as difference of two subsequent terms of a Wiener process. In this resspect we show the Q-Q plot in Figure 2.10 and provide the skewness and kurtosis values in Table **??**

Passing on to the other linear regression parameters, in Figure 2.11 we show the evolution over time of $\beta(t_i)$ and of $\rho(t_i)$. Note that both parameters show a stable behavior, which supports the interpretational choice of allocating the error terms arising in the linear regres-

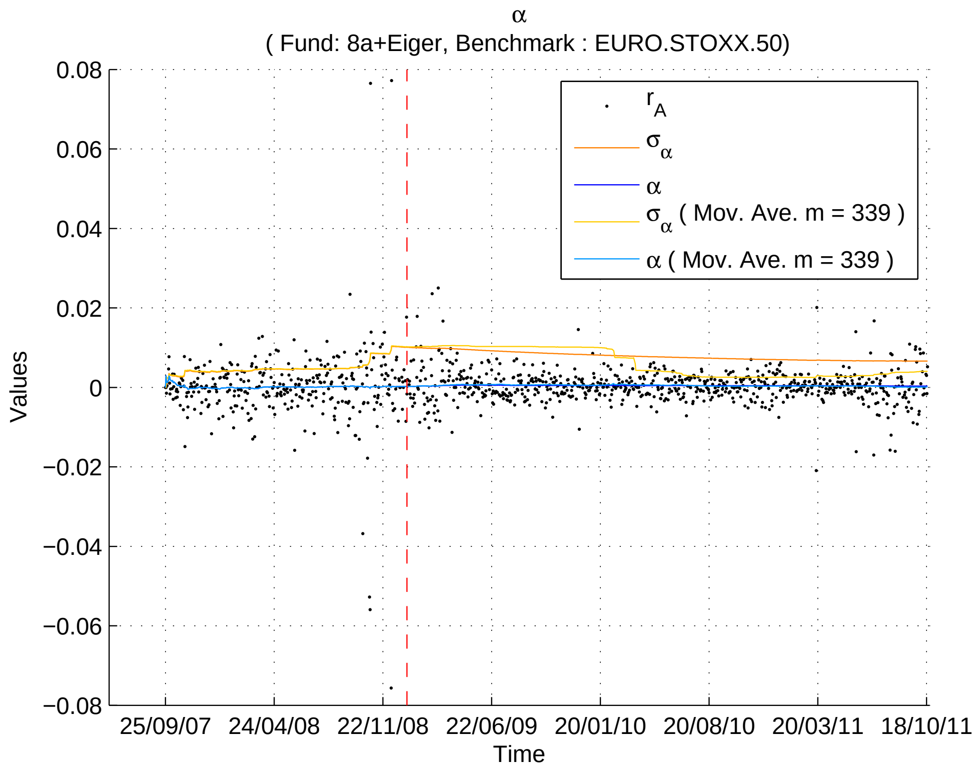


Figure 2.7: Evolution over time of $r_\alpha(t_{j+1})$ for Sample Case 1 fund, together with the sample mean return $\alpha(t_i)$ and the sample standard deviation $\sigma_\alpha(t_i)$, both computed with *full* and *partial* memory statistics. The red vertical line identifies the memory term m.

sion to the alpha dynamics, rather than considering them as possible errors occurring on beta. In table 2.4 we show the data relevant to the alpha return process for the three sample funds. In the first place, we outline that $\alpha = o(10^{-4})$ and that $\sigma_\alpha = o(10^{-3})$. Recalling that $\sigma_S = o(10^{-2})$, these funds are characterized by an alpha dynamics with positive α and volatility of an order of magnitude inferior to the fund overall one. For the first two funds, both β and ρ are close to the unit value, whereas the third fund has a lower beta.

2.3.2 Ex-post analysis

To summarize, from an ex-post point of view we have defined the alpha return process as

$$r_\alpha(t_{j+1}) := \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1})$$

where the sample mean

$$\alpha(t_i) = \mu_S(t_i) - \beta(t_i)\mu_X(t_i)$$

is obtained with the linear regression method. The sample standard deviation can be expressed starting from equation 2.15

$$\sigma_S^2(t_i) = \sigma_\alpha^2(t_i) + \beta^2(t_i)\sigma_X^2(t_i)$$

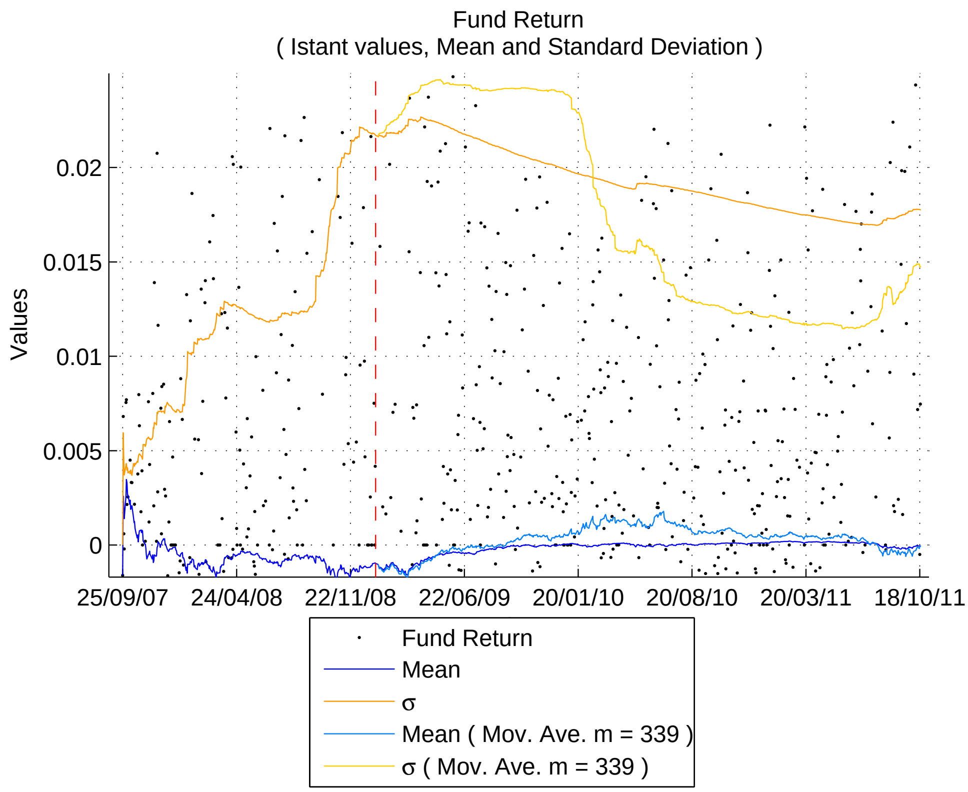


Figure 2.8: Evolution over time of $\alpha(t_i)$ and $\sigma_\alpha(t_i)$ for Sample Case 1 fund, both computed with *full* and *partial* memory statistics (zoom of figure 2.7).

$$\sigma^2_\alpha(t_i) = \sigma^2_S(t_i) - \beta^2(t_i)\sigma^2_X(t_i) = \sigma^2_S(t_i) - \rho^2(t_i)\frac{\sigma^2_S(t_i)}{\sigma^2_X(t_i)}\sigma^2_X(t_i) = \sigma^2_S(t_i)\Big[1 - \rho^2(t_i)\Big]$$

$$\sigma_\alpha(t_i) = \sigma_S(t_i)\sqrt{1-\rho^2(t_i)} \tag{2.16}$$

We can also introduce the compound return and growth for the alpha dynamics. Differently than for the other cases seen, now it is not possible to express these quantities in terms of ratio of values or prices from time t_0 to time t_i. For example, for the benchmark index we had

$$G_X(t_0,t_i) := \frac{X(t_i)}{X(t_0)} \qquad \rightarrow \qquad G_X(t_0,t_i) = \prod_{j=0}^{i-1}\Big[1 + r_X(t_{j+1})\Big]$$

The alpha return process has not been derived from a price or value process, which in fact does not exist by itself, but it originates as one component of the core-asset return process. For this reason, we define the compound growth and return in terms of each daily return only:

$$G_\alpha(t_0,t_i) := \prod_{j=0}^{i-1}\Big[1 + r_\alpha(t_{j+1})\Big]$$

$$R_\alpha(t_0,t_i) := \prod_{j=0}^{i-1}\Big[1 + r_\alpha(t_{j+1})\Big] - 1 = G_\alpha(t_0,t_i) - 1$$

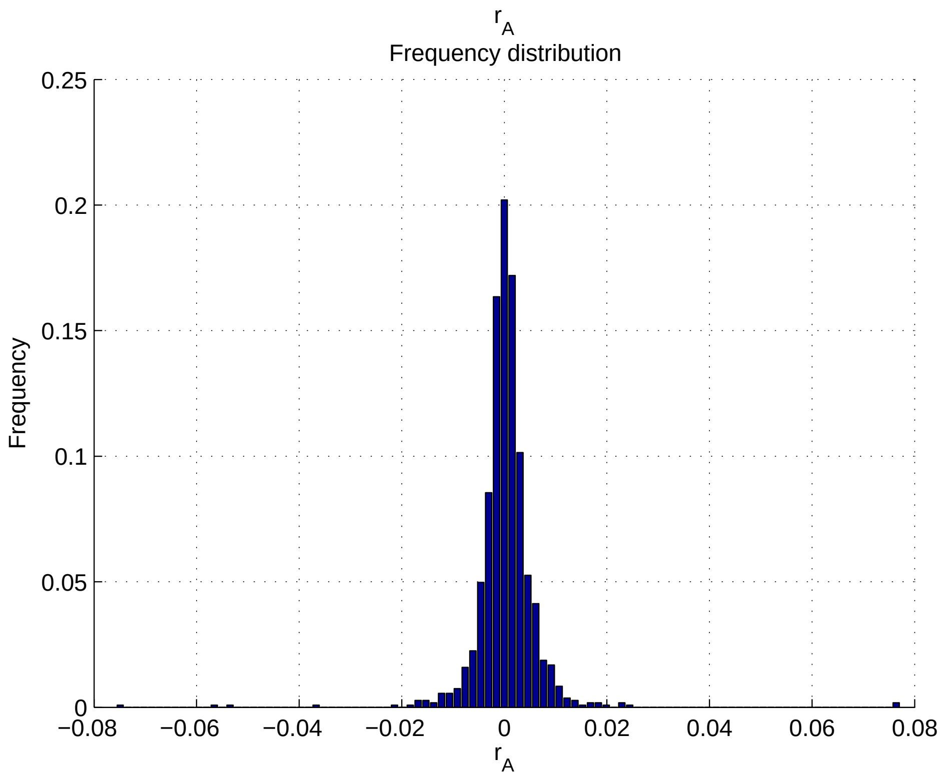


Figure 2.9: $r_\alpha(t_{j+1})$ frequency distribution for Sample Case 1 fund.

2.3.3 Ex-ante evaluation

At present time t_i with $i \in [1, N-1]$ we take a probabilistic view on the future evolution at times t_h with $h \in [i, N-1]$. Differently than before, we do not possess a value or price stochastic process directly connected to the alpha return. However, we may consider the $\mathcal{F}_{t_h}$-adapted stochastic process of the asset prices $\{S(t_h)\}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_{t_h}\}, \mathcal{P})$ considering that the alpha dynamics originates from the asset price process. As usual, Ω is the sample space, $\mathcal{F}$ a sigma-algebra of subsets of Ω, $\mathcal{P}$ a probability measure, and $\{\mathcal{F}_{t_h}\}$ is a filtration generated by $\{S(t_h)\}$). This allows us to write

$$r_\alpha(t_{h+1}) = E\big[r_\alpha(t_{h+1})|\mathcal{F}_{t_i}\big] + \sqrt{\mathrm{Var}\big[r_\alpha(t_{h+1})|\mathcal{F}_{t_i}\big]}\,\Delta W_\alpha(t_{h+1})$$

As previously done, we assume that the conditional drift and variance equal the sample mean and standard deviation respectively

$$E\big[r_\alpha(t_{h+1})|\mathcal{F}_{t_i}\big] = \alpha(t_i) \qquad \mathrm{Var}\big[r_\alpha(t_{h+1})|\mathcal{F}_{t_i}\big] = \sigma_\alpha^2(t_i)$$

so that we can express the future evolution of the index return as

$$r_\alpha(t_{h+1}) = \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{h+1}) \tag{2.17}$$

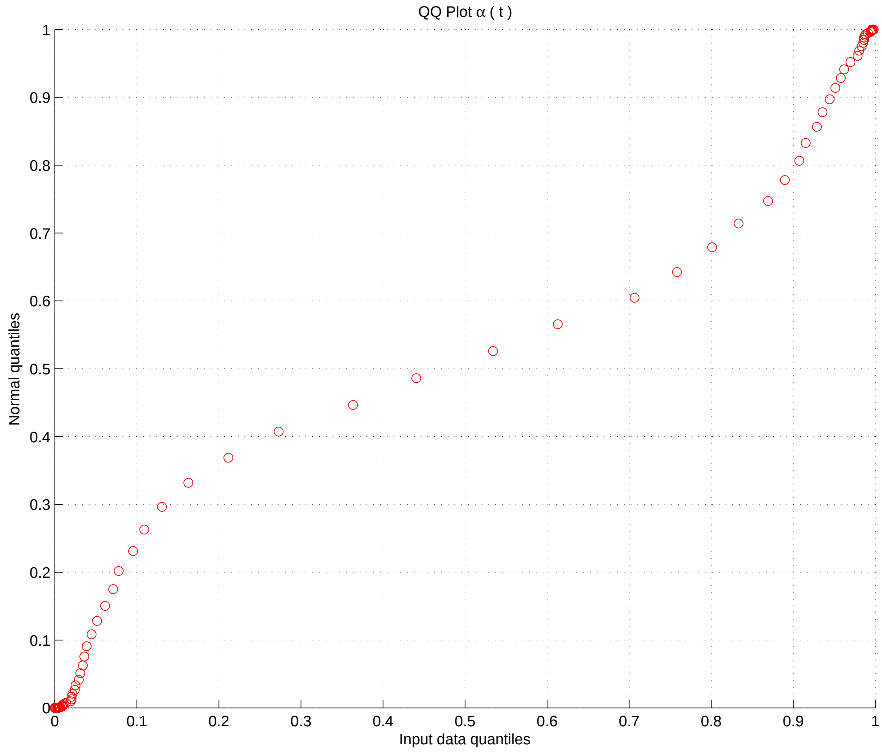


Figure 2.10: Q-Q plot for $\Delta W_\alpha(t_{j+1})$ for Sample Case 1 fund.

2.4 Estimation error

2.4.1 Ex-ante asset returns

We have decomposed the core-asset returns into two components, one given by the alpha dynamics and one given by the market returns, both of which we have analyzed from an ex-post point of view and then also from an ex-ante perspective. Before proceeding further, we are now interested in defining the asset return process from an ex-ante point of view.

To this scope, we consider at present time t_i with $i \in [1, N-1]$ the future evolution of the return process at times t_h with $h \in [i, N-1]$. Values $\{S(t_h)\}$ are seen as an $\mathcal{F}_{t_h}$-adapted stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_{t_h}\}, \mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ a sigma-algebra of subsets of Ω, $\mathcal{P}$ a probability measure, and $\{\mathcal{F}_{t_h}\}$ is a filtration generated by $\{S(t_h)\}$. Under such perspective, the index return process is given by

$$r_S(t_{h+1}) = E\big[r_S(t_{h+1})|\mathcal{F}_{t_i}\big] + \sqrt{\mathrm{Var}\big[r_S(t_{h+1})|\mathcal{F}_{t_i}\big]}\,\Psi_S(t_{h+1})$$

We can also express the future process in terms of the future processes of its two components

$$r_S(t_{h+1}) = r_\alpha(t_{h+1}) + \beta(t_i) r_X(t_{h+1})$$

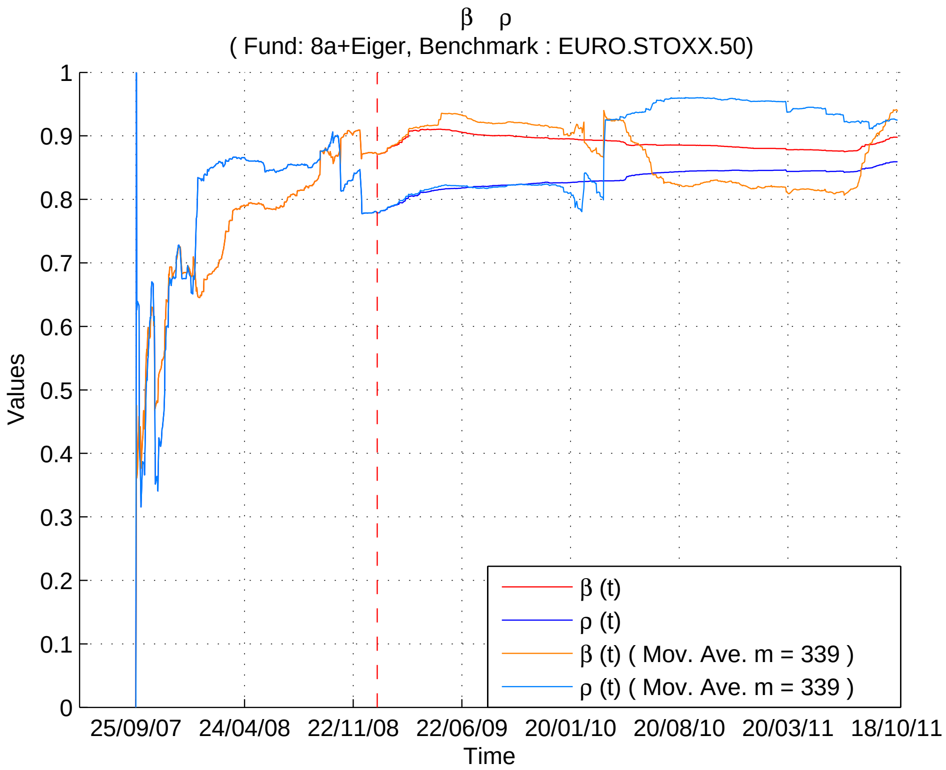


Figure 2.11: Evolution over time of $\beta(t_i)$ and $\rho(t_i)$ for Sample Case 1 fund, both computed with *full* and *partial* memory statistics. The red vertical line identifies the memory term m.

and, recalling equation 2.17, we may write either

$$r_S(t_{h+1}) = \Big[\alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{h+1})\Big] + \beta(t_i)\Big[\mu_X(t_i) + \sigma_X(t_i)\Psi_X(t_{h+1})\Big] \tag{2.18}$$

or

$$r_S(t_{h+1}) = \Big[\alpha(t_i) + \beta(t_i)\mu_X(t_i)\Big] + \Big[\sigma_\alpha(t_i)\Delta W_\alpha(t_{h+1}) + \beta(t_i)\sigma_X(t_i)\Psi_X(t_{h+1})\Big] \tag{2.19}$$

2.4.2 Ex-ante estimation error

We have decomposed the asset daily return into two components and analyzed them both from an ex-post and an ex-ante standpoint. Let's now focus on the implementation of the strategy on a daily basis see figure 5.1

1. At each present time t_i we will have statistical data deriving from past times t_j with $j \in [0, i-1]$, which will allow us to compute the linear regression and determine the values of $\alpha(t_i)$ and $\beta(t_i)$, assuming that $\rho(t_i)$ is sufficiently high to justify the return decomposition.

$$r_S(t_{j+1}) = \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1}) + \beta(t_i)\Big[\mu_X(t_i) + \sigma_X(t_i)\Psi_X(t_{j+1})\Big]$$

2. When $j = i-1$, reaches its maximum value, the previous equation gives us the linear

Table 2.4: **Alpha performance**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
From date	22.07.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
α	2.042E-04	2.257E-04	2.086E-04
σ_α	3.653E-03	3.785E-03	3.048E-03
Skewness	-2.874E-01	-3.426E-02	-2.874E-01
Kurtosis	9.869E+00	5.120E+00	6.270E+00
β	8.984E-01	9.331E-01	6.894E-01
ρ	8.627E-01	9.069E-01	8.580E-01
R_α	1.200E-01	1.348E-01	9.435E-02
Annualized R_α	5.182E-02	5.699E-02	5.345E-02

regression for the returns at present time

$$r_S(t_i) = \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_i) + \beta(t_i)\Big[\mu_X(t_i) + \sigma_X(t_i)\Psi_X(t_i)\Big]$$

$$r_S(t_i) = r_\alpha(t_i) + \beta(t_i) r_X(t_i)$$

3. Based on these parameters, at time t_i we will act on the portfolio and apply the hedge, as it will be shown in the following chapters. To do so, we will estimate the return at the following time step t_{i+1} and the linear regression parameters at that time.

$$E[r_S(t_{i+1})|\mathcal{F}_{t_i}] = E[r_\alpha(t_{i+1})|\mathcal{F}_{t_i}] + E[\beta(t_{i+1})r_X(t_{i+1})|\mathcal{F}_{t_i}] = \alpha(t_i) + \beta(t_i)\mu_X(t_i)$$

4. Then we move the present time from t_i to t_{i+1} and observe from an *ex-post* point of view what has actually been the return from time t_i to time t_{i+1}.

$$r_S(t_{i+1}) = \alpha(t_{i+1}) + \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1}) + \beta(t_{i+1}) r_X(t_{i+1})$$

5. Finally, we consider the difference between this value and the one we had previously estimated. Before doing so, we add and subtract to the latter equation the term $\beta(t_i)r_X(t_{i+1})$ so that the difference becomes

$$\zeta_S(t_{i+1}) := r_S(t_{i+1}) - E[r_S(t_{i+1})|\mathcal{F}_{t_i}] = \alpha(t_{i+1}) - \alpha(t_i) + \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1}) +$$

$$\beta(t_{i+1})r_X(t_{i+1}) - \beta(t_i)r_X(t_{i+1}) + \beta(t_i)r_X(t_{i+1}) - \beta(t_i)\mu_X(t_i)$$

which we can rearrange into

$$\zeta_S(t_{i+1}) = \Big[r_X(t_{i+1}) - \mu_X(t_i)\Big]\beta(t_i) + \Big[\alpha(t_{i+1}) - \alpha(t_i) + \big(\beta(t_{i+1}) - \beta(t_i)\big)r_X(t_{i+1})\Big] +$$

$$\sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})$$

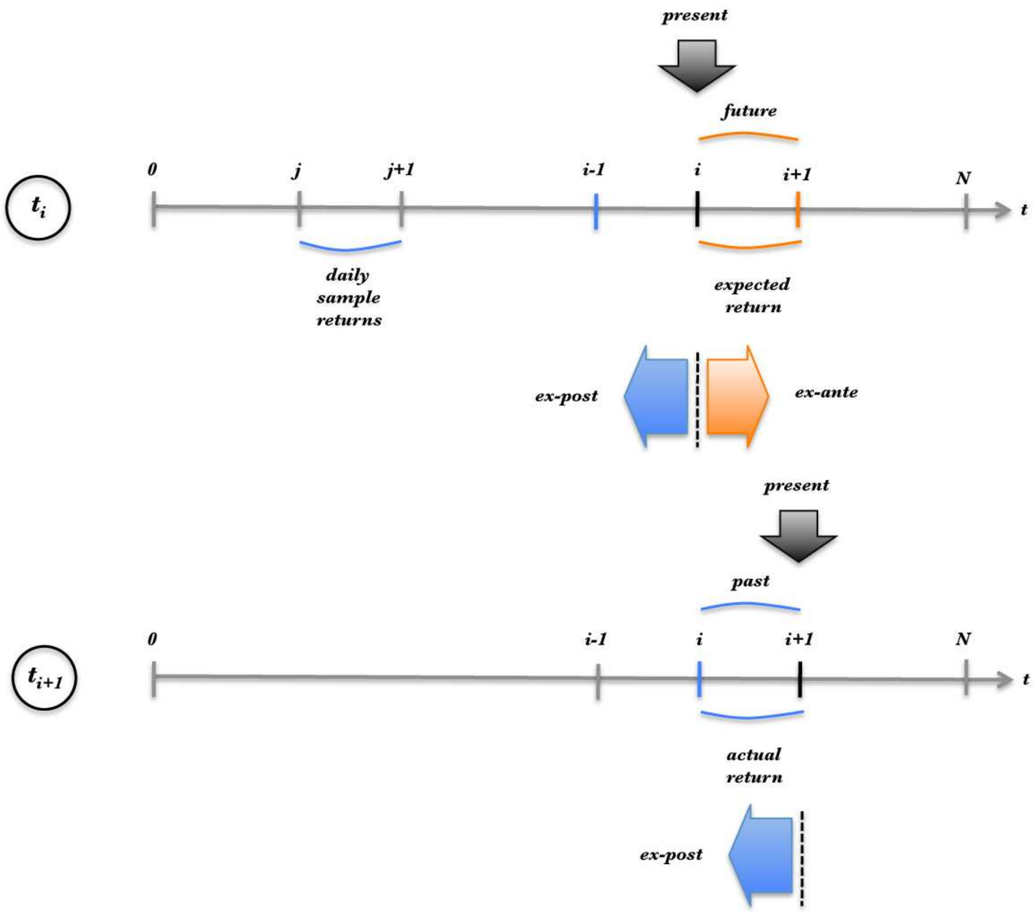


Figure 2.12: At present time t_i we assess from an ex-ante point of view which will be the return at the next time step t_{i+1}, and we do so based upon the data we have from past times t_j which allow us to compute the linear regression. Then, we move the present time to t_{i+1} and assess, now from an ex-post point of view, which has effectively been the return during the last time step.

and express as it composed of following three terms (Figure 2.13)

$$\zeta_S(t_{i+1}) = \zeta_{\text{hedge}}(t_{i+1}) + \zeta_x(t_{i+1}) + \zeta_\alpha(t_{i+1}) \tag{2.20}$$

$$\zeta_{\text{hedge}}(t_{i+1}) := \Big[r_X(t_{i+1}) - \mu_X(t_i)\Big]\beta(t_i) \tag{2.21}$$

$$\zeta_x(t_{i+1}) := \alpha(t_{i+1}) - \alpha(t_i) + \big(\beta(t_{i+1}) - \beta(t_i)\big)r_X(t_{i+1}) \tag{2.22}$$

$$\zeta_\alpha(t_{i+1}) := \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1}) \tag{2.23}$$

- $\zeta_{\text{hedge}}(t_{i+1}) := \Big[r_X(t_{i+1}) - \mu_X(t_i)\Big]\beta(t_i)$
  This first component represents an error which is *hedgable*, in the sense that this term will vanish from the portfolio performance as an effect of the implementation of the hedge. It is equal to the difference between the actual market return and the estimated one multiplied by the coefficient β known at time t_i. As long as the asset return at time t_{i+1} lies along the original mean line, the difference between this value and the mean one is fully hedgable. In other words, in order to

Table 2.5: **Estimation error**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
From date	22.07.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
Mean	-1.205E-06	1.173E-08	-1.877E-07
Std. dev.	1.865E-05	1.557E-05	2.066E-05

execute a perfect hedge it is not necessary that the actual return be equal to the expected one. It may be different so long as it remains on the linear regression line.

- $\zeta_x(t_{i+1}) := \alpha(t_{i+1}) - \alpha(t_i) + \big(\beta(t_{i+1}) - \beta(t_i)\big) r_X(t_{i+1})$
  This second term represents the *estimation error* originated by the difference in the mean regression parameters from time t_i to time t_{i+1}. It is equal to the difference in α plus the difference in β multiplied by the index return at time t_{i+1}. In the case of daily returns, this quantity is negligible as shown in Table 5.2, however when passing to compound returns it will assume an important role. This error quantifies the accuracy of our estimation with respect to the regression line, or likewise, the entity of the regression variation during the considered time step.

- $\zeta_\alpha(t_{i+1}) := \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})$
  Finally, this last term represents the extra premium return of each $r_S(t_{i+1})$ with respect to the mean regression line at time t_{i+1}, or equivalently, the stochastic term in the alpha dynamics at that time. We define this as *alpha error*.

The difference between the realized sample return $r_S(t_{i+1})$ and the estimated one $\mu_S(t_i) = \alpha(t_i) + \beta(t_i)\mu_X(t_i)$ is shown in figure 2.13 and decomposed into three error terms.

1. Starting at time t_i from the sample mean point $[\mu_X(t_i), \mu_S(t_i)]$, we move at the next time step t_{i+1} along the r_X axis of a quantity given by the difference $r_X(t_{i+1}) - \mu_X(t_i)$ which represents the stochastic movement of the index return with respect to the estimated mean. Along the r_S axis we move of the corresponding distance, which represents the *hedgable error*, determined by the proportionality coefficient $\beta(t_i)$.

2. From here, we then continue along the r_S axis and reach the point lying on the new mean line given at time t_{i+1}. The distance covered represents the *estimation error*, which vanishes if both mean lines are equal.

3. Finally, we add along the r_S axis a further distance given the *alpha error* and equal to $\sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})$.

In Appendix A we summarize the main formulas introduced in this chapter and throughout the thesis.

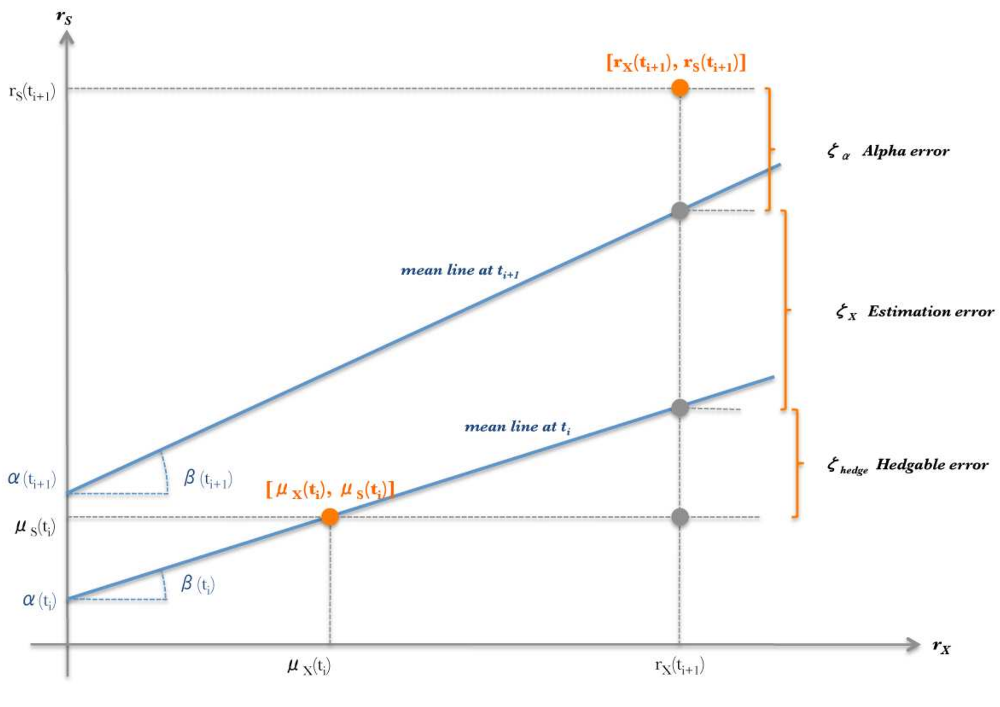


Figure 2.13: Geometric representation of the three components into which the overall difference between expected return and actual one may be decomposed. These are indicated as *hedgable error*, *estimation error* and *alpha error*.

Chapter 3

Core-asset compound returns

3.1 Alpha growth evolution

We are now interested in the ex-ante evaluation of the alpha dynamics *growth* or its *compound return*. The reason for doing so is that once the hedge against market return has been properly implemented, the portfolio overall return will be originated by the alpha dynamics, so that the portfolio growth will be determined by the alpha growth. It is therefore crucial to understand the properties of such dynamics, because these will also be the properties of hedged portfolio.
We have seen from figure 2.8 the evolution over time of the alpha returns $r_\alpha(t_{j+1})$ compared with the sample mean $\alpha(t_i)$ and the sample standard deviation $\sigma_\alpha(t_i)$. From Table 2.4, it results that $\alpha(t_i) > 0$ and that $\alpha(t_i) = o(10^{-4})$ whereas $\sigma_\alpha(t_i) = o(10^{-3})$. In other words, we do have a small but positive mean for r_α, so that on average we gain a daily premium with respect to the benchmark index, however the variance is one order of magnitude higher than the mean, and single realizations fall within a range of values which goes from highly positive to highly negative.
It would appear as if this circumstance creates a fundamental problem for the success of this strategy. However, this is not the case, and the explanation can be given looking into the evolution over time of the expected growth and of its variance.

$$E\big[G_\alpha(t_i, t_N)|\mathcal{F}_{t_i}\big] \tag{3.1}$$

$$\mathrm{Var}\big[G_\alpha(t_i, t_N)|\mathcal{F}_{t_i}\big] \tag{3.2}$$

3.1.1 Binomial formula

Before proceeding with the computation of the expected growth and variance, it is useful to recall the general first binomial formula [8] and highlight some of its properties, which will be exploited in the calculations and will allow for convenient approximations of otherwise complex expressions.

$$(a+b)^n = \sum_{g=0}^{n} \binom{n}{g} a^{n-g} b^g \tag{3.3}$$

valid for all $n \in \mathcal{N}$ and a, b real or complex, where the binomial coefficient and its symmetry law may be written as

$$\binom{n}{g} = \frac{n(n-1)(n-2)\cdots(n-g+1)}{g!} = \binom{n}{n-g} = \frac{n!}{g!(n-g)!}$$

When looking for simplifications of this equation, it will be necessary to assess the order of magnitude of each term appearing in the sum. A general way to do so is to consider the ratio of two subsequent terms in the sum for $g \in [0, n-1]$

$$\epsilon(g) := \frac{\binom{n}{g}}{\binom{n}{g+1}} \frac{a^{n-g} b^g}{a^{n-g-1} b^{g+1}} = \frac{\binom{n}{g}}{\binom{n}{g+1}} \frac{a}{b}$$

where the following coefficient may be simplified into

$$\frac{\binom{n}{g}}{\binom{n}{g+1}} = \frac{n!}{g!(n-g)!} \frac{(g+1)!(n-g-1)!}{n!} = \frac{g+1}{n-g}$$

so that

$$\epsilon(g) = \frac{g+1}{n-g} \frac{a}{b}$$

Note that such coefficient is a monotonic increasing function of g. The minimum is given when $g = 0$ and the maximum when $g = n-1$. We can thus write that

$$\frac{1}{n} \leq \frac{g+1}{n-g} \leq n$$

This implies that the function $\epsilon(g)$, which represents the ratio between two subsequent terms in equation 3.3 is bounded as follows

$$\frac{1}{n} \frac{a}{b} \leq \epsilon(g) \leq n \frac{a}{b} \tag{3.4}$$

There are cases in which rather than dealing with equation 3.3 we consider

$$(a+b)^n = a^n + \sum_{g=1}^{n} \binom{n}{g} a^{n-g} b^g \tag{3.5}$$

in which the term given by $g = 0$ has been extracted from the summation, which therefore starts with index $g = 1$. In this case, the coefficient $\frac{g+1}{n-g}$ for $g \in [1, n-1]$ is bounded as follows

$$\frac{2}{n-1} \leq \frac{g+1}{n-g} \leq n$$

so that

$$\frac{2}{n-1} \frac{a}{b} \leq \epsilon(g) \leq n \frac{a}{b} \tag{3.6}$$

Figure 3.1 shows in logarithmic scale the value of binomial coefficient $\binom{n}{g}$ for $g \in [0, n]$ and $n = 100$. For $g = 0$ and $g = n$ the coefficient is equal to one. Note its symmetry around

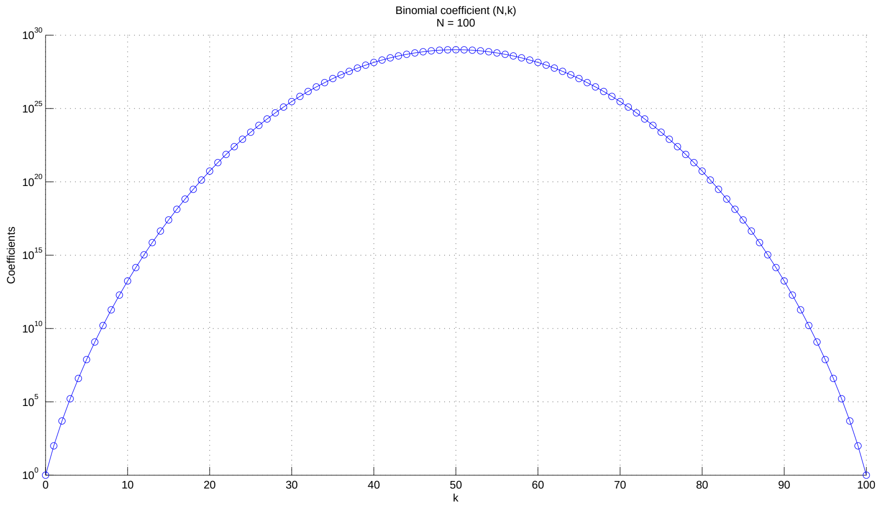


Figure 3.1: Binomial coefficients for $n = 100$.

$g = \frac{n}{2}$ where it reaches its maximum value in the order of 10^{28}.

3.1.2 Expected growth: exact value

We can now start with the expected value of growth given in equation 3.1.

$$E\big[G_\alpha(t_i, t_N)|\mathcal{F}_{t_i}\big] = E\Bigg[\prod_{h=i}^{N-1}\big(1 + r(t_{h+1})\big)|\mathcal{F}_{t_i}\Bigg]$$

We will adopt in this section a simplified notation, where pedices indicate the dependance on time $[]_h = [](t_h)$ and all quantities refer to the alpha dynamics:

$$r_{h+1} := r_\alpha(t_{h+1}) \qquad \alpha_i := \alpha(t_i) \qquad \sigma_i := \sigma_\alpha(t_i) \qquad \Delta W_{h+1} := \Delta W_\alpha(t_{h+1})$$

$$G_N := G_\alpha(t_i, t_N)$$

The approach used for this computation arises from the consideration that each ΔW_{h+1} appearing in the equation through r_{h+1} is assumed to be independent one to another, being differences of values of a Wiener process. In general [7], if two random variables X and Y are independent, then so are two corresponding functions $f(X)$ and $g(Y)$. In our case, for $h \in [i, N-1]$ we consider the $N - i$ independent variables ΔW_{h+1} and for each the function

$$f(\Delta W_{h+1}) = 1 + r_{h+1} = 1 + \alpha_i + \sigma_i \Delta W_{h+1}$$

Each $f(\Delta W_{h+1})$ is independent one to another, so that

$$E\big[\prod_{h=i}^{N-1}\big(1 + r_{h+1}\big)\big] = \prod_{h=i}^{N-1} E(1 + r_{h+1})$$

and consequently

$$E\Big[\prod_{h=i}^{N-1}\big(1+r_{h+1}\big)|\mathcal{F}_i\Big] = \prod_{h=i}^{N-1}E[(1+r_{h+1})|\mathcal{F}_i]$$

For each $h \in [i, N-1]$

$$E\big[r_{h+1}|\mathcal{F}_i\big] = E[\alpha_i|\mathcal{F}_i] + E[\sigma_i\Delta W_{h+1})|\mathcal{F}_i] = \alpha_i + \sigma_i E[\Delta W_{h+1}|\mathcal{F}_i] = \alpha_i$$

since

$$E[\Delta W_{h+1}|\mathcal{F}_i] = 0$$

This allows to write

$$E\Big[\prod_{h=i}^{N-1}\big(1+r_{h+1}\big)|\mathcal{F}_i\Big] = \prod_{h=i}^{N-1}(1+\alpha_i) = (1+\alpha_i)^{N-i}$$

$$E\big[G_N|\mathcal{F}_i\big] = \big(1+\alpha_i\big)^{N-i}$$

and, readopting full notation,

$$E\big[G_\alpha(t_i,t_N)|\mathcal{F}_{t_i}\big] = \big(1+\alpha(t_i)\big)^{N-i} \tag{3.7}$$

In words, the expected value at present time t_i of the terminal growth at the future time t_N, is equal to the compound return with $N-i$ periods of the mean return $\alpha(t_i)$ known at time t_i.
Since,

$$R_\alpha(t_i,t_N) = G_\alpha(t_i,t_N) - 1$$

we can also write

$$E\big[R_\alpha(t_i,t_N)|\mathcal{F}_{t_i}\big] = \big(1+\alpha(t_i)\big)^{N-i} - 1 \tag{3.8}$$

3.1.3 Expected growth: approximation

We can also find an approximated expression for the expected value of growth. Let's introduce the parameter $n := N-i$, which indicates the time horizon of the ex-ante evaluation. Applying the binomial formula 3.3

$$E\big[G_N|\mathcal{F}_i\big] = (1+\alpha_i)^n = \sum_{g=0}^{n}\binom{n}{g}\alpha_i^g \tag{3.9}$$

This sum can be truncated to the first g^* terms, in function of the level of approximation desired. To this scope, we define each term appearing in the sum as follows

$$u(g) := \binom{n}{g}\alpha_i^g$$

Table 3.1: **Expected growth approximation**
(data relevant to Sample Case 1)

g	$\binom{n}{g}$	α_i^g	$u(g)$ (formula)	$u(g)$ (value)	sum
0	1	1	1	1,00E+00	1,00E+00
1	2,50E+02	2,04E-04	$n\alpha_i$	5,11E-02	5,11E-02
2	3,11E+04	4,17E-08	$\binom{n}{2}\alpha_i^2$	1,30E-03	-
3	2,57E+06	8,51E-12	$\binom{n}{3}\alpha_i^3$	2,19E-05	-
			Truncated sum		1,05E+00
			Approx. value		1,05E+00
			Exact value		1,05E+00
			Approx. error		**-1,32E-03**
			$\epsilon(g)$ min		3,93E+01
			$\epsilon(g)$ max		1,22E+06

and adopt $n = 250$, which is an assumption coherent with the results that will follow. In Table 3.1 we show for index $g = [0, 3]$ the corresponding values of the binomial coefficient, the power terms α_i^g and $u(g)$, all referring to the fund data of Sample Case 1. As the index g increases, the power terms of $\alpha_i = o(10^4)$, assume very small values, whereas the binomial terms (in the first half of the index range) increase, but not enough to compensate the effect of the said powers, so that $u(g)$ becomes negligible very quickly. In other words, α_i power terms dominate over the binomial ones, and it appears acceptable to truncate the sum after index $g = 1$, which leads to an approximation error of order 10^{-3}.
The truncation effectivity is also guaranteed by the fact that the ratio between two subsequent terms from $g = 1$ onwards, given by equation 3.6

$$\frac{2}{n-1}\frac{1}{\alpha_i} \leq \epsilon(g) \leq n\frac{1}{\alpha_i}$$

ranges from $3.93 \cdot 10^1$ to $o(10^6)$, which means that each term is approximately between forty and one million times smaller than the previous one.
We can therefore write

$$E\big[G_N|\mathcal{F}_i\big] = \sum_{g=0}^{n} \binom{n}{g} \alpha_i^g \approx 1 + n\alpha_i \tag{3.10}$$

and

$$E\big[R_N|\mathcal{F}_i\big] \approx n\alpha_i \tag{3.11}$$

3.1.4 Growth variance: exact value

Similarly to the method adopted for the expected growth, we compute the variance starting from the definition of the following functions of ΔW_{h+1} for each $h \in [i, N-1]$

$$g(\Delta W_{h+1}) := (1 + r_{h+1})^2 = (1 + \alpha_i + \sigma_i \Delta W_{h+1})^2$$

As already stated, each ΔW_{h+1} is assumed to be independent one to another so that also the functions $g(\Delta W_{h+1})$ are independent one to another, which allows us to write

$$E[\prod_{h=i}^{N-1}\big(1+r_{h+1}\big)^2|\mathcal{F}_i] = \prod_{h=i}^{N-1} E[(1+r_{h+1})^2|\mathcal{F}_i]$$

For each $h \in [i, N-1]$

$$E\big[(1+r_{h+1})^2|\mathcal{F}_i\big] = E\big[(1+2r_{h+1}+r_{h+1}^2)|\mathcal{F}_i\big] = 1+2\alpha_i + E[\alpha_i^2|\mathcal{F}_i]+$$

$$+E[\sigma_i^2\Delta W_{h+1}^2|\mathcal{F}_i] + E[2\alpha_i\sigma_i\Delta W_{h+1}|\mathcal{F}_i] = 1+2\alpha_i+\alpha_i^2+\sigma_i^2 = (1+\alpha_i)^2+\sigma_i^2$$

since

$$E[\Delta W_{h+1}|\mathcal{F}_i] = 0 \qquad E[\Delta W_{h+1}^2|\mathcal{F}_i] = 1$$

Let's now consider the variance of growth

$$\mathrm{Var}\big[G_N|\mathcal{F}_i\big] = E\bigg[\big(G_N - E(G_N)\big)^2|\mathcal{F}_i\bigg] = E\big[G_N^2|\mathcal{F}_i\big] - E\big[G_N|\mathcal{F}_i\big]^2$$

Looking into the first term

$$E\big[G_N^2|\mathcal{F}_i\big] = E\bigg[\bigg(\prod_{h=i}^{N-1}(1+r_{h+1})\bigg)^2|\mathcal{F}_i\bigg] = E\bigg[\prod_{h=i}^{N-1}\big(1+r_{h+1}\big)^2|\mathcal{F}_i\bigg] = \prod_{h=i}^{N-1} E\big[(1+r_{h+1})^2|\mathcal{F}_i\big] =$$

$$= \prod_{h=i}^{N-1}\big[(1+\alpha_i)^2+\sigma_i^2\big] = \big[(1+\alpha_i)^2+\sigma_i^2\big]^{N-i}$$

Therefore,

$$\mathrm{Var}\big[G_N|\mathcal{F}_i\big] = \big[(1+\alpha_i)^2+\sigma_i^2\big]^{N-i} - (1+\alpha_i)^{2(N-i)} \tag{3.12}$$

so that

$$\mathrm{Var}\big[G_N|\mathcal{F}_i\big] = \big[(1+\alpha_i)^2+\sigma_i^2\big]^{n} - (1+\alpha_i)^{2n} \tag{3.13}$$

Finally, we show that both the compound growth and return share the same conditional variance. Given that

$$G_N = R_N + 1$$

we can write

$$\mathrm{Var}\big[G_N|\mathcal{F}_i\big] = \mathrm{Var}\big[R_N+1|\mathcal{F}_i\big] = E\bigg[\big(R_N+1-E[R_N+1|\mathcal{F}_i]\big)^2|\mathcal{F}_i\bigg] =$$

$$E\bigg[\big(R_N+1-E[R_N|\mathcal{F}_i]-1\big)^2|\mathcal{F}_i\bigg] = E\bigg[\big(R_N-E[R_N|\mathcal{F}_i]\big)^2|\mathcal{F}_i\bigg] = \mathrm{Var}\big[R_N|\mathcal{F}_i\big]$$

so that

$$\mathrm{Var}\big[G_N|\mathcal{F}_i\big] = \mathrm{Var}\big[R_N|\mathcal{F}_i\big] \tag{3.14}$$

3.1.5 Growth variance: first approximation

In this chapter and elsewhere in the work, we shall often look for approximated expressions of some quantities with the intent of finding the best interpretable formula with a given level of approximation. When possible, this will allow us to focus on the most important features and understand in more depth the results presented. Note however that in general in the numerical simulations we shall not use the simplified expressions but the exact ones, so that no approximation error is introduced when not essential.

We can now use the binomial formula 3.5 to express the first term in equation 3.13 as follows

$$\left[(1+\alpha_i)^2+\sigma_i^2\right]^n = (1+\alpha_i)^{2n} + \sum_{g=1}^{n}\binom{n}{g}(1+\alpha_i)^{2(n-g)}\sigma_i^{2g}$$

so that

$$\mathrm{Var}\left[G_N|\mathcal{F}_i\right] = \sum_{g=1}^{n}\binom{n}{g}(1+\alpha_i)^{2(n-g)}\sigma_i^{2g} \tag{3.15}$$

Looking for a simplification of this expression, we can apply again the methodology adopted for the approximation of the expected growth. To this scope, we define each term appearing in the sum as follows

$$v(g) := \binom{n}{g}(1+\alpha_i)^{2(n-g)}\sigma_i^{2g}$$

Table 3.2 shows for sample Case 1, the values of the binomial coefficient, the power terms σ_i^{2g} and $v(g)$ for index $g=[1,3]$. The term $(1+\alpha_i)^{2(n-g)} = o(10^0)$ is not shown since it assumes always values close to one. The power terms of $\sigma_i = o(10^{-3})$ dominate over the binomial ones. Again, it appears acceptable to truncate the sum after the term of index $g=1$, which leads to an approximation error of order 10^{-6}.
Applying equation 3.6 to determine the ratio between two subsequent terms from $g=1$ onwards, it results that

$$\frac{2}{n-1}\frac{(1+\alpha_i)^2}{\sigma_i^2} \leq \epsilon(g) \leq n\frac{(1+\alpha_i)^2}{\sigma_i^2}$$

In this case, $\epsilon(g)$ ranges from $6,02\cdot 10^2$ to $o(10^7)$, which means that each term is approximately between one six hundred and ten million times smaller than the previous one.
It is important to point out that we are now dealing with the *variance* of the growth, which is not directly comparable to the *expected value*. In order to compare the approximation errors, we need to consider the *volatility* of growth, which is equal to the square root of the variance and for which the error is in the order of 10^{-5}.
As an effect of this approximation, we write

$$\mathrm{Var}\left[G_N|\mathcal{F}_i\right] \approx n\sigma_i^2(1+\alpha_i)^{2(n-1)} \tag{3.16}$$

Table 3.2: **Growth variance first approximation**
(data relevant to Sample Case 1)

g	$\binom{n}{g}$	σ_i^{2g}	$v(g)$ (formula)	$v(g)$ (value)	sum	
1	2,50E+02	1,33E-05	$n(1+\alpha_i)^{2(n-1)}\sigma_i^2$	3,69E-03	3,69E-03	
2	3,11E+04	1,78E-10	$\binom{n}{2}(1+\alpha_i)^{2(n-2)}\sigma_i^4$	6,13E-06	-	
3	2,57E+06	2,38E-15	$\binom{n}{3}(1+\alpha_i)^{2(n-3)}\sigma_i^6$	6,76E-09	-	
			Truncated sum		3,69E-03	
					Variance	**Volatility**
			Approx. value		3,69E-03	6,08E-02
			Exact value		3,70E-03	6,08E-02
			Approx. error		**-6,14E-06**	**-5,05E-05**
			$\epsilon(g)$ min		6,02E+02	
			$\epsilon(g)$ max		1,87E+07	

3.1.6 Growth variance: further approximations

It is however possible to repeat once more this method applying formula 3.3 to equation 3.16

$$\mathrm{Var}\big[G_N|\mathcal{F}_i\big] \approx n\sigma_i^2(1+\alpha_i)^{2(n-1)} = n\sigma_i^2 \sum_{g=0}^{2(n-1)} \binom{2(n-1)}{g}\alpha_i^g \tag{3.17}$$

We define each term of the sum as

$$w(g) := n\sigma_i^2 \binom{2(n-1)}{g}\alpha_i^g$$

In Table 3.3 we show, with reference to Sample Case 1, the values of the binomial coefficient, the power terms α_i^g and $w(g)$ for index $g = [0, 3]$. Again, as the index increases, the power terms of $\alpha_i = o(10^4)$, become dominant versus the binomial coefficients, so that $w(g)$ becomes negligible very quickly. For this reason, we truncate the sum after the term of index $g = 1$, which leads to an approximation error of order 10^{-5} in the variance and 10^{-4} in the volatility estimation.
We can apply equation 3.4 by substituting n with $2(n-1)$ to obtain

$$\frac{2}{2n-3}\frac{1}{\alpha_i} \leq \epsilon(g) \leq 2(n-1)\frac{1}{\alpha_i}$$

which indicates us that the ratio of two subsequent terms ranges from about 20 to $o(10^6)$. As a consequence, the approximated expression of the variance is

$$\mathrm{Var}\big[G_N|\mathcal{F}_i\big] \approx n\sigma_i^2\big(1+2(n-1)\alpha_i\big) = n\sigma_i^2\big(1+2n\alpha_i-2\alpha_i\big) \tag{3.18}$$

By computing the values of each of the three terms in the equation, we see that further approximations may be applied. A first possibility is to neglect the term $2n\sigma_i^2\alpha_i = o(10^{-6})$, as shown in Table 3.4:

$$\mathrm{Var}\big[G_N|\mathcal{F}_i\big] \approx n\sigma_i^2\big(1+2n\alpha_i\big) \tag{3.19}$$

Table 3.3: **Growth variance second approximation**
(data relevant to Sample Case 1)

g	$\binom{2(n-1)}{g}$	α_i^g	$w(g)$ (formula)	$w(g)$ (value)	sum	
0	1	1	$n\sigma_i^2$	3,34E-03	3,34E-03	
1	4,98E+02	2,04E-04	$2n(n-1)\sigma_i^2\alpha_i$	3,39E-04	3,39E-04	
2	1,24E+05	4,17E-08	$n\binom{2(n-1)}{2}\sigma_i^2\alpha_i^2$	1,72E-05	-	
3	2,05E+07	8,51E-12	$n\binom{2(n-1)}{3}\sigma_i^2\alpha_i^3$	5,81E-07	-	
			Truncated sum		3,68E-03	
					Variance	**Volatility**
			Approx. value		3,68E-03	6,06E-02
			Exact value		3,70E-03	6,08E-02
			Approx. error		**-2,40E-05**	**-1,97E-04**
			$\epsilon(g)$ min		1,97E+01	
			$\epsilon(g)$ max		2,44E+06	

Table 3.4: **Growth variance third and fourth approximations**
(data relevant to Sample Case 1)

item	value	sum	
$n\sigma_i^2$	3,34E-03	3,34E-03	
$n\sigma_i^2 \cdot 2n\alpha_i$	3,41E-04	3,41E-04	
$n\sigma_i^2 \cdot 2\alpha_i$	1,36E-06	-	
Truncated sum		3,68E-03	
		Variance	**Volatility**
Approximated value		3,68E-03	6,06E-02
Exact value		3,70E-03	6,08E-02
Approximation error		**-2,26E-05**	**-1,86E-04**
$n\sigma_i^2$	13,34E-03	3,34E-03	
$n\sigma_i^2 \cdot 2n\alpha_i$	3,41E-04	-	
$n\sigma_i^2 \cdot 2\alpha_i$	1,36E-06	-	
Truncated sum		3,34E-03	
		Variance	**Volatility**
Approximated value		3,34E-03	5,78E-02
Exact value		3,70E-03	6,08E-02
Approximation error		**-3,63E-04**	**-3,06E-03**

A second possibility is to neglect both this and the term $2n^2\sigma_i^2\alpha_i = o(10^{-4})$ which leads to the following simplified expression, as shown in Table 3.4:

$$\mathrm{Var}\big[G_N|\mathcal{F}_i\big] \approx n\sigma_i^2 \tag{3.20}$$

3.1.7 Growth variance: expressions

Wrapping up, we have obtained different expressions of alpha expected growth, its variance and volatility, depending on the level of accuracy desired for the computation, which we summarize in Table 3.5.

If we compare the expressions (re-adopting full notation)

$$E\big[G_\alpha(t_i,t_N)|\mathcal{F}_{t_i}\big] = \big[1+\alpha(t_i)\big]^n$$

Table 3.5: **Ex-ante alpha growth expressions**
(data relevant to Sample Case 1)

approximation	*equation*	*error*
Expected value		
Exact	$(1+\alpha_i)^n$	0
First	$1+n\alpha_i$	-1,32E-03
Variance		
Exact	$\left[(1+\alpha_i)^2+\sigma_i^2\right]^n-(1+\alpha_i)^{2n}$	0
First	$n\sigma_i^2(1+\alpha_i)^{2(n-1)}$	-6,14E-06
Second	$n\sigma_i^2\left(1+2n\alpha_i-2\alpha_i\right)$	-2,40E-05
Third	$n\sigma_i^2\left(1+2n\alpha_i\right)$	-2,26E-05
Fourth	$n\sigma_i^2$	-3,63E-04
Volatility		
Exact	$\sqrt{\left[(1+\alpha_i)^2+\sigma_i^2\right]^n-(1+\alpha_i)^{2n}}$	0
First	$\sqrt{n}\sigma_i(1+\alpha_i)^{(n-1)}$	-5,05E-05
Second	$\sqrt{n}\sigma_i\sqrt{1+2n\alpha_i-2\alpha_i}$	-1,97E-04
Third	$\sqrt{n}\sigma_i\sqrt{1+2n\alpha_i}$	-1,86E-04
Fourth	$\sqrt{n}\sigma_i$	-3,06E-03

$$\sqrt{\mathrm{Var}\big[G_\alpha(t_i,t_N)|\mathcal{F}_{t_i}\big]}\approx\sqrt{n}\sigma_\alpha(t_i)\big[1+\alpha(t_i)\big]^{(n-1)}$$

it results that:

- The expected value is given by the compound return for n periods of the sample mean return $\alpha(t_i)$ at starting time, whereas the volatility is proportional to a compound return of $n-1$ periods;
- The volatility is also proportional to the sample standard deviation $\sigma_\alpha(t_i)$ of the underlying alpha return process at time t_i; and
- it is proportional to the square root of n.

To have a better understanding of which quantities are more important, we now compare the approximated expressions relevant to the compound return and growth (see equations 3.8 and 3.14)

$$E\big[G_\alpha(t_i,t_N)|\mathcal{F}_{t_i}\big]\approx 1+n\alpha(t_i)\qquad\rightarrow\qquad E\big[R_\alpha(t_i,t_N)|\mathcal{F}_{t_i}\big]\approx n\alpha(t_i)$$

$$\sqrt{\mathrm{Var}\big[G_\alpha(t_i,t_N)|\mathcal{F}_{t_i}\big]}=\sqrt{\mathrm{Var}\big[R_\alpha(t_i,t_N)|\mathcal{F}_{t_i}\big]}\approx\sqrt{n}\sigma_\alpha(t_i)$$

which tell us that the expected return is proportional to the length of the time horizon n by means of the sample mean return $\alpha(t_i)$ at starting time, whereas the volatility is a function of the square root $\sqrt{n}$ and proportional to the sample standard deviation $\sigma_\alpha(t_i)$ at starting time. This implies that, once determined the starting mean and standard deviation, both the expected value and volatility increase with time, but the volatility increases less, since for $n>1$ it results that $\sqrt{n}<n$.
If we introduce a parameter ξ indicating the ratio of the daily drift and volatility, and compare it with an equivalent parameter Ξ on a longer time period (indicating the ratio of the expected compound return and its volatility)

$$\xi(t_i):=\frac{E\big[r_\alpha(t_{h+1})|\mathcal{F}_{t_i}\big]}{\mathrm{Var}\big[r_\alpha(t_{h+1})|\mathcal{F}_{t_i}\big]]}=\frac{\alpha(t_i)}{\sigma(t_i)}$$

$$\Xi(t_i) := \frac{E\big[R_\alpha(t_i, t_N)|\mathcal{F}_{t_i}\big]}{\text{Var}\big[R_\alpha(t_i, t_N)|\mathcal{F}_{t_i}\big]]} = \frac{\sqrt{n}\alpha(t_i)}{\sigma(t_i)} = \sqrt{n}\xi(t_i)$$

it results that $\Xi(t_i) = \sqrt{n}\xi(t_i)$, which means that an improvement of the risk profile, passing from a daily time frame to a longer period one, can be achieved by allowing the process to last for a sufficient period. As an example, suppose that we have $\xi = 10^{-1}$ on a daily basis. If we allow for $n = 100$ days the compound effect of a small positive drift will result in parameter $\Xi = 10\xi = 1$.

Let's get back to Figure 2.8. Each point in the graph represents a *local* realization of the daily return r_α. This data set has a positive mean α, not large but still positive, which clearly is a necessary condition to achieve a positive outcome from the strategy and represents one of the criteria adopted to select the core-asset. The fact that these realizations have a standard deviation σ_α much bigger than their mean, does not compromise the *compound* effect of the dynamics. Passing from a *local* to a *global* perspective, it results that daily returns with small positive mean and high variance will produce a large positive compound effect on the growth in the medium term.

3.1.8 Waiting time

We define as *waiting time* the period necessary for the process to deliver a certain pre-defined target growth. To this scope we introduce a quantity Z, which is function of the information available at time t_i, of the length of the projection time $n := N - i$ and of a given parameter z, defined as the sum of the expected growth and a multiple z of the growth volatility

$$Z(\mathcal{F}_i, n, z) := E\big[G_\alpha(t_i, t_N)|\mathcal{F}_i\big] + z\sqrt{\text{Var}\big[G_\alpha(t_i, t_N)|\mathcal{F}_i\big]} \tag{3.21}$$

Suppose now that at time t_i we have enough data relevant to the alpha dynamics so that the sample mean $\alpha(t_i)$ and the sample standard deviation $\sigma_\alpha(t_i)$ can be considered meaningful. Starting from t_i and for increasing values of n, we consider a process which represents the alpha growth minus a given multiple of its volatility. In other words, we consider the function Z for $z \leq 0$. To determine the *waiting time*, we impose this quantity to equal a pre-defined target return μ^* as follows

$$Z(\mathcal{F}_i, n, z) = \big(1 + \mu^*\big)^n \tag{3.22}$$

Adopting the exact expressions of the expected growth and volatility, we look for a solution of the following transcendental equation in n

$$(1 + \alpha_i)^n + z\sqrt{\big[(1 + \alpha_i)^2 + \sigma_i^2\big]^n - (1 + \alpha_i)^{2n}} = \big(1 + \mu^*\big)^n$$

When existing, such solution gives an indication of the expected time required by the process to achieve a certain return with pre-defined confidence and in function of the sample data available. In particular, when $z = -1$ and $\mu^* = 0$, the equation simplifies into

$$(1 + \alpha_i)^n - \sqrt{\big[(1 + \alpha_i)^2 + \sigma_i^2\big]^n - (1 + \alpha_i)^{2n}} = 1$$

which, if solved in n, tells us the time necessary for the growth process to provide a positive return with the confidence level of $\sqrt{\mathrm{Var}\big[G_\alpha(t_i, t_N)|\mathcal{F}_i\big]}$.

It is possible to derive an analytical solution for the waiting time, adopting the simplified expressions for the expected value and volatility, in function of the level of accuracy desired. Suppose for example that we accept for these quantities errors equal to the ones given in the "Third" approximation of Table 3.5. Simplifying accordingly also the target compound return, equation 3.22 would become

$$1 + n\alpha_i + z\sqrt{n}\sigma_i\sqrt{1 + 2n\alpha_i} = 1 + n\mu^*$$

which can be treated as follows

$$n(\alpha_i - \mu^*) = -z\sigma_i\sqrt{n}\sqrt{1 + 2n\alpha_i}$$

$$n^2(\alpha_i - \mu^*)^2 = z^2\sigma_i^2 n(1 + 2n\alpha_i)$$

$$n\big[(\alpha_i - \mu^*)^2 - 2\alpha_i z^2\sigma_i^2\big] = z^2\sigma_i^2$$

to find the solution given by

$$n = \frac{z^2\sigma_i^2}{(\alpha_i - \mu^*)^2 - 2\alpha_i z^2\sigma_i^2}$$

which, if $z = -1$ and $\mu^* = 0$, becomes

$$n = \frac{\sigma_i^2}{\alpha_i^2 - 2\alpha_i\sigma_i^2} \tag{3.23}$$

If we accept even higher approximation errors, such as the ones given in the "Fourth" approximation of Table 3.5, then we can simplify equation 3.22 and look for solutions in n of equation

$$1 + n\alpha_i + z\sqrt{n}\sigma_i = 1 + n\mu^*$$

By simple passages

$$n(\alpha_i - \mu^*) = -z\sigma_i\sqrt{n}$$

$$n^2(\alpha_i - \mu^*)^2 = z^2\sigma_i^2 n$$

we obtain the solution

$$n = \frac{z^2\sigma_i^2}{(\alpha_i - \mu^*)^2}$$

which, in case $z = -1$ and $\mu^* = 0$, is given by

$$n = \frac{\sigma_i^2}{\alpha_i^2} = o(10^2) \tag{3.24}$$

This equation gives us an approximation of the waiting time necessary for the growth to achieve a positive value with confidence level given by the volatility of alpha. It also determines the order of magnitude of such time, which as we can see depends on the squared ratio of the process volatility and its drift. In Table 3.6, we summarize the different results

Table 3.6: **Waiting time** $(z = -1, \mu^* = 0)$
(data relevant to Sample Case 1)

approx.	*equation*	*anal. solution*	*solution*	*error*
Exact	$(1+\alpha_i)^n - \sqrt{\left[(1+\alpha_i)^2 + \sigma_i^2\right]^n - (1+\alpha_i)^{2n}} = 1$	n.a.	$n = 344$	0
Third	$n\alpha_i - n\sigma_i^2(1 + 2n\alpha_i) = 0$	$n = \frac{\sigma_i^2}{\alpha_i^2 - 2\alpha_i\sigma_i^2}$	$n = 368$	+24
Fourth	$n\alpha_i - n\sigma_i^2 = 0$	$n = \frac{\sigma_i^2}{\alpha_i^2}$	$n = 320$	-24

for Sample Case 1. In Figure 3.2 we give a graphical representation of the methodology previously described. The black line plots the growth for Sample Case 1 over the period of available data. Suppose now that we have data for a shorter period of time, from t_0 to a given t_i falling inside the period considered. At this time t_i we compute $\alpha(t_i)$ and $\sigma_\alpha(t_i)$ with the data available up to that moment. In function of these quantities we obtain the expected growth and variance, the first of which, from that moment onwards is plotted in the blue line. The two red lines represent instead the quantities $Z(\mathcal{F}_i, n, z = 1)$ and $Z(\mathcal{F}_i, n, z = -1)$ for increasing values of n. The waiting time would be defined, if existing, as the value n for which the lower red line for $z = -1$ intercepts a pre-defined growth value starting from time t_i. The grey lines represent different outcomes of simulated growth trajectories all depending on $\alpha(t_i)$ and $\sigma_\alpha(t_i)$. At terminal time, the difference in value between the blue line and the black one, represents the error made by assuming the final value of growth with data available only up to time t_i in place of the value actually known with data available up to terminal time. Such estimation depends crucially on the time selected t_i and on the accuracy of $\alpha(t_i)$ and $\sigma_\alpha(t_i)$ as proxies of the future behavior. It also depends on the methodology used to compute such quantities, if with *full* or *partial* memory statistics for example.

3.2 Linear regression on non-daily period

Up to this point we have examined the correlation between the asset return and the market one exclusively on a daily time frame. In order to be effective, hedging operations must be implemented considering the same time horizon for which we know the correlation. Therefore, if the only information we have is on a daily basis, then we will only be able to hedge with daily frequency.
Differently than with Short ETFs, for which the only possibility is to hedge at the end of each trading day, when using Futures we can potentially decrease the frequency of the hedging operations so long as we have sufficient confidence on what the correlation of returns will be, relevant to that specific time scale.
What we intend to do in this section, is to analyze the correlation between asset return and market return on longer time scales, starting from the knowledge of the correlation on a daily basis. As mentioned, this information will be necessary to implement an effective hedge on such longer period.

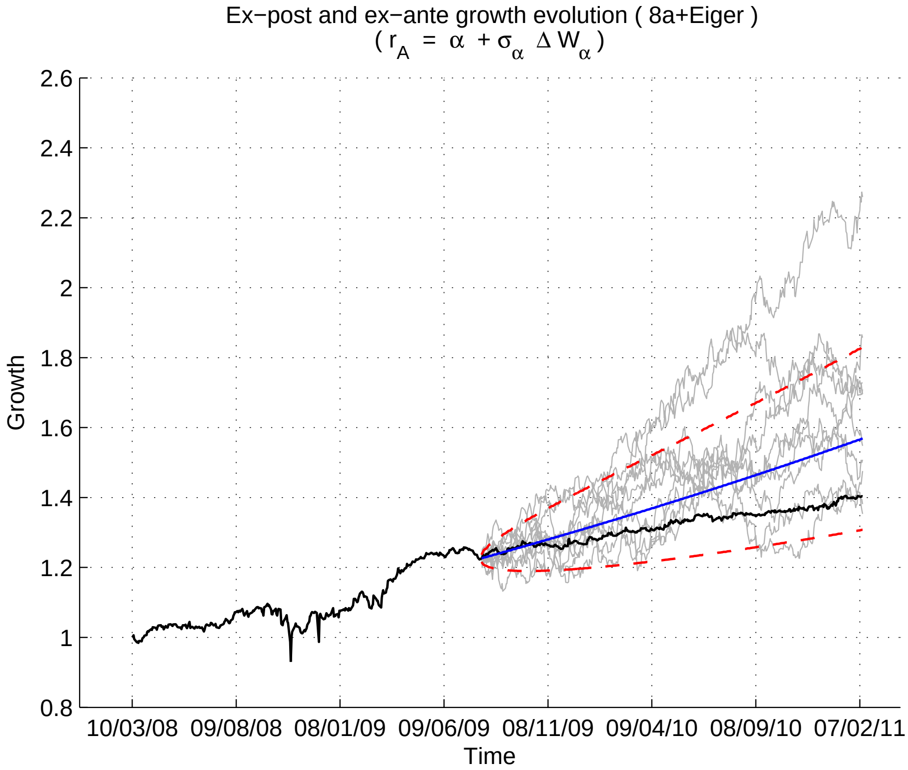


Figure 3.2: Alpha growth projection for Sample Case 1 fund. The black line represents the known growth trajectory. Starting from time t_i fixed at half of the potting period, the blue line represents the projection for increasing n of the expected growth, whereas the red lines represent the expected growth plus and minus the volatility. Each grey line is a simulation of the growth trajectory with fixed $\alpha(t_i)$ and $\sigma_\alpha(t_i)$.

In the previous chapter, we have defined the linear regression between daily returns r_S and r_X. Recalling equations 2.7, 2.8 and 2.9, for $j \in [0, i-1]$

$$r_S(t_{j+1}) := \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1}) + \beta(t_i) r_X(t_{j+1})$$

where

$$\alpha(t_i) = \mu_S(t_i) - \beta(t_i)\mu_X(t_i)$$

$$\beta(t_i) = \frac{\sigma^2_{SX}(t_i)}{\sigma^2_X(t_i)}$$

The same methodology can be applied considering returns R_S and R_X based on different time scales, such as on a weekly, monthly, or quarterly basis. At this point we can outline two possibilities:

1. We possess enough data of returns on that specific time frame, so that we can perform the linear regression and obtain the correlation parameters;

2. We not have sufficient sample points to validate a correlation analysis. This typically happens when we expand the time frame to long periods.

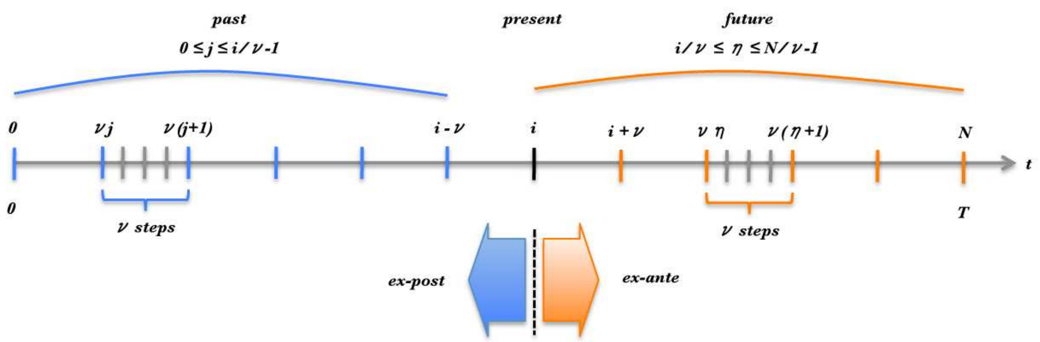


Figure 3.3: Ex-ante and ex-post time frame for compound returns

3.2.1 Case 1

Let's consider the first case, and assume that at present time t_i we have enough data to compute the correlation between the returns R_S and R_X given on a time frame of $\nu \in \mathcal{N}$ days (Figure 3.3).

$$R_S(t_{\nu(j+1)}) := \frac{S(t_{\nu(j+1)}) - S(t_{\nu j})}{S(t_{\nu j})}$$

$$R_X(t_{\nu(j+1)}) := \frac{X(t_{\nu(j+1)}) - X(t_{\nu j})}{X(t_{\nu j})}$$

Assume also that $\frac{i}{\nu} \in \mathcal{N}$ so that $j \in [0, \frac{i}{\nu} - 1]$. For instance, if $i = 1000$ and $\nu = 10$, then $\frac{i}{\nu} = 100$ and $j \in [0, 99]$. In the extreme cases,

$$j = 0 \qquad R_S(t_{10}) = \frac{S(t_{10}) - S(t_0)}{S(t_0)}$$

$$j = 99 \qquad R_S(t_{1000}) = \frac{S(t_{1000}) - S(t_{990})}{S(t_{990})}$$

The said correlation would be measured by the sample covariance defined as

$$\mathrm{Cov}_{\mathrm{sample}}[R_S, R_X] := \frac{1}{\frac{i}{\nu} - 1} \sum_{j=0}^{\frac{i}{\nu}-1} \Big[\big(R_S(t_{\nu(j+1)}) - M_S(t_i)\big)\big(R_X(t_{\nu(j+1)}) - M_X(t_i)\big) \Big]$$

where $M_S(t_i)$ and $M_X(t_i)$ indicate the sample means of these returns. This expression could also be normalized by dividing it with non-null standard deviations of the returns to obtain the *correlation coefficient*. At this point the following linear relation between R_S and R_X would be assumed, subject to having a sufficiently high correlation coefficient.

$$R_S(t_{\nu(j+1)}) = A(t_i) + B(t_i) R_X(t_{\nu(j+1)}) + \sigma_A(t_i) \Delta W_A(t_{\nu(j+1)}) \tag{3.25}$$

where $\sigma_A(t_i)\Delta W_A(t_{j+1})$ is the error term, which we associate to parameter A in analogy to what we did with α in the daily returns regression. Parameters $A(t_i)$ and $B(t_i)$ would be determined by the minimization of the mean square distance as previously seen and be equal to

$$A(t_i) = M_S(t_i) - B(t_i)M_X(t_i)$$
$$B(t_i) = \frac{\text{Cov}_{\text{sample}}[R_S, R_X]}{\text{Var}_{\text{sample}}[R_X]}$$

where $\text{Var}_{\text{sample}}[R_X]$ is the sample variance of returns R_X.

Ex-ante evaluation

Passing now from an ex-post to an ex-ante perspective, at time t_i we would define future returns as (Figure 3.3)

$$R_S(t_{\nu(\eta+1)}) := \frac{S(t_{\nu(\eta+1)}) - S(t_{\nu\eta})}{S(t_{\nu\eta})}$$
$$R_X(t_{\nu(\eta+1)}) := \frac{X(t_{\nu(\eta+1)}) - X(t_{\nu\eta})}{X(t_{\nu\eta})}$$

with $\eta \in [\frac{i}{\nu}, \frac{N}{\nu} - 1]$. Considering the same example of if $i = 1000$, $\nu = 10$ and $N = 2000$, then $\frac{i}{\nu} = 100$ and $\eta \in [100, 199]$, which in the extreme cases gives

$$\eta = 100 \qquad R_S(t_{1010}) = \frac{S(t_{1010}) - S(t_{1000})}{S(t_{1000})}$$

$$\eta = 199 \qquad R_S(t_{2000}) = \frac{S(t_{2000}) - S(t_{1990})}{S(t_{1990})}$$

At this point we would define the future correlation in terms of the equation

$$R_S(t_{\nu(\eta+1)}) = A(t_{\nu(\eta+1)}) + B(t_{\nu(\eta+1)})R_X(t_{\nu(\eta+1)}) + \sigma_A(t_{\nu(\eta+1)})\Delta W_A(t_{\nu(\eta+1)}) \tag{3.26}$$

where

$$A(t_{\nu(\eta+1)}) = E\big[R_S(t_{\nu(\eta+1)})|\mathcal{F}_{t_i}\big] - B(t_{\nu(\eta+1)})E\big[R_X(t_{\nu(\eta+1)})|\mathcal{F}_{t_i}\big]$$
$$B(t_{\nu(\eta+1)}) = \frac{\text{Cov}\big[R_S, R_X|\mathcal{F}_{t_i}\big]}{\text{Var}\big[R_X|\mathcal{F}_{t_i}\big]}$$

At this point, we would be in condition to make an ex-ante evaluation at time t_i of the future correlation by assuming that the sample means, variance and covariance of the returns will equal the conditional drift, variance and covariance in the future.

$$E\big[R_S(t_{\nu(\eta+1)})|\mathcal{F}_{t_i}\big] = M_S(t_i)$$

$$E\big[R_X(t_{\nu(\eta+1)})|\mathcal{F}_{t_i}\big] = M_X(t_i)$$

$$\text{Cov}\big[R_S, R_X|\mathcal{F}_{t_i}\big] = \text{Cov}_{\text{sample}}[R_S, R_X]$$

$$\text{Var}\big[R_X|\mathcal{F}_{t_i}\big] = \text{Var}_{\text{sample}}[R_X]$$

$$\text{Var}\big[R_\alpha|\mathcal{F}_{t_i}\big] = \text{Var}_{\text{sample}}[R_\alpha] := \sigma_A(t_i)^2 \tag{3.27}$$

which results in

$$A(t_{\nu(\eta+1)}) = M_S(t_i) - B(t_i)M_X(t_i) = A(t_i)$$

$$B(t_{\nu(\eta+1)}) = \frac{\mathrm{Cov}_{\mathrm{sample}}[R_S, R_X]}{\mathrm{Var}_{\mathrm{sample}}[R_X]]} = B(t_i)$$

so that

$$R_S(t_{\nu(\eta+1)}) = A(t_i) + B(t_i)R_X(t_{\nu(\eta+1)}) + \sigma_A(t_i)\Delta W_A(t_{\nu(\eta+1)}) \tag{3.28}$$

3.2.2 Case 2

Suppose now that at time t_i we do not possess sufficient data to perform a reliable regression analysis, so that we do not know the values of

$$M_S(t_i) \qquad M_X(t_i) \qquad \mathrm{Cov}_{\mathrm{sample}}[R_S, R_X] \qquad \mathrm{Var}_{\mathrm{sample}}[R_X] \qquad \mathrm{Var}_{\mathrm{sample}}[R_\alpha]$$

This means that we may not rely on equations 3.27 to find parameters $A(t_{\nu(\eta+1)})$ and $B(t_{\nu(\eta+1)})$. However we can still proceed with their calculation, by performing an ex-ante evaluation of the returns R_S and R_X seen as *compound* returns of the daily returns r_S and r_X, for which instead we possess enough data to compute an ex-post linear regression statistics. In this perspective, returns will be seen as

$$R_S(t_{\nu(\eta+1)}) = \prod_{h=\nu\eta}^{\nu(\eta+1)-1} \big(1 + r_S(t_{h+1})\big)$$

$$R_X(t_{\nu(\eta+1)}) = \prod_{h=\nu\eta}^{\nu(\eta+1)-1} \big(1 + r_X(t_{h+1})\big)$$

Let's simplify the overall notation with the following assumptions (Figure 3.4). We reduce the future time frame so that N falls just ν days ahead of present time t_i. This way $n := N - i = \nu$. Then, rather than using index η, we indicate future returns as

$$R_S(t_i, t_N) = \prod_{h=i}^{N-1} \big(1 + r_S(t_{h+1})\big)$$

$$R_X(t_i, t_N) = \prod_{h=i}^{N-1} \big(1 + r_X(t_{h+1})\big)$$

The ex-post correlation equation 3.25 becomes

$$R_S(t_i, t_N) = A(t_N) + B(t_N)R_X(t_i, t_N) + \sigma_A(t_N)\Delta W_A(t_i, t_N) \tag{3.29}$$

where

$$A(t_N) = M_S(t_N) - B(t_N)M_X(t_N) \tag{3.30}$$

$$B(t_N) = \frac{\mathrm{Cov}_{\mathrm{sample}}[R_S, R_X]}{\mathrm{Var}_{\mathrm{sample}}[R_X]} \tag{3.31}$$

$$\sigma_A(t_N) = \mathrm{Var}_{\mathrm{sample}}[R_\alpha] \tag{3.32}$$

Now, if we assume at time t_i an ex-ante perspective, this time we may not rely on sample values to determine A and B, but we can compute them directly by exploiting the information we have on the daily return processes. To avoid confusion, such values will be indicated

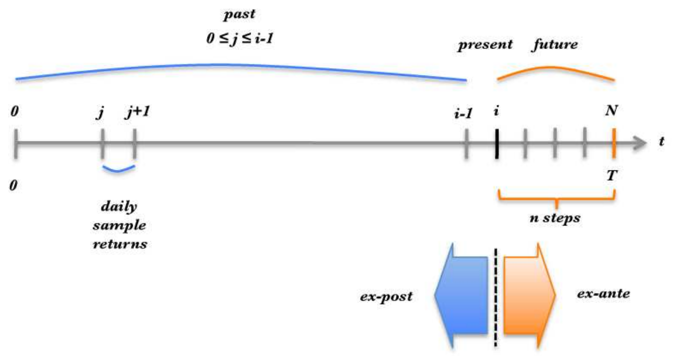


Figure 3.4: Ex-ante view at time t_i of the compound return up to terminal time t_N, based on the daily return data available.

in terms of expectations, since they are obtained at present time, not on the basis of past data as elsewhere done, but on the basis of expected future compound returns.

$$E\big[A(t_N)|\mathcal{F}_{t_i}\big] := E\big[R_S(t_i,t_N)|\mathcal{F}_{t_i}\big] - E\big[B(t_N)|\mathcal{F}_{t_i}\big]E\big[R_X(t_i,t_N)|\mathcal{F}_{t_i}\big] \tag{3.33}$$

$$E\big[B(t_N)|\mathcal{F}_{t_i}\big] := \frac{\text{Cov}\big[R_S,R_X|\mathcal{F}_{t_i}\big]}{\text{Var}\big[R_X|\mathcal{F}_{t_i}\big]} \tag{3.34}$$

$$E\big[\sigma_A(t_N)|\mathcal{F}_{t_i}\big] := \text{Var}\big[R_\alpha|\mathcal{F}_{t_i}\big] \tag{3.35}$$

This way, the ex-ante correlation equation 3.26 becomes

$$R_S(t_i,t_N) = E\big[A(t_N)|\mathcal{F}_{t_i}\big] + E\big[B(t_N)|\mathcal{F}_{t_i}\big]R_X(t_i,t_N) + E\big[\sigma_A(t_N)|\mathcal{F}_{t_i}\big]\Delta W_A(t_i,t_N) \tag{3.36}$$

3.2.3 Parameter $E\big[B(t_N)|\mathcal{F}_{t_i}\big]$

Let's start with the computation of $E\big[B(t_N)|\mathcal{F}_{t_i}\big]$ and adopt the simplified notation where the dependence on time is represented as $[]_i = [](t_i)$. In the fist place, we recall the exact and approximated equations for the expected compound returns of the index and the alpha dynamics

$$E\big[R_X(t_i,t_N)|\mathcal{F}_i\big] = (1+\mu_{X_i})^n - 1 \approx n\mu_{X_i}$$

$$E\big[R_\alpha(t_i,t_N)|\mathcal{F}_i\big] = (1+\alpha_i)^n - 1 \approx n\alpha_i$$

whereas for the asset return we consider the exact expression

$$E\big[R_S(t_i,t_N)|\mathcal{F}_i\big] = (1+\alpha_i+\beta_i\mu_{X_i})^n - 1$$

Then we recall the exact and approximated expression for the index compound return variance

$$\text{Var}\big[R_X|\mathcal{F}_{t_i}\big] = \big[(1+\mu_i)^2+\sigma^2_{X_i}\big]^n - (1+\mu_i)^{2n} \approx n\sigma^2_{X_i}(1+\mu_{X_i})^{2(n-1)}$$

Passing on to the covariance of compound returns appearing in the equation of $B(t_i)$, it may also be expressed in terms of growth

$$\text{Cov}\big[R_S,R_X|\mathcal{F}_{t_i}\big] = \text{Cov}\big[G_S,G_X|\mathcal{F}_{t_i}\big] = E\big[G_SG_X|\mathcal{F}_{t_i}\big] - E\big[G_S|\mathcal{F}_{t_i}\big]E\big[G_X|\mathcal{F}_{t_i}\big]$$

We consider first the product $G_S G_X$

$$E\big[G_S G_X|\mathcal{F}_{t_i}\big] = E\Bigg[\prod_{h=i}^{N-1}\big(1 + r_{\alpha_{h+1}} + \beta_i r_{X_{h+1}}\big)\big(1 + r_{X_{h+1}}\big)|\mathcal{F}_{t_i}\Bigg]$$

At the same time step, returns $r_{\alpha_{h+1}}$ and $r_{X_{h+1}}$ are independent one to another by definition. Each $r_{\alpha_{h+1}}$ and each $r_{X_{h+1}}$ is also independent to the returns at other times, thus all the quantities appearing in the product are independent one to another. As we have already mentioned, functions of independent variables are themselves independent, which allows us to write

$$E\big[G_S G_X|\mathcal{F}_{t_i}\big] = \prod_{h=i}^{N-1} E\Bigg[\big(1 + r_{\alpha_{h+1}} + \beta_i r_{X_{h+1}}\big)\big(1 + r_{X_{h+1}}\big)|\mathcal{F}_{t_i}\Bigg] =$$

$$\prod_{h=i}^{N-1} E\big[1 + r_{\alpha_{h+1}} + \beta_i r_{X_{h+1}} + r_{X_{h+1}} + r_{\alpha_{h+1}} r_{X_{h+1}} + \beta_i r^2_{X_{h+1}}|\mathcal{F}_{t_i}\big] =$$

$$\big[1 + \alpha_i + \beta_i \mu_{X_i} + \mu_{X_i} + \alpha_i \mu_{X_i} + \beta_i \mu^2_{X_i} + \beta_i \sigma^2_{X_i}\big]^n$$

$$E\big[G_S G_X|\mathcal{F}_{t_i}\big] = \Bigg[\big(1 + \alpha_i + \beta_i \mu_{X_i}\big)\big(1 + \mu_{X_i}\big) + \beta_i \sigma^2_{X_i}\Bigg]^n$$

We can use the binomial formula 3.17 to develop the power

$$E\big[G_S G_X|\mathcal{F}_{t_i}\big] = a^n + \sum_{g=1}^{n}\binom{n}{g} a^{n-g} b^g$$

where

$$a := \big(1 + \alpha_i + \beta_i \mu_{X_i}\big)\big(1 + \mu_{X_i}\big)$$

$$b := \beta_i \sigma^2_{X_i}$$

An approximation is now possible with the methodology seen in detail for the alpha growth variance. We will omit all the passages therein performed and limit here to the observation that all powers of a are of order 10^0, that $b = o(10^{-4})$ since $\sigma_{X_i} = o(10^{-2})$, so that we can neglect the terms of the sum with index $g > 1$. The reason is that the small terms given by the powers of the variance dominate over the binomial coefficients as the index increases.

$$\sum_{g=1}^{n}\binom{n}{g} a^{n-g} b^g \approx n a^{n-1} b$$

$$E\big[G_S G_X|\mathcal{F}_{t_i}\big] \approx a^n + n\beta_i \sigma^2_{X_i} a^{n-1}$$

Let's now consider the second term appearing in the covariance expression,

$$E\big[G_S|\mathcal{F}_{t_i}\big] E\big[G_X|\mathcal{F}_{t_i}\big] = \big(1 + \alpha_i + \beta_i \mu_{X_i}\big)^n \big(1 + \mu_{X_i}\big)^n = a^n$$

so that the covariance may be expressed as

$$\mathrm{Cov}\big[R_S, R_X|\mathcal{F}_{t_i}\big] \approx n\beta_i \sigma^2_{X_i}\big(1 + \alpha_i + \beta_i \mu_{X_i}\big)^{n-1}\big(1 + \mu_{X_i}\big)^{n-1}$$

At this point we can pass on to $E\big[B(t_N)|\mathcal{F}_{t_i}\big]$ and write

$$E\big[B(t_N)|\mathcal{F}_{t_i}\big] = \frac{\mathrm{Cov}\big[R_S, R_X|\mathcal{F}_{t_i}\big]}{\mathrm{Var}\big[R_X|\mathcal{F}_{t_i}\big]} \approx \frac{n\beta_i\sigma^2_{X_i}\big(1+\alpha_i+\beta_i\mu_{X_i}\big)^{n-1}\big(1+\mu_{X_i}\big)^{n-1}}{n\sigma^2_{X_i}(1+\mu_{X_i})^{2(n-1)}}$$

$$E\big[B(t_N)|\mathcal{F}_{t_i}\big]) \approx \beta_i\big(1+\alpha_i+\beta_i\mu_{X_i}\big)^{n-1}\big(1+\mu_{X_i}\big)^{-(n-1)}$$

A second approximation is now possible, made with the intent to 'extract' the alpha return component from the expression. We may apply again the binomial formula to write

$$\big(1+\alpha_i+\beta_i\mu_{X_i}\big)^{n-1} = (1+\alpha_i)^{n-1} + \sum_{g=1}^{n-1}\binom{n-1}{g}(1+\alpha_i)^{n-1-g}(\beta_i\mu_{X_i})^g$$

Again, if we consider that all the power terms of $(1+\alpha_i)$ are of order 10^0 and that $\beta_i\mu_{X_i} = o(10^{-4})$, we may neglect the sum terms with index $g > 1$, with a similar approach to the one extensively seen in the previous sections.

$$\big(1+\alpha_i+\beta_i\mu_{X_i}\big)^{n-1} \approx (1+\alpha_i)^{n-1} + (n-1)\beta_i\mu_{X_i}(1+\alpha_i)^{n-2} =$$

$$(1+\alpha_i)^{n-1}\left[1+(n-1)\mu_{X_i}\frac{\beta_i}{1+\alpha_i}\right]$$

Similarly, we now approximate the term

$$(1+\mu_{X_i})^{n-1} = 1 + \sum_{g=1}^{n-1}\binom{n-1}{g}(\mu_{X_i})^g \approx 1+(n-1)\mu_{X_i}$$

Getting back to $E\big[B(t_N)|\mathcal{F}_{t_i}\big]$,

$$E\big[B(t_N)|\mathcal{F}_{t_i}\big] \approx \beta_i(1+\alpha_i)^{n-1}\frac{1+(n-1)\mu_{X_i}\frac{\beta_i}{1+\alpha_i}}{1+(n-1)\mu_{X_i}}$$

Looking at

$$\frac{1+(n-1)\mu_{X_i}\frac{\beta_i}{1+\alpha_i}}{1+(n-1)\mu_{X_i}} \approx 1$$

we can clearly state that is very close to the unit value, so that finally we can write

$$E\big[B(t_N)|\mathcal{F}_{t_i}\big] \approx \beta_i(1+\alpha_i)^{n-1} \tag{3.37}$$

$E\big[B(t_N)|\mathcal{F}_{t_i}\big]$ represents the proportionality factor between the compound returns of R_S and R_X as estimated at time t_i. It is proportional to $\beta(t_i)$, which represents the equivalent parameter for the daily returns r_S and r_X. It is also proportional to $(1+\alpha_i)^{n-1}$, equivalent to the compound return of the alpha dynamics over $n-1$ time steps.

3.2.4 Parameter $E\big[A(t_N)|\mathcal{F}_{t_i}\big]$

We can now determine $E\big[A(t_N)|\mathcal{F}_{t_i}\big]$ from the relation

$$E\big[A(t_N)|\mathcal{F}_{t_i}\big] = E\big[R_S(t_i,t_N)|\mathcal{F}_{t_i}\big] - E\big[B(t_N)|\mathcal{F}_{t_i}\big]E\big[R_X(t_i,t_N)|\mathcal{F}_{t_i}\big]$$

We can exploit the results obtained in the previous section to approximate the estimated asset compound return

$$E\big[R_S(t_i,t_N)|\mathcal{F}_{t_i}\big] = (1+\alpha_i+\beta_i\mu_{X_i})^n - 1$$

$$\big(1+\alpha_i+\beta_i\mu_{X_i}\big)^n = (1+\alpha_i)^n + \sum_{g=1}^{n}\binom{n}{g}(1+\alpha_i)^{n-g}(\beta_i\mu_{X_i})^g \approx$$

$$(1+\alpha_i)^n + n\beta_i\mu_{X_i}(1+\alpha_i)^{n-1} = (1+\alpha_i)^n\left[1+n\mu_{X_i}\frac{\beta_i}{1+\alpha_i}\right]$$

Therefore,

$$E\big[R_S(t_i,t_N)|\mathcal{F}_{t_i}\big] \approx (1+\alpha_i)^n\left[1+n\mu_{X_i}\frac{\beta_i}{1+\alpha_i}\right] - 1$$

As for the expected value of index compound return,

$$E\big[R_X(t_i,t_N)|\mathcal{F}_i\big] = (1+\mu_{X_i})^n - 1 \approx n\mu_{X_i}$$

so that for $E\big[A(t_N)|\mathcal{F}_{t_i}\big]$ we can write

$$E\big[A(t_N)|\mathcal{F}_{t_i}\big] \approx (1+\alpha_i)^n\left[1+n\mu_{X_i}\frac{\beta_i}{1+\alpha_i}\right] - 1 - \beta_i(1+\alpha_i)^{n-1}n\mu_{X_i} =$$

$$(1+\alpha_i)^n + n\mu_{X_i}\beta_i(1+\alpha_i)^{n-1} - 1 - n\mu_{X_i}\beta_i(1+\alpha_i)^{n-1}$$

$$E\big[A(t_N)|\mathcal{F}_{t_i}\big] \approx (1+\alpha_i)^n - 1 = E\big[R_\alpha(t_i,t_N)|\mathcal{F}_i\big] \tag{3.38}$$

$E\big[A(t_N)|\mathcal{F}_{t_i}\big]$ can be approximated by the expected compound return of the alpha dynamics. As mentioned, $E\big[A(t_N)|\mathcal{F}_{t_i}\big]$ is the equivalent parameter of $\alpha(t_i)$, which is the expected value of the alpha return on a daily basis.

If we use this result in equation 3.33,

$$E\big[R_\alpha(t_i,t_N)|\mathcal{F}_{t_i}\big] = E\big[R_S(t_i,t_N)|\mathcal{F}_{t_i}\big] - E\big[B(t_N)|\mathcal{F}_{t_i}\big]E\big[R_X(t_i,t_N)|\mathcal{F}_{t_i}\big]$$

it results that

$$E\big[B(t_N)|\mathcal{F}_{t_i}\big] \approx \frac{E\big[R_S(t_i,t_N)|\mathcal{F}_{t_i}\big] - E\big[R_\alpha(t_i,t_N)|\mathcal{F}_{t_i}\big]}{E\big[R_X(t_i,t_N)|\mathcal{F}_{t_i}\big]} \tag{3.39}$$

3.2.5 Ex-ante estimation error

Similarly to the analysis put forward for the daily returns, we detail the steps of an implementation of the strategy on a non-daily time scale, as shown in Figure 3.5. Also in this case, we have decomposed the asset compound return into two components.

1. At each present time t_i we have statistical data deriving from past times t_j with $j \in [0, i-1]$, which allow us to compute the linear regression on a *daily* basis, but

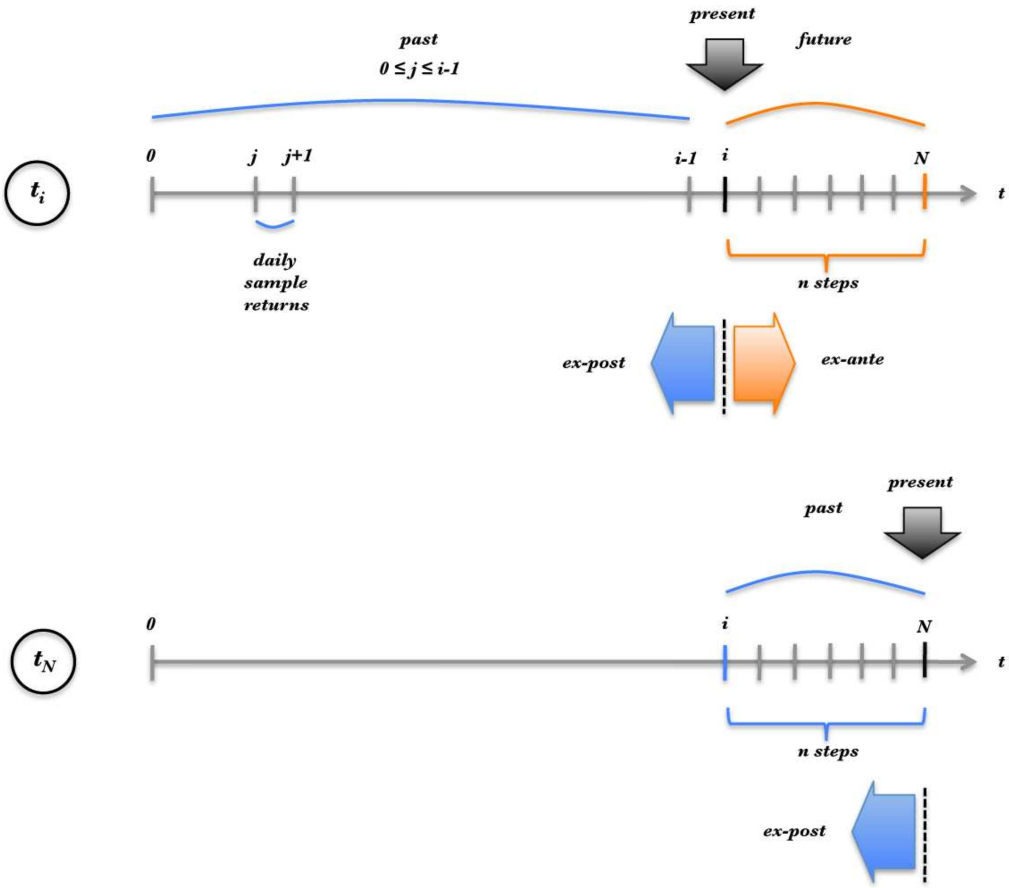


Figure 3.5: At present time t_i we assess from an ex-ante point of view which will be the *compound* return for the next n time steps up to terminal time t_N, and we do so based upon the data we have from past times t_j relevant to the *daily* returns. Then, we move the present time to t_N and assess, now from an ex-post perspective, which has effectively been the compound return during this period.

possibly not a longer time scale. Parameters $\alpha(t_i)$ and $\beta(t_i)$ are computed, verifying that $\rho(t_i)$ is high enough to validate the daily linear regression.

$$r_S(t_{j+1}) = \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1}) + \beta(t_i)\Big[\mu_X(t_i) + \sigma_X(t_i)\Psi_X(t_{j+1})\Big]$$

2. In particular, when $j = i-1$ reaches its maximum value, the previous equation gives us the linear regression for the *daily* returns at present time

$$r_S(t_i) = \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_i) + \beta(t_i)\Big[\mu_X(t_i) + \sigma_X(t_i)\Psi_X(t_i)\Big]$$

$$r_S(t_i) = r_\alpha(t_i) + \beta(t_i) r_X(t_i)$$

3. At time t_i, we assume that on a period given by the n next time steps the compound return will be given by the following equation, which represents the return decompo-

sition on such time frame

$$R_S(t_i, t_N) = E\big[A(t_N)|\mathcal{F}_{t_i}\big] + E\big[B(t_N)|\mathcal{F}_{t_i}\big]R_X(t_i, t_N) + E\big[\sigma_A(t_N)|\mathcal{F}_{t_i}\big]\Delta W_A(t_i, t_N)$$

where

$$E\big[A(t_N)|\mathcal{F}_{t_i}\big] \approx E\big[R_\alpha(t_i, t_N)|\mathcal{F}_i\big] = (1+\alpha_i)^n - 1$$

$$E\big[B(t_N)|\mathcal{F}_{t_i}\big] \approx \frac{E\big[R_S(t_i, t_N)|\mathcal{F}_{t_i}\big] - E\big[R_\alpha(t_i, t_N)|\mathcal{F}_{t_i}\big]}{E\big[R_X(t_i, t_N)|\mathcal{F}_{t_i}\big]} \approx \beta_i(1+\alpha_i)^{n-1}$$

4. At this point we apply the hedge for the next n time steps. Again, we perform an estimation of what the compound return will be at time t_N.

$$E[R_S(t_i, t_N)|\mathcal{F}_{t_i}] = E[A(t_N)|\mathcal{F}_{t_i}] + E\big[B(t_N)|\mathcal{F}_{t_i}\big]E[R_X(t_i, t_N)|\mathcal{F}_{t_i}]+$$

$$E\big[\sigma_A(t_N)\Delta W_A(t_i, t_N)|\mathcal{F}_{t_i}]$$

which is equal to

$$E[R_S(t_i, t_N)|\mathcal{F}_{t_i}] = E[A(t_N)|\mathcal{F}_{t_i}] + E[B(t_N)|\mathcal{F}_{t_i}]E[R_X(t_i, t_N)|\mathcal{F}_{t_i}] \tag{3.40}$$

5. Then we move move the present time from t_i to t_N and observe from an *ex-post* point of view, which has actually been the compound return from time t_i to time t_N.

$$R_S(t_i, t_N) = A(t_N) + B(t_N)R_X(t_i, t_N) + \sigma_A(t_N)\Delta W_A(t_i, t_N)$$

Before proceeding further, we add and subtract the term $E[B(t_N)|\mathcal{F}_{t_i}]R_X$ so that

$$R_S(t_i, t_N) = A(t_N) + B(t_N)R_X(t_i, t_N) - E[B(t_N)|\mathcal{F}_{t_i}]R_X+$$

$$E[B(t_N)|\mathcal{F}_{t_i}]R_X + \sigma_A(t_N)\Delta W_A(t_i, t_N)$$

6. Finally, we consider the difference between this value and the one we had estimated at the beginning of the period

$$\zeta_S(t_i, t_N) := R_S(t_i, t_N) - E[R_S(t_i, t_N)|\mathcal{F}_{t_i}] =$$

$$\Bigg[A(t_N) - E[A(t_N)|\mathcal{F}_{t_i}] + \Bigg(B(t_N) - E[B(t_N)|\mathcal{F}_{t_i})\Bigg)R_X(t_i, t_N)\Bigg]+$$

$$\Bigg[R_X - E[R_X(t_i, t_N)|\mathcal{F}_{t_i}]\Bigg]E[B(t_N)|\mathcal{F}_{t_i}] + \sigma_A(t_N)\Delta W_A(t_i, t_N)$$

Similarly to the result exposed in with the daily regression, also in this case we rearrange the equation to express it as the sum of the following terms (Figure 3.6)

$$\zeta_S(t_i, t_N) = \zeta_{\text{hedge}}(t_i, t_N) + \zeta_X(t_i, t_N) + \zeta_A(t_i, t_N) \tag{3.41}$$

where

$$\zeta_{\text{hedge}}(t_i, t_N) := \Bigg[R_X - E[R_X(t_i, t_N)|\mathcal{F}_{t_i}]\Bigg]E[B(t_N)|\mathcal{F}_{t_i}]$$

$$\zeta_X(t_i, t_N) := A(t_N) - E[A(t_N)|\mathcal{F}_{t_i}] + \Big(B(t_N) - E[B(t_N)|\mathcal{F}_{t_i})\Big) R_X(t_i, t_N)$$

$$\zeta_A(t_i, t_N) := \sigma_A(t_N)\Delta W_A(t_i, t_N)$$

Let's simplify notation and omit to indicate the dependance on time.

$$A := A(t_i, t_N) \qquad B := B(t_i, t_N)$$

$$R_S := R_S(t_i, t_N) \qquad R_X := R_X(t_i, t_N)$$

With such notation the difference between actual and estimated compound return is expressed by

$$\zeta_S = R_S - E[R_S|\mathcal{F}_i] = \Big[A - E[A|\mathcal{F}_i] + \Big(B - E[B|\mathcal{F}_i]\Big) R_X\Big] +$$

$$\Big[R_X - E[R_X|\mathcal{F}_i]\Big] E[B|\mathcal{F}_i] + \sigma_A \Delta W_A$$

and the error terms are

$$\zeta_{\text{hedge}} = \Big[R_X - E[R_X|\mathcal{F}_i]\Big] E[B|\mathcal{F}_i]$$

$$\zeta_X = A - E[A|\mathcal{F}_i] + \Big(B - E[B|\mathcal{F}_i]\Big) R_X$$

$$\zeta_A = \sigma_A \Delta W_A$$

Looking into each:

- $\zeta_{\text{hedge}} = \Big[R_X - E[R_X|\mathcal{F}_i]\Big] E[B|\mathcal{F}_i]$
  This first term is defined as *compound hedgable error*, for the reason that this term will vanish from the portfolio return as an effect of the implementation of the hedge. It is equal to the difference between the actual market compound return R_X and the estimated one $E[R_X|\mathcal{F}_i]$ multiplied by the coefficient $E[B|\mathcal{F}_i]$ estimated at time t_i. As long as the asset compound return at time t_N lies along the original mean line, the difference between this value and the mean one is fully hedgable. In other words, in order to execute a perfect hedge it is not necessary that the actual compound return be equal to the expected one. It may be different as long as it remains on the linear regression line.
- $\zeta_X = A - E[A|\mathcal{F}_i] + \Big(B - E[B|\mathcal{F}_i]\Big) R_X$
  This second term represents the *compound estimation error* originated by the difference in the estimated regression parameters at time t_i and the actual ones at time t_N. It is equal to the difference $A - E[A|\mathcal{F}_i]$ plus the difference in $B - E[B|\mathcal{F}_i]$ multiplied by the index compound return R_X. This error has a conceptual importance in the non-daily hedging and quantifies the accuracy of our estimation of the regression of future compound returns.

- $\zeta_A = \sigma_A \Delta W_A$

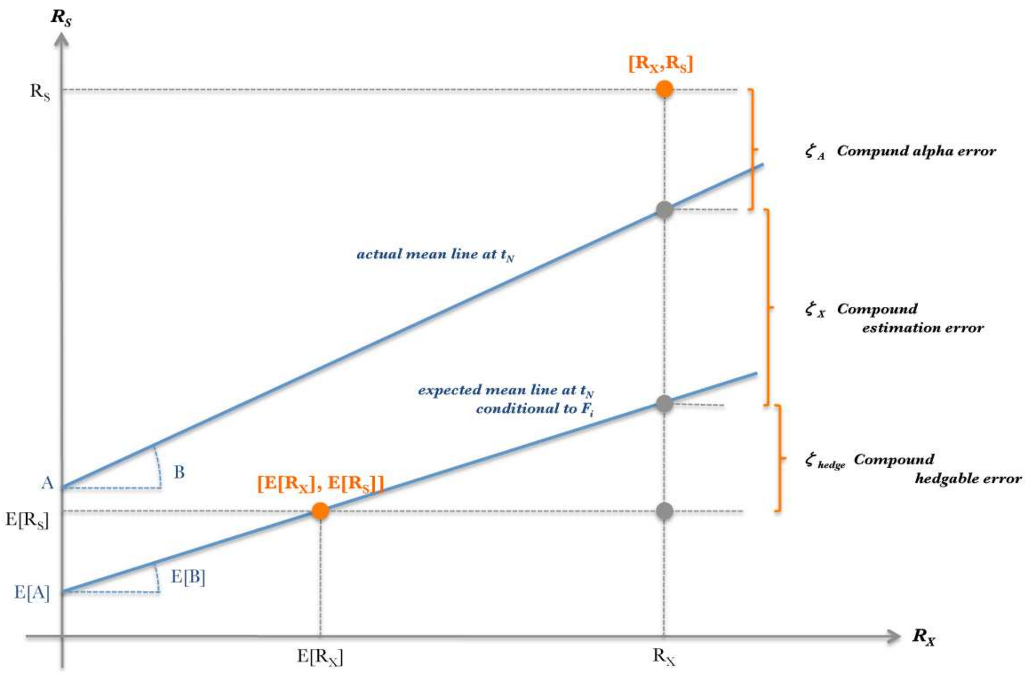


Figure 3.6: Geometric representation of the three components into which the overall difference between the expected compound return and the actual one may be decomposed. These are indicated as *compound hedgable error*, *compound estimation error* and *compound alpha error*.

> Finally, this last term represents the stochastic difference between the acutal return R_S and the value it would have if lying on the actual regression line, and is defined as *alpha compound error*.

The difference between the realized sample return R_S and the estimated one $E[R_S|\mathcal{F}_i] = E[A|\mathcal{F}_i] + E[B|\mathcal{F}_i]E[R_X|\mathcal{F}_i]$ is shown in figure 3.6 and decomposed into three error terms.

1. At time t_i we compute the expected compound return up to time t_N and identify the point $\big[E[R_X|\mathcal{F}_i], E[R_S|\mathcal{F}_i]\big]$. From here we move at time t_N along the R_X axis of a quantity given by the difference $R_X - E[R_X|\mathcal{F}_i]$ which represents the difference between expected and actual index compound return. Along the R_S axis we move of the corresponding distance, which represents the *compound hedgable error*, determined by the estimated proportionality coefficient $E[B|\mathcal{F}_i]$.

2. From here, we then continue along the R_S axis and reach the point lying on the actual mean line known at time t_N. The distance covered represents the *compound estimation error*, which vanishes if both this line is superimposed to the estimated one.

3. Finally, we add along the R_S axis a further distance given the *compound alpha error* and equal to $\sigma_A \Delta W_A$.

The main formulas introduced are summarized in Appendix A.

Chapter 4

Hedging tools

4.1 Short ETFs

Exchange Traded Funds are a relatively new type of financial security engineered in order to replicate the performance of an underlying index. We shall look into a subclass of these funds, named Short ETFs, which reproduce inversely the returns of their reference index. To this scope, we consider a benchmark index $\{X\}$ and a Short ETF $\{H\}$ priced $X(t_i)$ and $H(t_i)$ for $i \in [0, N-1]$. Their daily returns are defined as

$$r_X(t_{i+1}) := \frac{X(t_{i+1}) - X(t_i)}{X(t_i)} = \frac{\Delta X(t_{i+1})}{X(t_i)} \tag{4.1}$$

and

$$r_H(t_{i+1}) := \frac{H(t_{i+1}) - H(t_i)}{H(t_i)} = \frac{\Delta H(t_{i+1})}{H(t_i)} \tag{4.2}$$

ETFs are issued in order to deliver the index return with a pre-defined proportionality, which can be written as

$$r_H(t_{i+1}) = k r_X(t_{i+1}) \tag{4.3}$$

The proportionality is determined by k which is usually an integer number. If negative, the ETF delivers an inverse return and is named Short ETF. Typical values of k are $\{-3, -2, -1, 1, 2, 3\}$. In Table 4.1 the benchmark index performance over the implementation period is shown together with the k factor of the ETFs adopted in the three sample cases. Note that for Sample Cases 1 and 3 $k = -1$, whereas for Sample Case 2 $k = -3$.

4.1.1 ETF replication time frame

A central point of the replication features of ETFs is that the proportionality described by equation 4.1 is guaranteed exclusively on a specific time frame which is typically daily. In the case of Short ETFs, their compound return over a longer time period may be significantly different than the inverse index compound return. In other words, if we keep in portfolio an inverse ETF with $k = -1$ for one month and at the end of the period look into its overall return, we will discover that this is not equal to the inverse return of the index. This difference depends on the *path* of daily returns experienced by the index, and it may reach

Table 4.1: **Benchmark index performance**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
Benchmark	**Euro Stoxx 50**	**S&P 500**	**DAX**
From date	22.07.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
μ_X	-6.192E-05	6.307E-04	5.765E-04
σ_X	1.543E-02	1.223E-02	1.161E-02
R_X	-7.860E-02	3.756E-01	2.564E-01
Annualized R_X	-3.582E-02	1.499E-01	1.409E-01
k ideal ETF	-1	-3	-1

unexpectedly large values. Many operators have erroneously adopted these instruments for hedging purposes by holding them for a longer period than the one they were engineered for [3, 6]. Consequence of this feature is that it is necessary to trade the ETFs with daily frequency in order to execute an efficient hedge.
To prove this, let's consider the compound returns of both the benchmark index and the ETF from time t_0 to tine t_N.

$$X(t_N) = X(t_0)\prod_{i=0}^{N-1}\big[1 + r_X(t_{i+1})\big] = X(t_0)G_X(t_N) = X(t_0)\big(R_X(t_N) + 1\big)$$

where the compound growth and return are respectively defined as

$$G_X(t_N) := \prod_{i=0}^{N-1}\big[1 + r_X(t_{i+1})\big]$$

and the overall return as

$$R_X(t_N) := \prod_{i=0}^{N-1}\big[1 + r_X(t_{i+1})\big] - 1 = G_X(t_N) - 1$$

Since the ETF daily return is given by

$$r_H(t_{i+1}) = kr_X(t_{i+1})$$

its terminal value may be expressed in terms of the compound growth and return as follows

$$H(t_N) = H(t_0)G_H(t_N) = H(t_0)\big(R_H(t_N) + 1\big)$$

where

$$G_H(t_N) := \prod_{i=0}^{N-1}\big[1 + r_H(t_{i+1})\big] = \prod_{i=0}^{N-1}\big[1 + kr_X(t_{i+1})\big]$$

$$R_H(t_N) := \prod_{i=0}^{N-1}\big[1 + r_H(t_{i+1})\big] - 1 = \prod_{i=0}^{N-1}\big[1 + kr_X(t_{i+1})\big] - 1 = G_H(t_N) - 1$$

What we want to show is that after N periods, the overall return of the ETF $R_H(t_N)$, in general will not be proportional to the benchmark overall return $R_X(t_N)$

$$R_H(t_N) \neq kR_X(t_N) \tag{4.4}$$

In other terms,

$$R_H(t_N) = \prod_{i=0}^{N-1} \big(1 + kr_X(t_{i+1})\big) - 1 \neq kR_X(t_N) = k\big[\prod_{i=0}^{N-1} \big(1 + r_X(t_{i+1})\big) - 1\big]$$

For notation's sake, let's now indicate the benchmark returns as follows

$$r_{i+1} := r_X(t_{i+1}) \qquad R_N := R_X(t_N)$$

so that

$$\prod_{i=0}^{N-1} (1 + kr_{i+1}) - 1 \neq k\big[\prod_{i=0}^{N-1} (1 + r_{i+1}) - 1\big] \tag{4.5}$$

In order to see that in general 4.5 is true, let's consider its corresponding equation and look into solutions of it for the variable k.

$$\prod_{i=0}^{N-1} (1 + kr_{i+1}) - 1 = k\big[\prod_{i=0}^{N-1} (1 + r_{i+1}) - 1\big] \tag{4.6}$$

The product of the terms $(1 + r_{i+1})$ may be written as

$$\prod_{i=0}^{N-1} (1 + r_{i+1}) = \sum_{g=0}^{N-1} a_{g+1} + 1$$

where for $i \in [0, N-1]$ and $g \in [0, N-1]$ each coefficient a_{g+1} is the sum of all the $\binom{N}{g+1}$ possible combinations of product terms of order $g+1$ of the returns $r(t_{i+1})$. Similarly,

$$\prod_{i=0}^{N-1} (1 + kr_{i+1}) = \sum_{g=0}^{N-1} a_{g+1}k^{g+1} + 1$$

To find an expression for the coefficients a_{g+1}, let's start from a simple case and then by induction proceed towards the general one. Assume $N = 3$

$$R_3 = \prod_{i=0}^{2} \big(1 + r_{i+1}\big) = (1 + r_1)(1 + r_2)(1 + r_3) =$$

$$= 1 + (r_1 + r_2 + r_3) + (r_1r_2 + r_1r_3 + r_2r_3) + (r_1r_2r_3)$$

$$R_3 = \sum_{g=0}^{2} a_{g+1} + 1$$

where

$$a_1 = r_1 + r_2 + r_3 \qquad a_2 = r_1r_2 + r_1r_3 + r_2r_3 \qquad a_3 = r_1r_2r_3$$

Generalizing this result,

$$a_1 = \sum_{i=0}^{N-1} r_{i+1} = r_1 + r_2 + \cdots + r_N$$

$$a_2 = \sum_{i=0}^{N-1} \sum_{h \neq i}^{N-1} r_{i+1} r_{k+1}$$

$$\cdots$$

$$a_N = r_1 r_2 \cdots r_N$$

Getting back to equation 4.6, we can write

$$\sum_{g=0}^{N-1} a_{g+1} k^{g+1} + 1 - 1 = k\big[\sum_{g=0}^{N-1} a_{g+1} + 1 - 1\big]$$

$$k\big[\sum_{g=0}^{N-1} a_{g+1} k^g - \sum_{g=0}^{N-1} a_{g+1}\big] = 0 \tag{4.7}$$

$$\sum_{g=0}^{N-1} a_{g+1}\big(k^g - 1\big) = 0 \tag{4.8}$$

which is a polynomial equation in the variable k of order $(N-1)$. When allowing for complex solutions, there are $(N-1)$ values of k which depend on the coefficients a_{g+1} and thus on the combinations of the returns r_{i+1}, which therefore depend on the path of the index returns. If we consider equation 4.7, there are two trivial solutions. The first, is given by $k = 0$ which has no meaning in financial terms, whereas the second is $k = 1$ which indicates a unitary replication. As expected, in this case the compound return of the ETF will be equal to the index one, no matter of which the path of returns will be. In general however, the ETF overall return R_H over the period $t \in [t_0, t_N]$ will have achieved the desired proportionality with respect to the compound index return R_X only for specific values of k, namely those corresponding to the the real solutions of equation 4.8, if any. Moreover, with different time series even these specific values of k will not deliver the desired proportionality because they represent solutions only for that particular set of returns. Therefore, in the general case, ETFs engineered to deliver a proportionality on a given time scale, typically the daily scale, do not deliver it on a different period of time.

As a consequence, when adopting ETFs for hedging purposes, the hedge will be effective only on the daily basis and will not be so on a different period. In order to achieve a hedge on such time scale, a daily trade on the quantity of ETF held in the portfolio is required.

Let's look into the solutions of equation 4.8 for the first values of N. Starting with $N = 1$,

$$a_1 = r_1$$

$$\sum_{g=0}^{N-1} a_{g+1}\big(k^g - 1\big) = r_1(1-1) = 0$$

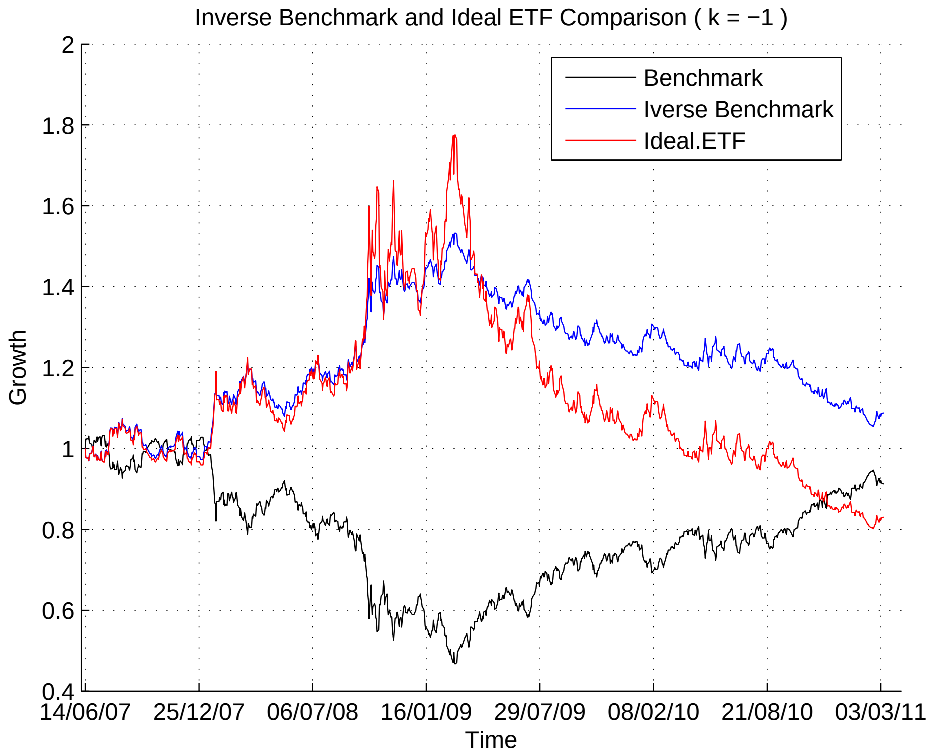


Figure 4.1: A comparison between the growth of an ideal Short ETF with $k = -1$ and an ideal inverse index.

The equation is true for all possible values of k.

For $N = 2$,

$$a_1 = r_1 + r_2 \qquad a_2 = r_1 r_2$$

$$\sum_{g=0}^{N-1} a_{g+1}\big(k^g - 1\big) = (r_1 + r_2)(1 - 1) + r_1 r_2 (k - 1) = 0$$

$$k^* = 1$$

and for $N = 3$,

$$a_1 = r_1 + r_2 + r_3 \qquad a_2 = r_1 r_2 + r_1 r_3 + r_2 r_3 \qquad a_3 = r_1 r_2 r_3$$

$$\sum_{g=0}^{N-1} a_{g+1}\big(k^g - 1\big) = a_1(1 - 1) + a_2(k - 1) + a_3(k^2 - 1) = 0$$

$$(k - 1)\big(a_2 + a_3(k + 1)\big) = 0$$

$$k_1^* = 1 \qquad k_2^* = -\big(\frac{a_2}{a_3} + 1\big) = -\big(\frac{r_1 r_2 + r_1 r_3 + r_2 r_3}{r_1 r_2 r_3} + 1\big)$$

Figure 4.1 compares the growth of an ideal ETF delivering an inverse return with respect to the benchmark index on a daily basis, with a theoretical growth $(1 + kR_X)$ equal to

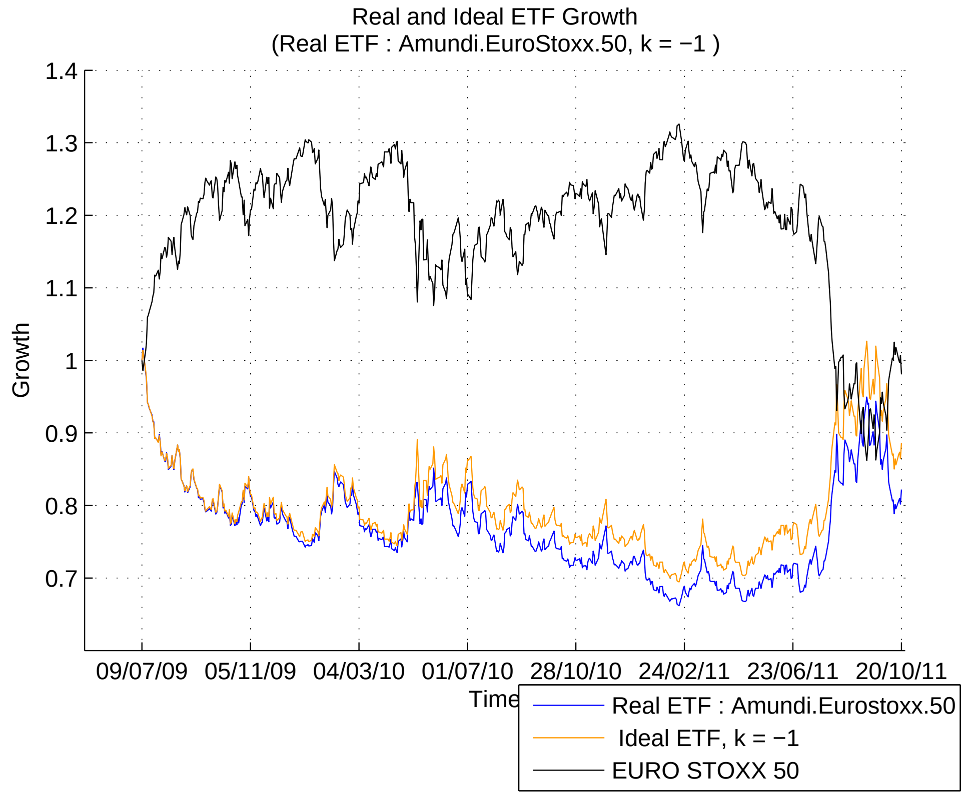


Figure 4.2: Ideal vs real growth for the ETF of Sample Case 3.

the *inverse* index. Note that the ideal ETF delivers in the long term a growth which is considerably different than the inverse one.

4.1.2 ETF tracking errors

In reality ETFs do not deliver the desired proportionality with respect to the benchmark index in a perfect way even on the daily time frame. Their returns are affected by so-called *tracking errors* which diminish importantly their effectiveness as hedging tools. A more realistic representation of their return process may be written, for $i \in [1, N-1]$ as

$$r_H(t_{i+1}) = \left[k + \delta k(t_{i+1})\right] r_X(t_{i+1}) \tag{4.9}$$

which differs from the *ideal* dynamics of equation (4.3)

$$r_H(t_{i+1}) = k r_X(t_{i+1})$$

because of the stochastic term $\delta k(t_{i+1})$ representative of the tracking error.
In Figure 4.2 we compare the growth in value of an *ideal* Short ETF with $k = -1$ on the DAX benchmark index and a *real* one affected by such errors. On the long run the compound error may be significant.
We now define the *real* proportionality factor as

$$\tilde{k}(t_{i+1}) := k + \delta k(t_{i+1}) = \frac{r_H(t_{i+1})}{r_X(t_{i+1})} \tag{4.10}$$

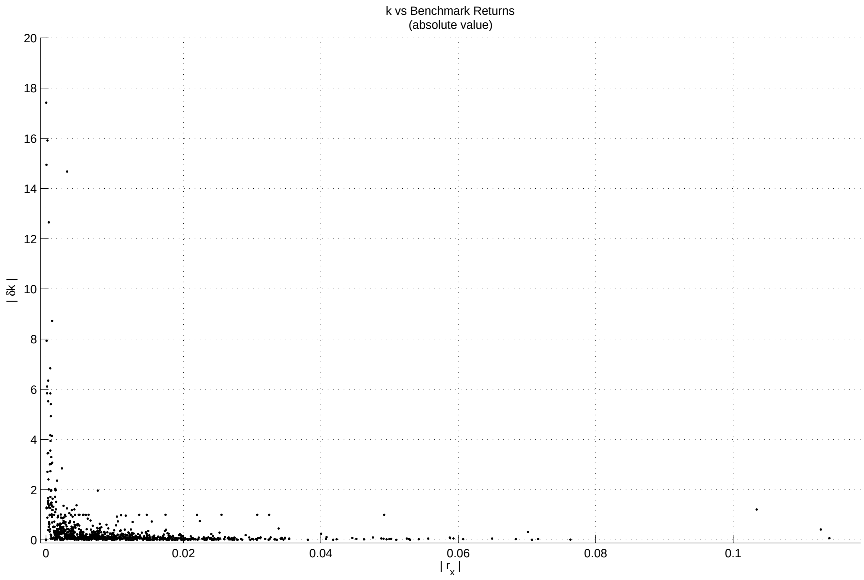


Figure 4.3: Corrleation between the index returns r_X and the ETF tracking errors δk.

and compare it for two different Short ETFs both acting on the same EuroStoxx 50 benchmark index. We first consider the ETF "DbX"[1] for which in Figure 4.4 we show the evolution over time of $\tilde{k}$ and in Figure 4.5 its frequency distribution. The same data (Figures 4.6 and 4.7) is also plotted for the other ETF "Amundi"[2] chosen as hedging tool in Sample Case 1. We can notice that the latter is characterized by a much better tracking accuracy than the first. The same hedging strategy implemented with different ETFs may deliver very different results, and an accurate selection of these need to be made with an eye on their tracking accuracy.

Figure 4.7 also shows that we can outline two different regions in the frequency distribution graph. The first is comprehensive of a vast majority of points falling very close to the nominal value $k = -1$, whereas the second is characterized by points with low frequency but distant to such value. In other words, we could identify two categories of $\tilde{k}$. The first, for which the tracking errors are negligible, and the second which instead is characterized by significant errors.
If we get back to equation 4.10, the proportionality factor is defined as the ratio of the actual ETF returns r_H with respect to the index ones r_X. When r_X is very small, a small error in r_H results in a very large tracking error δk. In Figure 4.3 we show the correlation between r_X and δk. Note that high values of the tracking error occur exclusively with very low values of the index return. This implies that the overall error on r_H, given by the product $\delta k r_X$ does not experience great variations.

[1] db x-trackers Euro Stoxx 50 Short Dly
[2] Amundi ETF Short Dow Jones Euro Stoxx 50

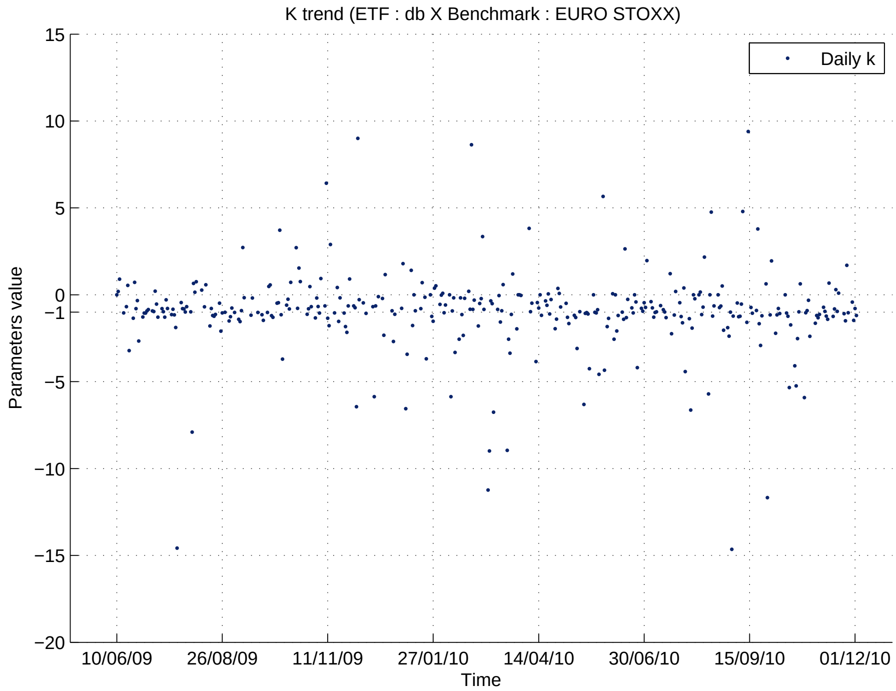


Figure 4.4: Evolution over time of $\tilde{k}(t_i)$ for the ETF DbX

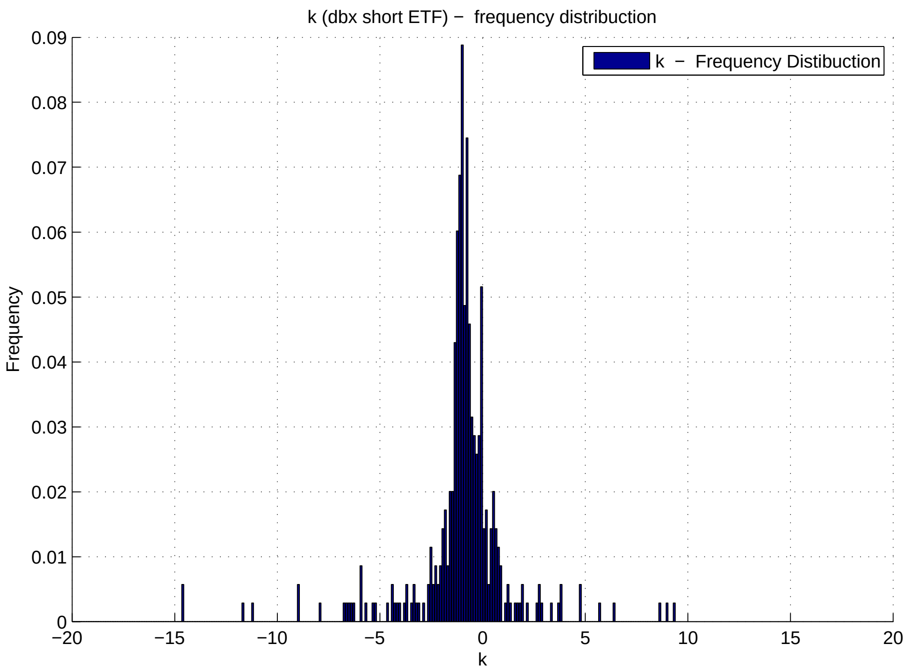


Figure 4.5: Frequency distribution of $\tilde{k}(t_i)$ for the ETF DbX

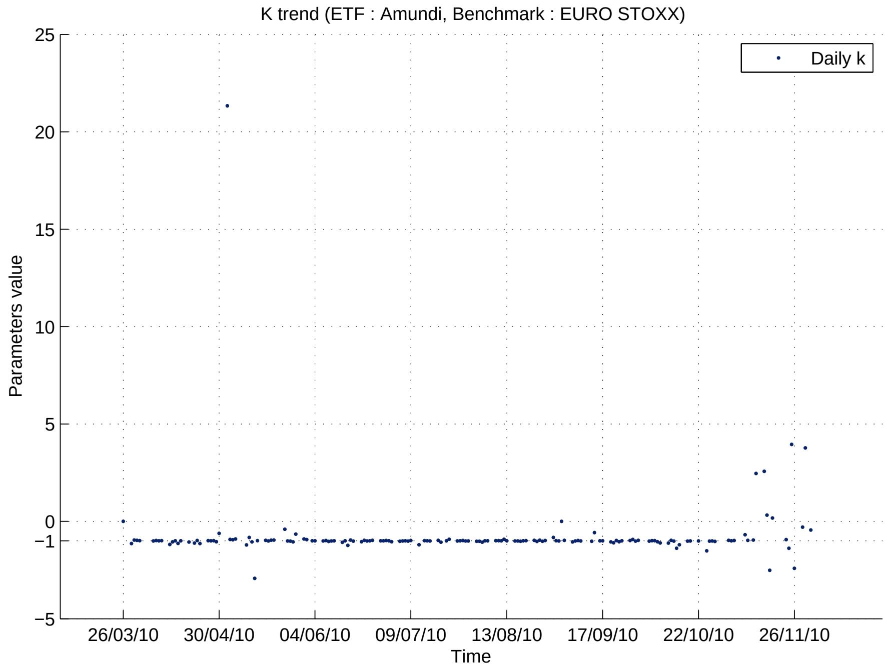


Figure 4.6: Evolution over time of $\tilde{k}(t_i)$ for the ETF Amundi

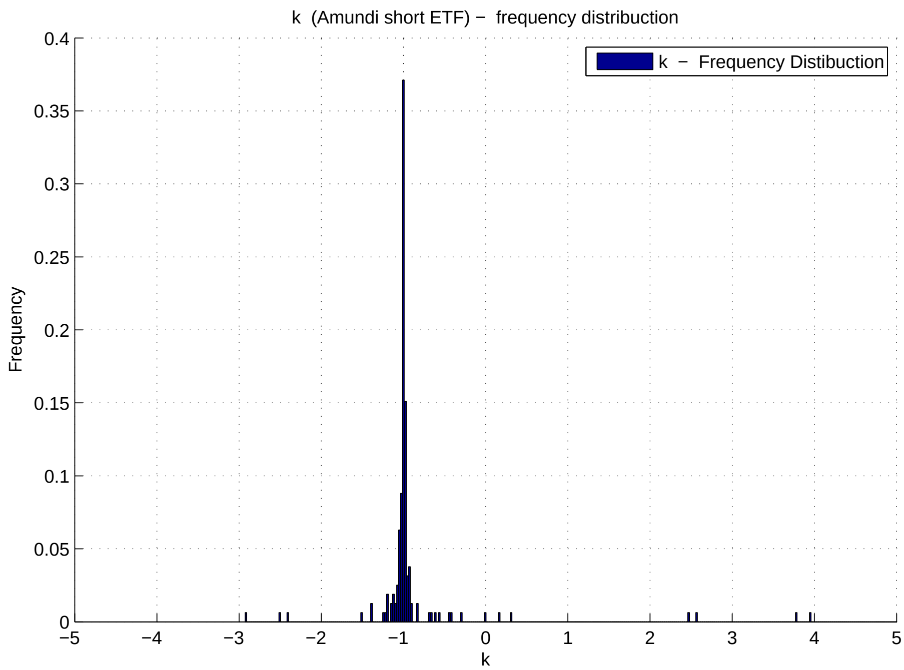


Figure 4.7: Frequency distribution of $\tilde{k}(t_i)$ for the ETF Amundi

4.2 Futures contracts

The second type of financial product that we consider for hedging purposes is the Futures contract, which is a standardized contract between two parties regulating the exchange in a future *delivery time* of a certain asset or other instrument for a pre-defined price determined at the moment of subscription. The party entering into a *long* position undertakes at present time t_i the obligation to *purchase* at delivery time t_N the security at a pre-defined price $F(t_i)$ equal to the price of the contract at present time. The party entering into a *short* position assumes on the contrary the obligation to *sell* at delivery the underlying security at the same pre-defined price. Differently than for ETFs which are financial securities that you may purchase or sell and which list among the portfolio assets, Futures are not assets themselves but contracts. For this reason, they are not directly part of the portfolio, however the contractual commitment will determine a daily cash flow up the time of delivery.

The underlying asset may be tangible or intangible, and also may represent currencies, interest rates and stock indexes, which is the type of product that we consider in this thesis. These contracts are characterized by standardized terms and are traded on specific Futures exchange markets. The exchange institutions act as an intermediary for both parties, so that when trading these contacts you only interact with the institution and will not be exposed to the default risk of the other party. The price for this guarantee is represented by the initial *margin*, which is an amount of cash that each party needs to deposit to the institution in order to have access to the contracts. With daily frequency, the intermediary will either draw money from the margin or pay an amount equal to the difference in the Futures price with respect to the previous day, according to the position of the subscriber. If the price of the contract has raised and the subscriber has a long position, then it will be credited of an amount equal to the variation of the contract price. On the contrary, for short positions this amount will be deducted from the margin and a *margin call* will be made by the institution which obliges the party to cover the difference. This mechanism, known as *marking to market*, assures that all positions are settled daily so that if one party does not maintain the required amount of margin it will automatically be cancelled from the contract [10, 17, 18].

Futures originate from a similar product, the *Forward* contract which are typically not standardized and exchanged over the counter rather than marked to market. However, it is useful to determine the price of these, because it is equal to the price of the corresponding Futures contract.

4.2.1 Pricing of a Forward contract

The pricing of a Forward and of a Futures contract can be determined via non-arbitrage argument, even though this is not the sole method. To do so, we consider the cash-flows associated with holding a Forward contract. The passage form Forward to Futures is immediate and does not imply any change in the price value. The Forward contract determines the same type of obligation between two parties, which agree on a commitment to either sell or buy a certain security at delivery time, at a price determined today.

Let's indicate with $F(t_i)$ the price of the contract at present time, which is also the settlement price agreed upon, and with $V(t_i)$ the price of the underlying financial security. At delivery time t_N, the parties agree to exchange the security $V(t_N)$ at the price $F(t_i)$. For a long position,

1. At present time t_i with $i \in [0, N-1]$:
    - No cash flow;
    - Obligation to buy the security at terminal time at price $F(t_i)$;
    - Net position: 0
2. At delivery time t_N:
    - Cash flow associated with the purchase of the security: $-F(t_i)$;
    - Value of the security $V(t_N)$;
    - Net position: $V(t_N) - F(t_i)$.

At delivery time,

- If $V(t_N) > F(t_i)$, the subscriber of a long position makes a profit;
- If $V(t_N) < F(t_i)$, the subscriber of a long position realizes a loss;

There is the possibility to build a specific portfolio in order to eliminate this uncertainty on the final outcome (profit or loss) which will determine a unique price for the Forward contract at time t_i for no-arbitrage reasons.
Suppose that at time t_i the subscriber enters into the obligation given by the Forward contract but that at the same time short sells the underlying security and deposits the corresponding revenues on a bank account to receive interests. At delivery time t_N, he will then pay an amount $F(t_i)$ to purchase the asset so that he can cancel the short position. In Table 4.2, we show a simulation of the balance sheet related to such operation, which we summarize as follows:

1. Present time t_i with $i \in [0, N-1]$:
    - Obligation to buy the security at terminal time at price $F(t_i)$;
    - Position deriving from the short sell of the security: $-V(t_i)$
    - Revenues from short selling the security: $V(t_i)$
    - Net position: 0
2. Terminal time t_N:
    - Capital and accrued interests on the cash deposit: $V(t_i)\big[1 + I(t_N)\big]$
    - Cash flow associated with the purchase of the security: $-F(t_i)$;
    - Value of the security: $V(t_N)$;
    - Cancel of the short position: $-V(t_N)$;
    - Net position: $V(t_i)\big[1 + I(t_N)\big] - F(t_i)$.

Table 4.2: **Arbitrage portfolio**

Assets		**Liabilities**	
	Time t_i		
Cash from short sell of security	$V(t_i)$	Debt for delivering security	$V(t_i)$
Total cash	$V(t_i)$	**Total debt**	$V(t_i)$
	Time t_N		
Initial cash	$V(t_i)$		
Accrued interests	$V(t_i)I(t_N)$	Debt for delivering security	$V(t_N)$
Cash to purchase the security	$-F(t_i)$	Cancel of debt	$-V(t_N)$
Total cash	$V(t_i)(1+I(t_N))-F(t_i)$	**Total debt**	0
Purchase of the security	$V(t_N)$	Profit from interests	$V(t_i)I(t_N)$
Delivery of the security	$-V(t_N)$	Plusvalue	$V(t_i) - F(t_i)$
Total holdings	0	**Total profit**	$V(t_i)(1+I(t_N))-F(t_i)$

where we have indicated the accrued interests with the notation

$$I(t_N) := \prod_{h=i}^{N-1} [1 + r(t_h)]$$

Note that with such scheme, there is no uncertainty as to which will be net positions at initial and expiry times. This determines the contract's price at time t_i as equal to

$$F(t_i) = V(t_i)E\big[I(t_n)\big|\mathcal{F}_{t_i}\big] \tag{4.11}$$

which, assuming that

$$E\big[I(t_n)\big|\mathcal{F}_{t_i}\big] = [1 + r(t_i)]^{N-i} = [1 + r(t_i)]^n \tag{4.12}$$

with $n := N - i$, becomes

$$F(t_i) = V(t_i)[1 + r(t_i)]^n \tag{4.13}$$

The correct price is given exactly by this value because it were bigger or smaller there would an arbitrage opportunity that once exploited would force the price to stabilize back around this value.

Suppose in the first place that $F(t_i) < V(t_i)[1 + r(t_i)]^n$. This would imply that the subscriber of a long position realizes a risk-less profit equal to $V(t_i)[1+r(t_i)]^n - F(t_i) > 0$. This would attract more subscribers which would make the price rise up to reaching the point given by equation 4.13.

Now suppose instead that $F(t_i) > V(t_i)[1 + r(t_i)]^n$. In this case, it would be convenient to enter into a short position on the Futures contract and at the same time to purchase the security by borrowing money. In this case we would have:

1. At present time t_i with $i \in [0, N - 1]$:

    - Obligation to sell the security at terminal time at price $F(t_i)$;

- Debt towards the bank for borrowing money: $-V(t_i)$;
- Cash from borrowing money: $V(t_i)$;
- Purchase of the security: $-V(t_i)$;
- Asset in portfolio: $V(t_i)$;
- Net position: 0

2. At delivery time t_N:

   - Capital and accrued interests on the debt: $-V(t_i)\big[1+I(t_N)\big]$
   - Cash flow associated with the sale of the security: $+F(t_i)$;
   - Asset in portfolio: $V(t_N)$;
   - Sale of the security: $-V(t_N)$;
   - Net position: $F(t_i) - V(t_i)\big[1+I(t_N)\big]$.

We have assumed that

$$F(t_i) - V(t_i)\big[1+I(t_N)\big] > 0$$

which implies also that

$$F(t_i) - V(t_i)[1+r(t_i)]^n > 0$$

Also this scheme has no uncertainty as to the net balance at initial and terminal times and thus would deliver a risk-less profit. Such arbitrage opportunity would then lead more and more subscribers to enter into short positions and this would lower the price until it reaches the value $F(t_i) = V(t_i)[1+r(t_i)]^n$.

4.2.2 Futures on index with no dividends

As mentioned, the price of a Forward and a Futures contract with same contractual terms is equivalent. To our scope, we are interested in Futures linked to the stock indexes which we use as benchmarks in the correlation analysis [11, 12, 13, 14, 15]The pricing formula therefore becomes:

$$F(t_i) = X(t_i)E\big[I(t_n)\big|\mathcal{F}_{t_i}\big] \tag{4.14}$$

and if we assume equation 4.12 to be valid,

$$F(t_i) = X(t_i)[1+r(t_i)]^n \tag{4.15}$$

In real life the price of the Futures contract is not equal to the theoretical one just described. In the first place, oscillations around the equilibrium value are considered in this pricing scheme, because through such oscillations the price if forced to get back to the non-arbitrage one if for any reason it has assumed a different value. But the price also depends on the expectation of the future interests that a deposit can produce, and this expectation may not be the same for different operators and may be different to the one indicated in equation 4.12. To justify the pricing formula, we have also assumed that the interest rate for positive deposits is the same than that of negative deposits, which is usually not the case. At any given time, interest rates may be different from operator to operator.

Table 4.3: **Futures price error**
(data relevant to Sample Case 3)

Item	*Value*
Mean	-5.717E-05
σ_F	6.406E-03

To take into account all these factors, we expand the previous price formula into a more *realistic* one as follows

$$F(t_i) = X(t_i)\Big[E\big[I(t_n)\big|\mathcal{F}_{t_i}\big] + \sigma_F(t_i)\Psi_F(t_i)\Big] \tag{4.16}$$

where $\Psi_F(t_i)$ is a stochastic variable with zero mean and unit variance and $\sigma_F(t_i)$ indicates the standard deviation of such error. If we assume that the expected future interest rates are equal to the present ones as given in equation 4.12, then

$$F(t_i) = X(t_i)\Big[[1 + r(t_i)]^n + \sigma_F(t_i)\Psi_F(t_i)\Big] \tag{4.17}$$

In Figures 4.8 and 4.9 we show the ratio

$$\frac{F(t_i)}{X(t_i)} = [1 + r(t_i)]^n + \sigma_F(t_i)\Psi_F(t_i)$$

for the Futures on the DAX index. Note that the slope of the ideal value (blue lines) depends on the interest rates and that the range of oscillation of the real prices (green lines) is approximately constant over the period of life of the contracts. As with most financial securities, the standard deviation experiences a significant increase during the periods of financial crises, but this is an aspect which goes beyond the purposes of this work and in any case does not affect importantly the results. In Table 4.3 we show the mean and standard deviation σ_F for the Futures of Sample Case 3, which is the only one for which we can compare the real prices with the ideal ones, since as we have mentioned the DAX index is the only out of the three sample indexes which reflects in its price any dividend distribution [16].

The *ideal* value we have indicated with a blue line however is not directly observed from market data. In Figures 4.8 and 4.9 we have considered the Euribor 1 month interest rate, but as we have mentioned this might not be the rate adopted by all operators. What you get is the real prices, from which it is possible to obtain an implicit rate

$$r_{\text{impl}}(t_i) = \left[\frac{F(t_i)}{X(t_i)}\right]^{\frac{1}{n}} - 1$$

which as n decreases to zero becomes increasingly unstable, as plotted in Figure 4.10 together with the risk free rate.

If we consider the Euro Stoxx 50 and the DAX indexes, adopted in Sample Cases 1 and 3 respectively, at any given date three different Futures contracts are available, each being issued three quarters before delivery time. The contract which is closest to delivery is

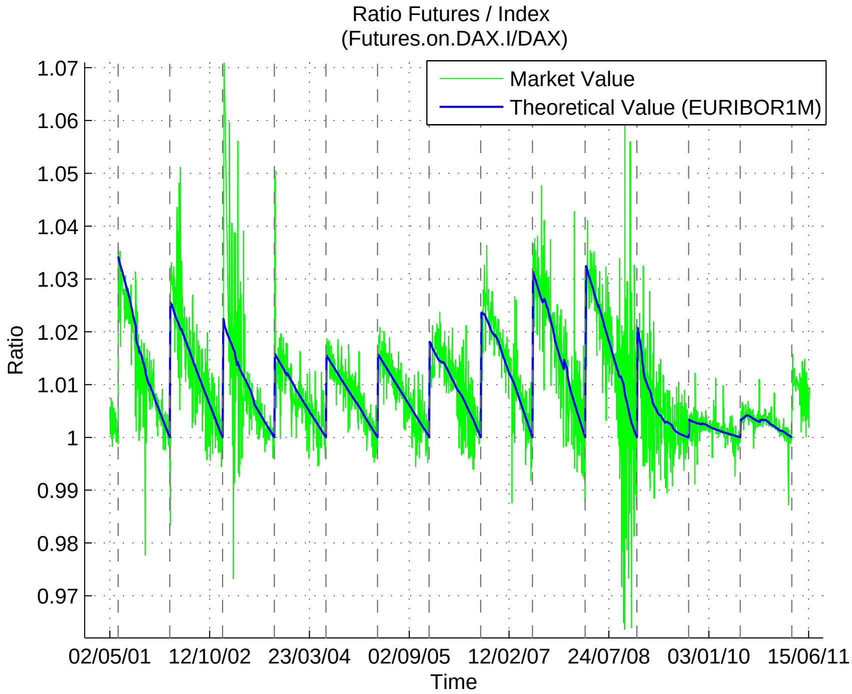


Figure 4.8: Over ten years of pricing for the Futures on the DAX index (Sample Case 3). The graph shows the ratio between contract value $F(t_i)$ and index value $X(t_i)$. The vertical lines indicate the delivery date of the contracts which occur every three quarters. The blue lines indicate the ideal value considering the Euribor 1 month interest rate, which determines the slope. The green line shows instead the real data.

typically traded with higher volumes than the other two, and has a price which is inferior, given that the interest factor is computed for a shorter period of time. This implies that at every quarter there is a contract expiring which is substituted by a new one. The S&P 500 (Sample Case 2) has instead eight contracts contemporarily on the market each with a life time of two years. Again, at the end of every quarter there will be a contract expiring and a new one being issued. In Figure 4.11 we show the ratio between the three Futures contracts in existence and the index for the DAX.

4.2.3 Futures on index with dividends

When a financial security V such as a stock distributes dividends, its market price experiences a decrease of exactly the amount of the dividends, which can be easily shown with a no-arbitrage argument. Indicating time t_d with $d \in [i, N-1]$ the moment in which an amount $d(t_d)$ is distributed, and with t_d^- and t_d^+ the times immediately before and after the distribution, it results that

$$V(t_d^-) = V(t_d^+) + d(t_d)$$

This has also an effect on the Forward and Futures contracts linked to the asset, which also experience a decrease in value.

$$F(t_d^-) = V(t_d^-)[1 + r(t_i)]^n = \left[V(t_d^+) + d(t_d)\right][1 + r(t_i)]^n = F(t_d^+) + d(t_d)[1 + r(t_i)]^n$$

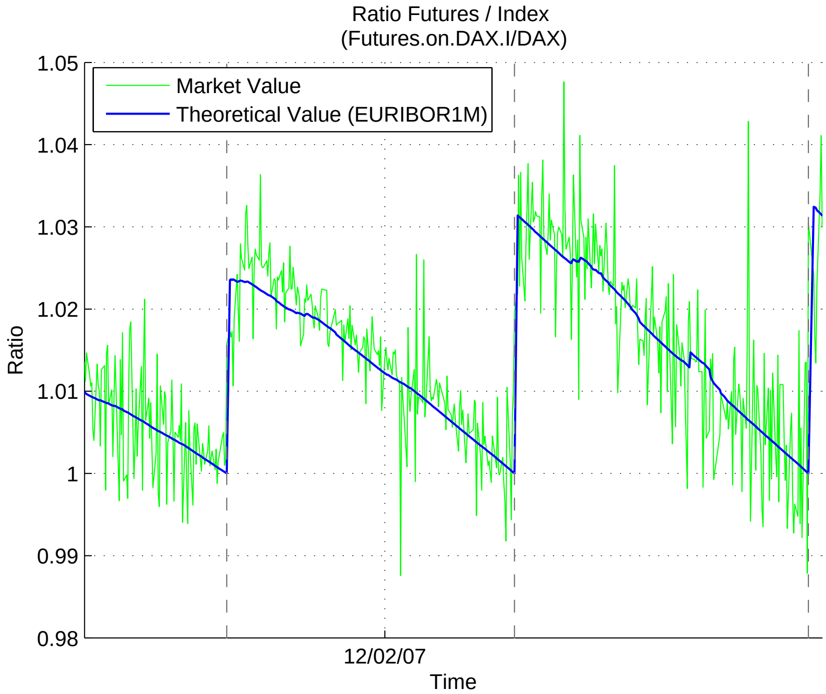


Figure 4.9: Zoom of the pervious figure in a central area. Note that the oscillation of the real data remains approximately constant over the two periods shown.

$$F(t_d^-) = F(t_d^+) + d(t_d)[1 + r(t_i)]^n$$

A correct pricing of these contracts needs to take into account the possibility that the underlying security might distribute dividends, and this is done by discounting the contract by an amount equal to the expected present value of the received dividend.

$$D(t_i) := E\Big[d(t_d)[1 + r(t_i)]^{-(d-i)}|\mathcal{F}_{t_i}\Big]$$

$$F(t_i) = \big[V(t_i) - D(t_i)\big]E\big[I(t_n)\big|\mathcal{F}_{t_i}\big]$$

or

$$F(t_i) = \big[V(t_i) - D(t_i)\big][1 + r(t_i)]^n$$

Passing now to the Futures contracts on an index, we have to consider that each stock comprised in it might distribute a dividend. If we indicate the total number of stocks with $Z \in \mathcal{N}$ and each stock V_z with $z \in [1, Z]$, then the dividends distributed by each stock can de indicated as d_z occurring at times t_{d_z} and the discount factor may thus be written as

$$D(t_i) := E\Bigg[\sum_{z=1}^{Z} d_z(t_{d_z})\big[1 + r(t_i)\big]^{-(d_z-i)}|\mathcal{F}_{t_i}\Bigg]$$

The Futures contract price then becomes

$$F(t_i) = \big[X(t_i) - D(t_i)\big]\Big[E\big[I(t_n)\big|\mathcal{F}_{t_i}\big] + \sigma_F(t_i)\Psi_F(t_i)\Big] \tag{4.18}$$

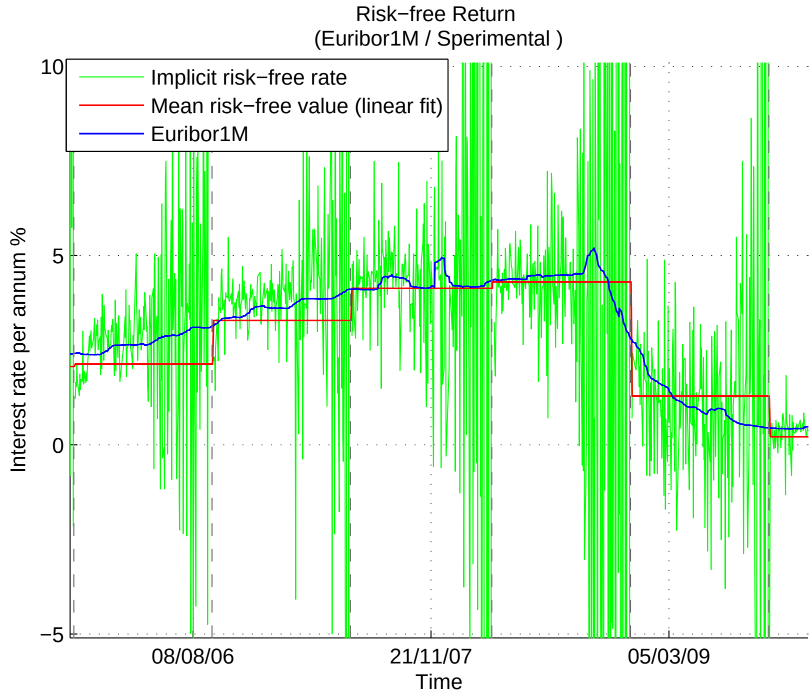


Figure 4.10: The green lines represent the implicit risk-free rate which becomes very unstable close to the delivery times, but always oscillates always around the blue line given by the Euribor 1 month rate, as shown also by the red line obtained through linear fit.

or

$$F(t_i) = \big[X(t_i) - D(t_i)\big]\Big[[1 + r(t_i)]^n + \sigma_F(t_i)\Psi_F(t_i)\Big] \tag{4.19}$$

We can also adopt a simplified notation by introducing a new quantity which collects all the terms that do not appear in the ideal pricing

$$F(t_i) = X(t_i)[1 + r(t_i)]^n + X(t_i)\sigma_F(t_i)\Psi_F(t_i) - D(t_i)\Big[[1 + r(t_i)]^n + \sigma_F(t_i)\Psi_F(t_i)\Big]$$

$$dF(t_i) := \big[X(t_i) - D(t_i)\big]\sigma_F(t_i)\Psi_F(t_i) - D(t_i)[1 + r(t_i)]^n \tag{4.20}$$

so that

$$F(t_i) = X(t_i)[1 + r(t_i)]^n + dF(t_i) \tag{4.21}$$

Figure 4.12 shows the ratio between Futures price and index value for the Euro Stoxx 50 (Sample Case 1). This index does not discount the dividends distribution in its value, as other indexes such as the DAX do. For this reason, the green line representing the sample data is at most times positioned under the ideal value shown in blue, which is not comprehensive of both the stochastic fluctuation and the dividend discount factor. This was not the case for the DAX as shown in Figure 4.8, because this index takes into account the dividend distribution directly into its value. To verify that the difference in the values is attributable to the dividend discount, in Figure 4.13 we have used a time series relevant to the Euro Stoxx 50 dividend distributions to track a dividend-adjusted line starting from the green one and adding to it the amount of dividends actually distributed. By doing so, the resulting blue line becomes very close to the ideal one colored in red, and the remaining

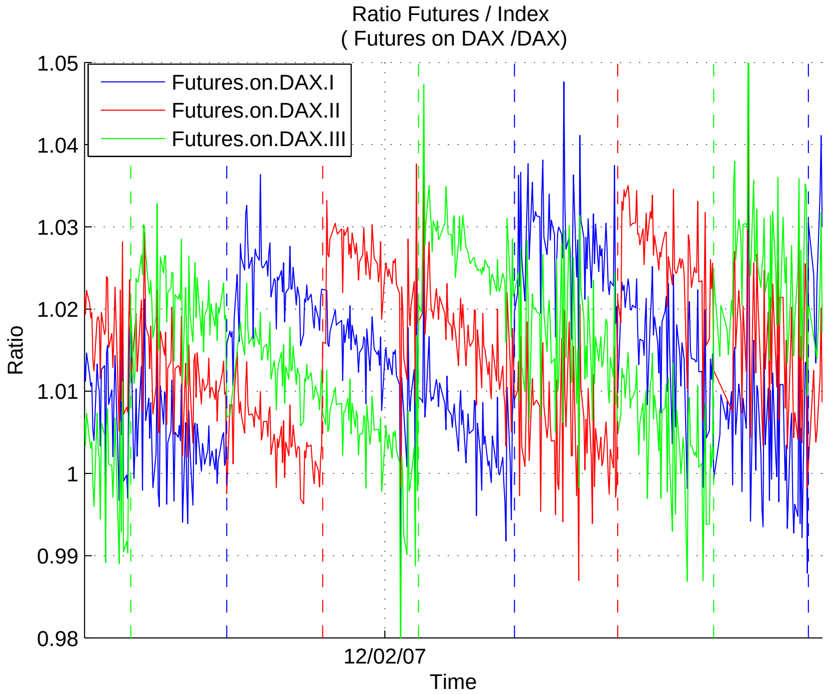


Figure 4.11: At any given time, there are three different Futures contracts on the DAX index, each valid for three quarters, so that there is a contract expiring every quarter.

distance represents the difference between the dividend estimation implicit into the price and the realized distribution.

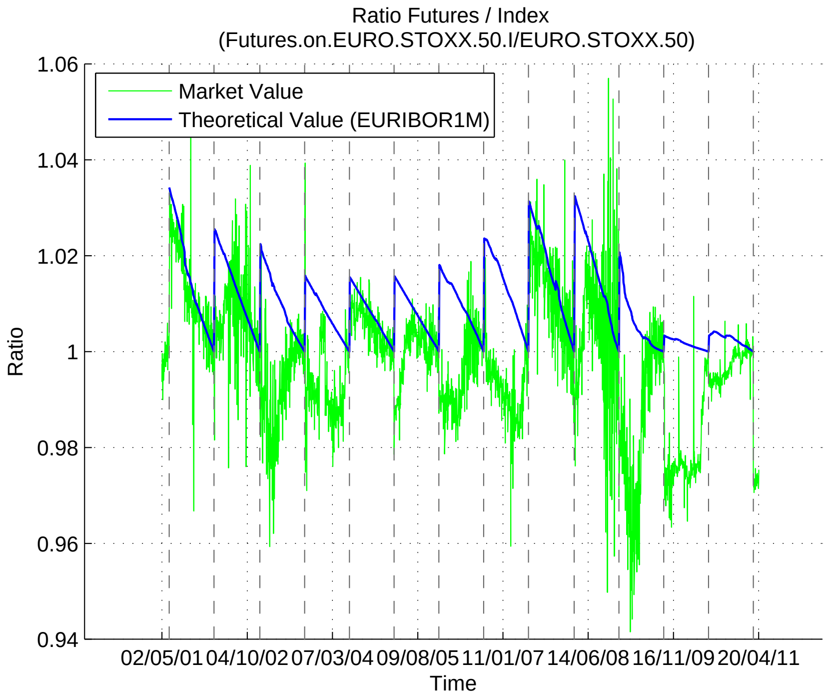


Figure 4.12: Ratio between price $F(t_i)$ and $X(t_i)$ for the Euro Stoxx 50 index. Note that the green line has an inferior value with respect to the no-dividend ideal blue line.

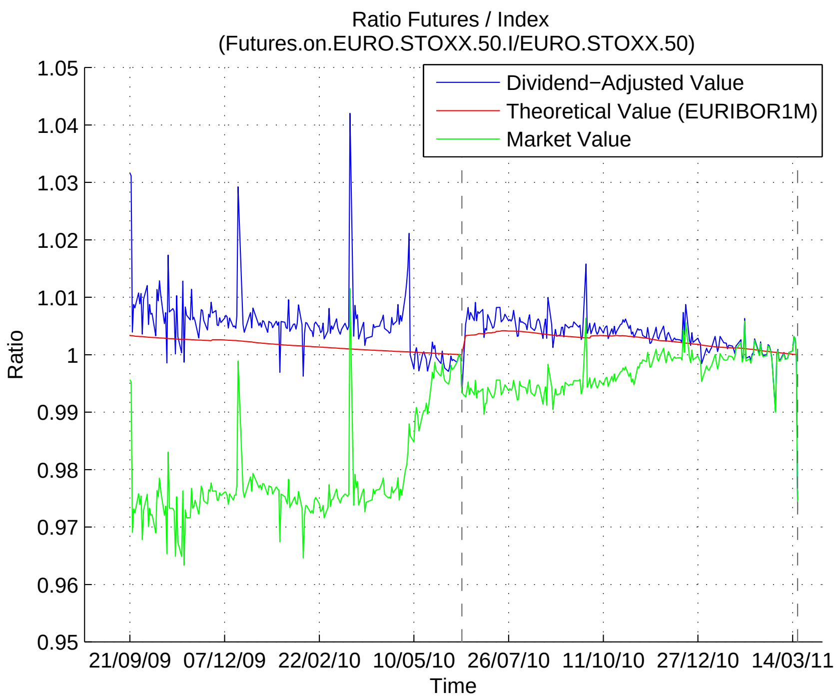


Figure 4.13: The blue line is obtained by adding the known distributed dividends on the Euro Stoxx 50 to the green line which represents the market price ratio. In this way, it gets very close to the ideal ratio shown in red.

Chapter 5

Portfolios with daily hedging

We are now ready to discuss the implementation of the investment strategy. In this chapter we will consider the first two of the three portfolios indicated in Section 2.1.2, namely the portfolio hedged on a daily basis with Short ETFs and the one hedged with same frequency by the use of Futures.

5.1 Portfolio hedged with Short ETFs

5.1.1 Expected portfolio return

Let's start with the portfolio hedged with Short ETFs. Present time is fixed at t_i with $i \in [1, N-1]$ and the portfolio $\{\Pi_H\}$ considered is composed of a fixed quantity λ of the core-asset $\{S\}$ priced $S(t_i)$, a deterministically variable quantity $q_H(t_i)$ of the short ETF $\{H\}$ priced $H(t_i)$ and cash $C_H(t_i)$. As mentioned, λ is a constant since we are treating the core-asset as non-tradable.

$$\Pi_H(t_i) = \lambda S(t_i) + q_H(t_i)H(t_i) + C_H(t_i) \tag{5.1}$$

At this same time, we evaluate from an ex-ante point of view what the portfolio value will be at the following time step t_{i+1}, and according to this evaluation we determine the quantity $q_H(t_i)$ of Short ETF to hold so that the portfolio value will vary independently from the benchmark index.

$$E[\Pi_H(t_{i+1})|\mathcal{F}_{t_i}] = \lambda E[S(t_{i+1})|\mathcal{F}_{t_i}] + q_H(t_i)E[H(t_{i+1})|\mathcal{F}_{t_i}] + E[\Delta q_H(t_{i+1})H(t_{i+1})|\mathcal{F}_{t_i}] +$$

$$C_H(t_i)\big(1 + r(t_i)\big) - E[\Delta q_H(t_{i+1})H(t_{i+1})|\mathcal{F}_{t_i}]$$

where we consider that the quantity $q_H(t_i)$ of ETF currently held will vary of the expected traded quantity $\Delta q_H(t_{i+1})$ valued at price $H(t_{i+1})$. The future value of cash will be equal to the current balance increased by the interests given by the risk-free interest rate $r(t_i)$ and diminished by the quantity of cash $E[\Delta q_H(t_{i+1})H(t_{i+1})|\mathcal{F}_{t_i}]$ necessary to trade the ETF.

As a result,

$$E[\Pi_H(t_{i+1})|\mathcal{F}_{t_i}] = \lambda E[S(t_{i+1})|\mathcal{F}_{t_i}] + q_H(t_i)E[H(t_{i+1})|\mathcal{F}_{t_i}] + C_H(t_i)\big(1 + r(t_i)\big)$$

Now we consider the expected variation of the portfolio value

$$E[\Pi_H(t_{i+1}) - \Pi_H(t_i)|\mathcal{F}_{t_i}] = E[\Pi_H(t_{i+1})|\mathcal{F}_{t_i}] - \Pi_H(t_i) =$$

$$\lambda\Big[E[S(t_{i+1})|\mathcal{F}_{t_i}] - S(t_i)\Big] + q_H(t_i)\Big[E[H(t_{i+1})|\mathcal{F}_{t_i}] - H(t_i)\Big] + C_H(t_i)r(t_i)$$

The first term may be written as the expected value of the asset price difference as

$$E[S(t_{i+1})|\mathcal{F}_{t_i}] - S(t_i) = E[S(t_{i+1}) - S(t_i)|\mathcal{F}_{t_i}] = S(t_i)E[r_S(t_{i+1})|\mathcal{F}_{t_i}] =$$

$$S(t_i)\big[\alpha(t_i) + \beta(t_i)\mu_X(t_i)\big]$$

Similarly, for the second term we can write

$$E[H(t_{i+1})|\mathcal{F}_{t_i}] - H(t_i) = E[H(t_{i+1}) - H(t_i)|\mathcal{F}_{t_i}] = H(t_i)E[r_H(t_{i+1})|\mathcal{F}_{t_i}] = H(t_i)k\mu_X(t_i)$$

given that

$$r_H(t_{i+1}) = \big(k + \delta k(t_{i+1})\big)r_X(t_{i+1})$$

which has expected value of

$$E[r_H(t_{i+1})|\mathcal{F}_{t_i}] = kE[r_X(t_{i+1})|\mathcal{F}_{t_i}] + E[\delta k(t_{i+1})r_X(t_{i+1})|\mathcal{F}_{t_i}] = k\mu_X(t_i)$$

since

$$E[\delta k(t_{i+1})r_X(t_{i+1})|\mathcal{F}_{t_i}] = 0$$

Getting back to the expected portfolio variation,

$$E[\Pi_H(t_{i+1})|\mathcal{F}_{t_i}] - \Pi_H(t_i) = \lambda S(t_i)\big[\alpha(t_i) + \beta(t_i)\mu_X(t_i)\big] + q_H(t_i)H(t_i)k\mu_X(t_i) + C_H(t_i)r(t_i)$$

which we can rearrange into

$$E[\Pi_H(t_{i+1})|\mathcal{F}_{t_i}] - \Pi_H(t_i) = \lambda S(t_i)\alpha(t_i) + \Big[\lambda S(t_i)\beta(t_i) + q_H(t_i)H(t_i)k\Big]\mu_X(t_i) + C_H(t_i)r(t_i)$$

At this point, if we impose to hold at time t_i the following quantity of Short ETF

$$q(t_i) = -\beta(t_i)\frac{\lambda S(t_i)}{kH(t_i)} \tag{5.2}$$

then the term proportional to the market drift $\mu_X(t_i)$ will vanish and the expected portfolio variation will be

$$E[\Pi_H(t_{i+1})|\mathcal{F}_{t_i}] - \Pi_H(t_i) = \lambda S(t_i)\alpha(t_i) + C_H(t_i)r(t_i) \tag{5.3}$$

By adopting this specific quantity of ETF, the expected variation in the portfolio value will not depend on the market return and it will be composed of two terms. The first, equal to the alpha mean return multiplied by the value of asset withheld, and the second equal to the interests on the cash balance. Note that this quantity is proportional to the ratio $\frac{S(t_i)}{H(t_i)}$ of the asset price to the ETF price and that for positive $\beta(t_i)$ and negative k this quantity is positive, which means that we hold a *long* position on a Short ETF. For increasing values of $|k|$ the ETF delivers a higher proportionality factor with respect to the market and so q_H diminishes because a smaller amount of ETF is required to obtain the same hedge.

$$k := \frac{r_H(t_{i+1})}{r_X(t_{i+1})} - \delta k(t_{i+1})$$

The ETF quantity is also proportional to $\beta(t_i)$ which represents the proportionality between the mean asset and the market returns

$$\beta(t_i) = \frac{\mu_S(t_i) - \alpha(t_i)}{\mu_X(t_i)}$$

An increasing $\beta(t_i)$ implies a higher sensitivity of the asset towards the market and therefore a higher value of ETF is required to implement the hedge.
In order to consider the expected *return* of the portfolio, rather than the expected variation, we introduce the following quantities which indicate the proportions of asset, ETF and cash held in the portfolio.

$$\phi_H(t_i) := \frac{q(t_i)H(t_i)}{\Pi_H(t_i)} \qquad \phi_S(t_i) := \frac{\lambda S(t_i)}{\Pi_H(t_i)} \qquad \phi_C(t_i) := \frac{C_H(t_i)}{\Pi_H(t_i)} \tag{5.4}$$

where

$$\phi_S(t_i) + \phi_H(t_i) + \phi_C(t_i) = 1$$

If we divide each term of equation 5.3 by $\Pi_H(t_i)$, we obtain

$$E[r_{\Pi_H}(t_{i+1})|\mathcal{F}_{t_i}] = \phi_S(t_i)\alpha(t_i) + \phi_C(t_i)r(t_i) \tag{5.5}$$

We can also express the expected return in function of only one allocation variable $\theta_H(t_i)$, defined as the fraction of portfolio allocated jointly into the core-asset and into the ETF.

$$\theta_H(t_i) := \phi_S(t_i) + \phi_H(t_i) \tag{5.6}$$

To do so, we write ϕ_C and ϕ_S in function of θ_H. For the first, it results that

$$\phi_C(t_i) = 1 - \theta_H(t_i)$$

whereas we can express the second as a sub-fraction γ of θ_H

$$\gamma(t_i) := \frac{\phi_S(t_i)}{\theta_H(t_i)} \tag{5.7}$$

which is equal to

$$\gamma(t_i) = \frac{\lambda S(t_i)}{\lambda S(t_i) + q_H(t_i)H(t_i)}$$

or, if we substitute

$$q_H(t_i)H(t_i) = -\lambda S(t_i)\frac{\beta(t_i)}{k}$$

is equal to

$$\gamma(t_i) = \frac{1}{1 - \frac{\beta(t_i)}{k}} = \frac{k}{k - \beta(t_i)} \tag{5.8}$$

so that

$$\phi_S(t_i) = \gamma(t_i)\theta_H(t_i) = \frac{k}{k - \beta(t_i)}\theta_H(t_i)$$

Note that for

$$\beta(t_i) \geq 0 \qquad \text{and} \qquad k < 0 \qquad \rightarrow \qquad 0 < \gamma(t_i) \leq 1$$

Getting back to the expected daily return, it may be written as

$$E[r_{\Pi_H}(t_{i+1})|\mathcal{F}_{t_i}] = \Big[\gamma(t_i)\alpha(t_i)\Big]\theta_H(t_i) + \Big[r(t_i)\Big]\big(1 - \theta_H(t_i)\big) \tag{5.9}$$

The component $\theta_H(t_i)$ representing the invested part of the portfolio is expected to deliver a fraction $\gamma(t_i)$ of the mean alpha return $\alpha(t_i)$, whereas the remaining part $1 - \theta_H(t_i)$ invested in cash is expected to drift with the risk-free interest rate $r(t_i)$.

5.1.2 Actual portfolio return

At this point we move the present time from t_i to t_{i+1} and consider the *actual* portfolio variation and return (Figure 5.1).

$$\Pi_H(t_{i+1}) - \Pi_H(t_i) = \lambda\big[S(t_{i+1}) - S(t_i)\big] + q_H(t_i)\big[H(t_{i+1}) - H(t_i)\big] + C_H(t_i)r(t_i)$$

We can write the asset and ETF variation terms as follows

$$S(t_{i+1}) - S(t_i) = S(t_i)r_S(t_{i+1}) = S(t_i)\Big[\alpha(t_{i+1}) + \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1}) + \beta(t_{i+1})r_X(t_{i+1})\Big]$$

$$H(t_{i+1}) - H(t_i) = H(t_i)r_H(t_{i+1}) = H(t_i)\big[k + \delta k(t_{i+1})\big]r_X(t_{i+1})$$

so that

$$\Pi_H(t_{i+1}) - \Pi_H(t_i) = \lambda S(t_i)\Big[\alpha(t_{i+1}) + \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1}) + \beta(t_{i+1})r_X(t_{i+1})\Big] +$$

$$+ q_H(t_i)H(t_i)\big[k + \delta k(t_{i+1})\big]r_X(t_{i+1}) + C_H(t_i)r(t_i)$$

which, rearranged is

$$\Pi_H(t_{i+1}) - \Pi_H(t_i) = \lambda S(t_i)\Big[\alpha(t_{i+1}) + \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})\Big] +$$

$$\Big[\lambda S(t_i)\beta(t_{i+1}) + q_H(t_i)H(t_i)\big[k + \delta k(t_{i+1})\big]\Big]r_X(t_{i+1}) + C_H(t_i)r(t_i)$$

Since $q(t_i)$ was determined at the previous time step t_i such that

$$q_H(t_i)H(t_i)k = -\lambda S(t_i)\beta(t_i)$$

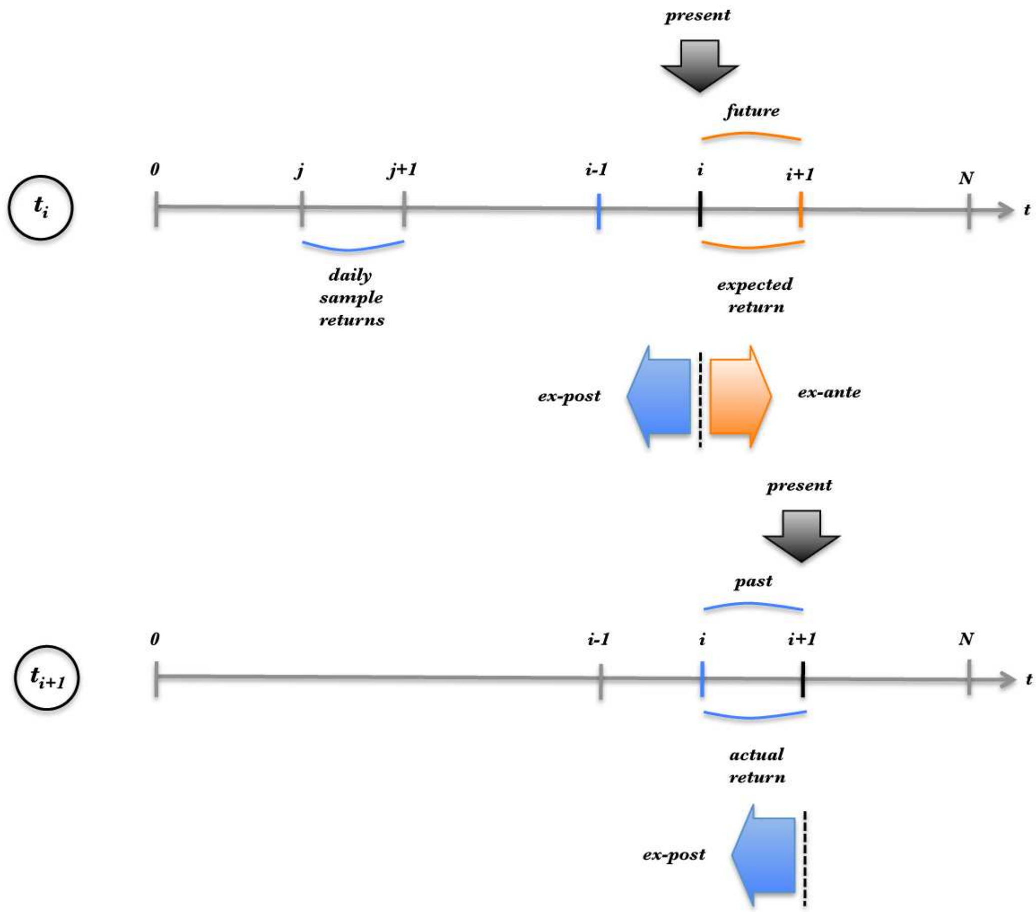


Figure 5.1: At present time t_i we assess from an ex-ante point of view which will be the return at the next time step t_{i+1}, and we do so based upon the data we have from past times t_j which allow us to compute the linear regression. Then, we move the present time to t_{i+1} and assess, now from an ex-post point of view, which has effectively been the return during the last time step.

it results that

$$\Pi_H(t_{i+1}) - \Pi_H(t_i) = \lambda S(t_i)\Big[\alpha(t_{i+1}) + \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1}) +$$

$$\big(\beta(t_{i+1}) - \beta(t_i)\big)r_X(t_{i+1}) - \beta(t_i)\frac{\delta k(t_{i+1})}{k}r_X(t_{i+1})\Big] + C_H(t_i)r(t_i)$$

Now, if we consider the difference between this value and the one we had estimated at the previous time step (equation 5.3)

$$\Delta\Pi_H(t_{i+1}) - E[\Delta\Pi_H(t_{i+1})|\mathcal{F}_{t_i}] =$$

$$\lambda S(t_i)\Big[\alpha(t_{i+1}) - \alpha(t_i) + \big(\beta(t_{i+1}) - \beta(t_i)\big)r_X(t_{i+1}) - \beta(t_i)\frac{\delta k(t_{i+1})}{k}r_X(t_{i+1}) + \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})\Big]$$

we note that it may also be written in function of the *estimation error*,

$$\zeta_x(t_{i+1}) := \alpha(t_{i+1}) - \alpha(t_i) + \big(\beta(t_{i+1}) - \beta(t_i)\big)r_X(t_{i+1})$$

of the *alpha error*,

$$\zeta_\alpha(t_{i+1}) := \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})$$

and of a new term which we define as *etf error*

$$\zeta_h(t_{i+1}) := -\frac{\delta k(t_{i+1})}{k} r_X(t_{i+1}) \tag{5.10}$$

so that

$$\Delta\Pi_H(t_{i+1}) - E[\Delta\Pi_H(t_{i+1})|\mathcal{F}_{t_i}] = \lambda S(t_i)\Big[\zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big] \tag{5.11}$$

This way, the portfolio actual variation may be expressed as

$$\Pi_H(t_{i+1}) - \Pi_H(t_i) = \lambda S(t_i)\Big[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big] + C_H(t_i)r(t_i) \tag{5.12}$$

Dividing each term by $\Pi_H(t_i)$ and adopting the portfolio allocation quantities defined in equation 5.4, we express the portfolio return as

$$r_{\Pi_H}(t_{i+1}) = \Big[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big]\phi_S(t_i) + \Big[r(t_i)\Big]\phi_C(t_i) \tag{5.13}$$

As we can see, it is composed of two terms. The first, is proportional to the fraction of portfolio allocated in the core-asset, and has a return given by the mean alpha plus the alpha, estimation and etf errors. The second is instead given by the interests maturing on the cash balance.

Note that when the estimation error is null, the sum of mean alpha and alpha error is equal to the real alpha return. For $\alpha(t_i) = \alpha(t_{i+1})$ and $\beta(t_i) = \beta(t_{i+1})$

$$\zeta_x(t_{i+1}) = 0$$

$$\alpha(t_i) + \zeta_\alpha(t_{i+1}) = r_\alpha(t_{i+1})$$

If we also assume the etf error to be null, the portfolio return is equal to

$$r_{\Pi_H}(t_{i+1}) = r_\alpha(t_{i+1})\phi_S(t_i) + r(t_i)\phi_C(t_i) \tag{5.14}$$

We can express equation 5.13 in function of the fraction $\theta_H(t_i)$ of portfolio jointly invested in the core-asset and in the ETF.

$$r_{\Pi_H}(t_{i+1}) = \Big[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big]\gamma(t_i)\theta_h(t_i) + \Big[r(t_i)\Big]\big(1 - \theta_H(t_i)\big) \tag{5.15}$$

which, if the estimation and etf errors are null is equal to

$$r_{\Pi_H}(t_{i+1}) = r_\alpha(t_{i+1})\gamma(t_i)\theta_h(t_i) + r(t_i)\big(1 - \theta_h(t_i)\big) \tag{5.16}$$

5.1.3 Portfolio performance

Errors

If we take the difference between equations 5.13 and 5.5, the overall error on the return will be equal to

$$r_{\Pi_H}(t_{i+1}) - E[r_{\Pi_H}(t_{i+1})|\mathcal{F}_{t_i}] = \Big[\zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big]\phi_S(t_i) \tag{5.17}$$

or, looking at the difference between equations 5.15 and 5.9,

$$r_{\Pi_H}(t_{i+1}) - E[r_{\Pi_H}(t_{i+1})|\mathcal{F}_{t_i}] = \Big[\zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big]\gamma(t_i)\theta_H(t_i) \tag{5.18}$$

which as we can see is composed of the alpha, the estimation and etf errors. Note that the last term appearing in the asset variation error (equation 2.20), which we had indicated as *hedgable error* and equal to

$$\zeta_{\text{hedge}}(t_{i+1}) := \Big[r_X(t_{i+1}) - \mu_X(t_i)\Big]\beta(t_i)$$

does not appear in the portfolio variation error, since it has been eliminated by applying the hedge. We also point out that the sign of the estimation error depends on the index return

$$\zeta_x(t_{i+1}) \geq 0 \qquad \text{for} \qquad r_X(t_{i+1}) \geq -\frac{\alpha(t_{i+1}) - \alpha(t_i)}{\beta(t_{i+1}) - \beta(t_i)}$$

$$\zeta_x(t_{i+1}) < 0 \qquad \text{for} \qquad r_X(t_{i+1}) < -\frac{\alpha(t_{i+1}) - \alpha(t_i)}{\beta(t_{i+1}) - \beta(t_i)}$$

As to the etf error $\zeta_h(t_{i+1})$, it originates from the tracking error of the ETF and its impact can either determine an increase or a decrease of the portfolio return depending on the signs of δk and r_X and in proportion to $\beta(t_i)$ as follows

- If $\delta k \geq 0$ and $r_X \geq 0 \quad \rightarrow \quad \zeta_h(t_{i+1}) \geq 0$
  In this case, the ETF under-hedges the market which has a positive return. Such residual exposure to the market results in an increase of the portfolio return with respect to the ideal case.

- If $\delta k \geq 0$ and $r_X < 0 \quad \rightarrow \quad \zeta_h(t_{i+1}) \leq 0$
  Now the residual exposure to the market penalizes the portfolio return since the market return is negative.

- If $\delta k < 0$ and $r_X < 0 \quad \rightarrow \quad \zeta_h(t_{i+1}) > 0$
  In this case, the ETF over-hedges the market's negative return and the portfolio return benefits from an increase.

- If $\delta k < 0$ and $r_X \geq 0 \quad \rightarrow \quad \zeta_h(t_{i+1}) \leq 0$
  Now the ETF is under-exposed to the market's positive return, thus penalizing the portfolio return.

Leverage

Getting back to equation 5.8,

$$\gamma(t_i) := \frac{\phi_S(t_i)}{\theta_H(t_i)} = \frac{\lambda S(t_i)}{\lambda S(t_i) + q_H(t_i)H(t_i)} = \frac{k}{k - \beta(t_i)}$$

this parameter represents the portion of portfolio invested in the core-asset with respect to the fraction invested jointly in the asset and in the ETF. Suppose for a moment that $\beta = 1$, which implies a unitary proportionality between asset and market. In this case, for decreasing values of k, it results that

$$k = -1 \qquad \gamma = \frac{1}{2} \qquad 1 - \gamma = \frac{1}{2}$$

$$k = -2 \qquad \gamma = \frac{2}{3} \qquad 1 - \gamma = \frac{1}{3}$$

$$k = -3 \qquad \gamma = \frac{3}{4} \qquad 1 - \gamma = \frac{1}{4}$$

If $k = -1$, Π_{SH} defined as

$$\Pi_{SH}(t_i) := \lambda S(t_i) + q_H(t_i)H(t_i)$$

will be equally divided in value into the asset and the ETF. If $k = -2$, the ETF required to achieve the hedge is half of the asset in value, so that the asset will cover two thirds of Π_{SH} and the ETF one third. Similarly, if $k = -3$, the ETF needed is in value one third of the asset, which therefore represents three quarters of Π_{SF} and the ETF itself the remaining quarter.
We also point out that by increasing the leverage through k, we achieve a higher proportion of the alpha return but at the price of a higher fraction of its volatility because a smaller quantity of ETF is required to hedge the benchmark index as shown in Figure 5.2. To the limit, when tracking an etf errors are null,

$$\lim_{k\to\infty} \gamma(t_i) = 1 \qquad \lim_{k\to\infty} \phi_S(t_i) = \theta_H(t_i)$$

so that

$$\lim_{k\to\infty} r_{\Pi_H}(t_{i+1}) = r_\alpha(t_{i+1})\theta_H(t_i) + r(t_i)\big(1 - \theta_H(t_i)\big)$$

We now introduce a new parameter defined as the ratio between the cash necessary to hedge the portfolio and the overall portfolio value, the inverse of which measures the *leverage* of the portfolio.

$$l_H(t_i) := \frac{q_H(t_i)H(t_i)}{\Pi_H(t_i)} = -\frac{\beta(t_i)}{k}\gamma(t_i)\theta_H(t_i) \tag{5.19}$$

Portfolio process

The portfolio daily return may also be written in terms of a stochastic process as follows

$$r_{\Pi_H}(t_{i+1}) = \mu_{\Pi_H}(t_{i+1}) + \sigma_{\Pi_H}(t_{i+1})\Psi_{\Pi_H}(t_{i+1}) \tag{5.20}$$

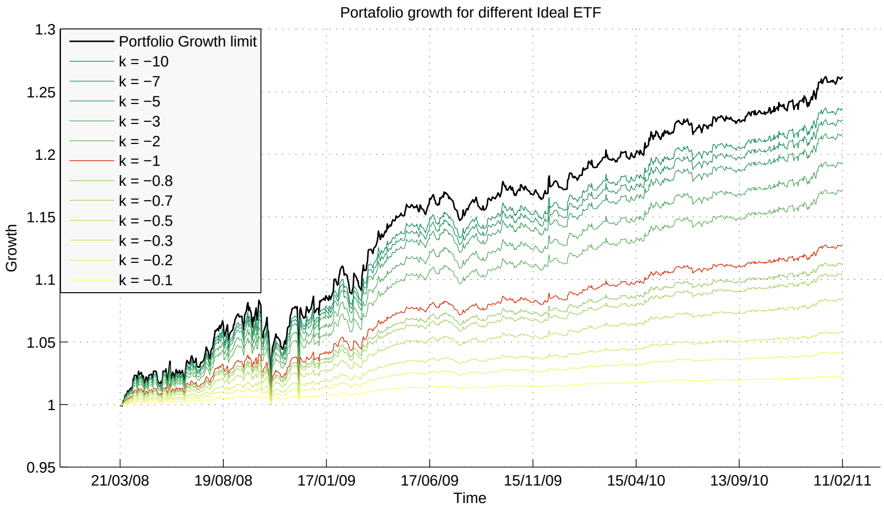


Figure 5.2: Ideal ETF portfolio growth for increasing values of k. The red line represents $k = -1$ and the black one the limit case of $k = \infty$ for which the portfolio growth is equal to the alpha growth.

We refer now to equation 5.15 insert the full expression of the alpha error

$$\zeta_\alpha(t_{i+1}) := \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})$$

If we neglect the estimation and etf errors

$$\zeta_x(t_{i+1}) \approx 0$$

$$\zeta_h(t_{i+1}) \approx 0$$

the alpha error may be written as

$$\zeta_\alpha(t_{i+1}) \approx \sigma_\alpha(t_i)\Delta W_\alpha(t_{i+1})$$

which allows us to write an approximated expression of the portfolio return

$$r_{\Pi_H}(t_{i+1}) \approx \Big[\alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{i+1})\Big]\gamma(t_i)\theta_H(t_i) + \Big[r(t_i)\Big]\big(1 - \theta_H(t_i)\big)$$

which can be rearranged as

$$r_{\Pi_H}(t_{i+1}) \approx \Big[\alpha(t_i)\gamma(t_i)\theta_H(t_i) + r(t_i)(1 - \theta_H(t_i))\Big] + \Big[\gamma(t_i)\theta_H(t_i)\sigma_\alpha(t_i)\Big]\Delta W_\alpha(t_{i+1})$$

If we compare this to the definition 5.20, we can express the portfolio drift as

$$\mu_{\Pi_H}(t_{i+1}) \approx \alpha(t_i)\gamma(t_i)\theta_H(t_i) + r(t_i)(1 - \theta_H(t_i))$$

and its volatility as

$$\sigma_{\Pi_H}(t_{i+1}) \approx \gamma(t_i)\theta_H(t_i)\sigma_\alpha(t_i)$$

We can also introduce the Sharpe ratio

$$\text{Sharpe}_H(t_{i+1}) := \frac{\mu_{\Pi_H}(t_{i+1}) - r(t_i)}{\sigma_{\Pi_H}(t_{i+1})} \approx \frac{\alpha(t_i)\gamma(t_i) - r(t_i)}{\gamma(t_i)\sigma_\alpha(t_i)}$$

which, if $r(t_i) = 0$ is equal to the ratio of the alpha mean drift and its volatility

$$\text{Sharpe}_H(t_{i+1}) \approx \frac{\alpha(t_i)}{\sigma_\alpha(t_i)}$$

Volatility reduction

Finally, we show how the portfolio hedged with ETFs experiences a reduction in the volatility if compared to an investment in only the core-asset. To do so, we take the ratio of the portfolio standard deviation

$$\sigma_{\Pi_H}(t_i) \approx \gamma(t_i)\theta_H(t_i)\sigma_\alpha(t_i)$$

over the asset one

$$\sigma_S(t_i) = \sqrt{\sigma_\alpha(t_i)^2 + \beta(t_i)^2\sigma_X(t_i)^2}$$

and impose

$$\frac{\gamma(t_i)\theta_H(t_i)\sigma_\alpha(t_i)}{\sqrt{\sigma_\alpha(t_i)^2 + \beta(t_i)^2\sigma_X(t_i)^2}} \leq 1$$

which is equal to

$$\gamma(t_i)\theta_H(t_i)\sigma_\alpha(t_i) \leq \sqrt{\sigma_\alpha(t_i)^2 + \beta(t_i)^2\sigma_X(t_i)^2}$$

$$\gamma(t_i)^2\theta_H(t_i)^2\sigma_\alpha(t_i)^2 \leq \sigma_\alpha(t_i)^2 + \beta(t_i)^2\sigma_X(t_i)^2$$

$$\left[1 - \gamma(t_i)^2\theta_H(t_i)^2\right]\sigma_\alpha(t_i)^2 + \beta(t_i)^2\sigma_X(t_i)^2 \geq 0$$

a condition which is always true.

Performance indexes

For a better evaluation of the risk-return profile, we shall consider some largely adopted performance indexes [19, 20, 21, 22]. Before doing so, we introduce for $i \in [0, N-1]$ the excess return of a portfolio Π defined as

$$\tilde{r}_\Pi(t_i) := r_\Pi(t_i) - r(t_i)$$

which for $j \in [0, i-1]$ is

$$\tilde{r}_\Pi(t_{j+1}) := r_\Pi(t_{j+1}) - r(t_{j+1})$$

together with its sample mean

$$\tilde{\mu}_\Pi(t_i) := \frac{1}{i}\sum_{j=0}^{i-1}\big(r_\Pi(t_{j+1}) - r(t_{j+1})\big)$$

and its sample variance

$$\tilde{\sigma}_\Pi(t_i)^2 := \frac{1}{i-1}\sum_{j=0}^{i-1}\big(\tilde{r}_\Pi(t_{j+1}) - \tilde{\mu}_\Pi(t_i)\big)^2$$

Numerical simulations relevant to the portfolio and the core-asset performances will be presented together with the following indexes:

1. In the first place we consider the *Sharpe ratio* as just anticipated. From an ex-post perspective, this index is defined as the ratio of the sample mean of the excess return over the standard deviation of such excess return.

$$\text{Sharpe}(t_i) := \frac{\tilde{\mu}_\Pi(t_i)}{\tilde{\sigma}_\Pi(t_i)}$$

2. There are a number of limitations in the effectivity of the Sharpe ratio, amongst which the fact that it penalizes equally both positive and negative excess returns. In order to introduce an asymmetry in this sense, we consider the *Sortino ratio* which is a modification of the Sharpe ratio that to the scope of this application may be defined as

$$\text{Sortino}(t_i) := \frac{\tilde{\mu}_\Pi(t_i)}{\tilde{\sigma}_{\text{neg}}(t_i)}$$

where

$$\tilde{\sigma}_{neg}(t_i)^2 := \frac{1}{i-1}\sum_{j=0}^{i-1}\xi(t_{j+1})\big(\tilde{r}_\Pi(t_{j+1}) - \tilde{\mu}_\Pi(t_i)\big)^2$$

with

$$\xi(t_{j+1}) = 0 \qquad \text{if} \qquad \tilde{r}_\Pi(t_i) \geq 0$$

$$\xi(t_{j+1}) = 1 \qquad \text{if} \qquad \tilde{r}_\Pi(t_i) < 0$$

This way, the Sortino ratio only takes into account the negative excess returns in the computation of the standard deviation at the denominator.

3. Another limit of the Sharpe ratio is given by the fact that its dimensionless value is not always easily interpretable. The *Modigliani risk-adjusted performance* or *RAP* is instead measured in units of percent return, which is easily understandable. This indicator measures the excess return of a portfolio and adjusts it in function of the risk it takes with respect to a certain benchmark. To our scope, we consider as benchmark the index and define its excess return

$$\tilde{r}_X(t_i) := r_X(t_i) - r(t_i)$$

sample mean

$$\tilde{\mu}_X(t_i) := \frac{1}{i}\sum_{j=0}^{i-1}\big(r_X(t_{j+1}) - r(t_{j+1})\big)$$

Table 5.1: **Performance of portfolio with Short ETF daily hedge**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
From date	22.07.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
Core-asset			
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
μ_S	1.486E-04	8.142E-04	6.061E-04
σ_S	1.422E-02	1.326E-02	9.767E-03
R_S	4.575E-02	5.158E-01	2.838E-01
Annualized R_S	2.014E-02	1.999E-01	1.552E-01
Sharpe ratio	8.956E-03	5.981E-02	6.015E-02
Sortino ratio	5.247E-04	3.551E-03	4.184E-03
RAP (benchmark $r\alpha$*)*	-1.503E-04	2.185E-05	-6.685E-06
RAP (benchmark r_X*)*	2.213E-04	1.217E-04	1.405E-04
Portfolio with Short ETF daily hedge			
μ_{Π_H}	9.882E-06	6.439E-05	9.768E-05
σ_{Π_H}	1.978E-03	3.166E-03	2.485E-03
R_Π	6.672E-03	3.814E-02	4.420E-02
Annualized R_{Π_H}	2.968E-03	1.654E-02	2.533E-02
Sharpe ratio	-5.670E-03	1.368E-02	3.179E-02
Sortino ratio	-3.242E-04	7.695E-04	2.117E-03
RAP (benchmark $r\alpha$*)*	-1.830E-04	-1.646E-04	-1.043E-04
RAP (benchmark r_X*)*	-4.734E-05	-4.192E-04	-1.836E-04

and sample variance

$$\tilde{\sigma}_X(t_i)^2 := \frac{1}{i-1}\sum_{j=0}^{i-1}\left(\tilde{r}_X(t_{j+1}) - \tilde{\mu}_X(t_i)\right)^2$$

With such notation, the RAP is defined as

$$\mathrm{RAP}_X(t_i) := \mathrm{Sharpe}(t_i)\tilde{\sigma}_X(t_i) - \tilde{\mu}_X(t_i)$$

which is equal to

$$\mathrm{RAP}_X(t_i) = \tilde{\mu}_\Pi(t_i)\frac{\tilde{\sigma}_X(t_i)}{\tilde{\sigma}_\Pi(t_i)} - \tilde{\mu}_X(t_i)$$

4. Finally, we also consider the RAP with benchmark the alpha excess return rather than the benchmark index.

In Table 5.1 we summarize the main performance data relevant to the portfolio and compare it with an equivalent investment in the sole core-asset.

5.1.4 Ex-post evolution of cash

It is clear that this type of hedging strategy requires a cash allowance in order to be implemented. It is therefore important to understand which is the evolution of cash both from an ex-post and an ex-ante perspective.

We fix the present time to t_i with $i \in [0, N-1]$ and look backwards assuming that the strategy has been implemented starting from time t_0 and consider the values of the portfolio Π_H, cash C_H and ETF quantity q_H. We now simplify the notation, adopting underscores to indicate the dependance on time $[]_i = [](t_i)$ and writing $q_i = q_H(t_i)$ and $C_i = C_H(t_i)$. Starting from time t_0,

$$\Pi_0 = \lambda S_0 + q_0 H_0 + C_0$$

whereas at time t_1

$$\Pi_1 = \lambda S_1 + q_1 H_1 + C_1$$

$$\Pi_1 = \lambda S_1 + q_0 H_1 + \Delta q_1 H_1 + C_0(1 + r_0) - \Delta q_1 H_1$$

$$q_1 = q_0 + \Delta q_1$$

$$C_1 = C_0(1 + r_0) - \Delta q_1 H_1$$

Similarly, at time t_2

$$\Pi_2 = \lambda S_2 + q_2 H_2 + C_2$$

$$\Pi_2 = \lambda S_2 + q_1 H_2 + \Delta q_2 H_2 + C_1(1 + r_1) - \Delta q_2 H_2$$

$$q_2 = q_1 + \Delta q_2$$

$$C_2 = C_1(1 + r_1) - \Delta q_2 H_2 = C_0(1 + r_0)(1 + r_1) - \Delta q_1 H_1(1 + r_1) - \Delta q_2 H_2$$

and at time t_3

$$\Pi_3 = \lambda S_3 + q_3 H_3 + C_3$$

$$\Pi_3 = \lambda S_3 + q_2 H_3 + \Delta q_3 H_3 + C_2(1 + r_2) - \Delta q_3 H_3$$

$$q_3 = q_2 + \Delta q_3$$

$$C_3 = C_2(1 + r_2) - \Delta q_3 H_3 = C_0(1 + r_0)(1 + r_1)(1 + r_2) - \Delta q_1 H_1(1 + r_1)(1 + r_2)$$
$$-\Delta q_2 H_2(1 + r_2) - \Delta q_3 H_3$$

Getting back to full notation, for $j \in [0, i-1]$, we can define the recursive expression

$$C_H(t_{j+1}) = C_H(t_j)(1 + r(t_j)) - \Delta q_H(t_{j+1}) H(t_{j+1}) \tag{5.21}$$

and by induction we can find the resulting cash in the portfolio and the quantity of ETF held at time t_i

$$C_H(t_i) = C_H(t_0) \prod_{j=0}^{i-1} (1 + r(t_j)) - \sum_{j=0}^{i-2} \left[\Delta q_H(t_{j+1}) H(t_{j+1}) \prod_{g=j+1}^{i-1} (1 + r(t_g)) \right] - \Delta q_H(t_i) H(t_i)$$
$$q_H(t_i) = q_H(t_0) + \sum_{j=0}^{i-1} \Delta q_H(t_{j+1}) \tag{5.22}$$

We can simplify the expression above by introducing the following compound growth parameters

$$G(t_0, t_i) := \prod_{j=0}^{i-1} (1 + r(t_j))$$

$$G(t_{j+1}, t_i) := \prod_{g=j+1}^{i-1} (1 + r(t_g))$$

to write

$$C_H(t_i) = C_H(t_0)G(t_0, t_i) - \sum_{j=0}^{i-2} \left[\Delta q_H(t_{j+1})H(t_{j+1})G(t_{j+1}, t_i)\right] - \Delta q_H(t_i)H(t_i) \tag{5.23}$$

A simplification of equation 5.22 may be obtained if we suppose that the risk-free rate is constant over time. In this case, for $j \in [0, i-1]$

$$r(t_j) = r$$

$$C_1 = C_0(1 + r) - \Delta q_1 H_1$$

$$C_2 = C_1(1 + r) - \Delta q_2 H_2 = C_0(1 + r)^2 - \Delta q_1 H_1(1 + r) - \Delta q_2 H_2$$

$$C_3 = C_2(1 + r) - \Delta q_3 H_3 = C_0(1 + r)^3 - \Delta q_1 H_1(1 + r)^2 - \Delta q_2 H_2(1 + r) - \Delta q_3 H_3$$

Again, by induction we can generalize

$$C_H(t_i) = C_H(t_0)(1 + r)^i - \sum_{j=0}^{i-1} \Delta q_H(t_{j+1})H(t_{j+1})(1 + r)^{i-j-1} \tag{5.24}$$

5.1.5 Purchase and sale of ETF

We have seen that the cash evolution depends on the trading sequence of the ETF, which we now discuss from a qualitative point of view. To do so, let's have a close look at the quantities $\Delta q(t_{j+1})$ for $j \in [0, i-1]$

$$\Delta q(t_{j+1}) = q(t_{j+1}) - q(t_j) = -\beta(t_{j+1})\frac{\lambda S(t_{j+1})}{kH(t_{j+1})} + \beta(t_j)\frac{\lambda S(t_j)}{kH(t_j)} \tag{5.25}$$

Let's assume for a moment that the variation in parameter β from one time step to another is negligible

$$\beta(t_{j+1}) = \beta(t_j) + \Delta\beta(t_{j+1}) \approx \beta(t_j)$$

$$\Delta q(t_{j+1}) \approx -\frac{\beta(t_j)\lambda}{k}\left[\frac{S(t_{j+1})}{H(t_{j+1})} - \frac{S(t_j)}{H(t_j)}\right]$$

For

$$\beta(t_j) \geq 0 \qquad \text{and} \qquad k < 0 \qquad \rightarrow \qquad -\frac{\beta(t_j)\lambda}{k} \geq 0$$

therefore the traded ETF quantity is either positive (purchase order) or negative (sale order) depending on the difference between the ratios at time t_{j+1} and t_j of the asset to the ETF. We can also write

$$\Delta q(t_{j+1}) \approx -\frac{\beta(t_j)\lambda}{k}\frac{S(t_j)}{H(t_j)}\left[\frac{\frac{S(t_{j+1})}{S(t_j)}}{\frac{H(t_{j+1})}{H(t_j)}} - 1\right] = -\frac{\beta(t_j)\lambda}{k}\frac{S(t_j)}{H(t_j)}\left[\frac{r_S(t_{j+1}) + 1}{r_H(t_{j+1}) + 1} - 1\right]$$

As a consequence of the correlation between r_S and r_X and of the fact that typically $o(\sigma_S) = o(\sigma_X) > o(\sigma_\alpha)$, we can state that in general, if the market return $r_X(t_{j+1})$ is positive, also the asset return will be positive whereas the ETF return will be negative, which leads to a positive quantity of traded ETF. In the opposite case, when the market return is negative, so will be the asset's one whereas the ETF will be positive, leading to a negative traded quantity of ETF. Analytically,

$$r_S(t_{j+1}) = r_\alpha(t_{j+1}) + \beta r_X(t_{j+1})$$

if we consider the *sign* of such returns, the alpha return r_α may be neglected so that

$$\text{sign}[r_S(t_{j+1})] = \text{sign}[r_X(t_{j+1})]$$

and

$$\text{sign}[r_H(t_{j+1})] = -\text{sign}[r_X(t_{j+1})]$$

so that, when the market return is positive,

$$r_X(t_{j+1}) > 0 \qquad \rightarrow \qquad r_S(t_{j+1}) > 0 \qquad r_H(t_{j+1}) < 0 \qquad \rightarrow \qquad \Delta q(t_{j+1}) > 0$$

whereas, on the contrary, when the market return is negative

$$r_X(t_{j+1}) < 0 \qquad \rightarrow \qquad r_S(t_{j+1}) < 0 \qquad r_H(t_{j+1}) > 0 \qquad \rightarrow \qquad \Delta q(t_{j+1}) < 0$$

Under periods of positive market, an increasing quantity of ETF is required to keep the portfolio hedged, which means that cash is being increasingly consumed. This might also lead to the consumption of all the available cash, a fact which represents a criticality in this strategy. On the contrary, during bear markets, less and less ETF is required, which to the limit may possibly lead to the complete sale of the ETF and a corresponding cumulation of cash.

5.1.6 Ex-ante evolution of cash

It is of interest to assess also from an ex-ante perspective which is the expected evolution of cash, since this will determine the cash allowance we need to procure in order to implement the strategy. Starting from present time t_i, we look into future times t_h with $h \in [i, N-1]$ and in particular asses the expected value at terminal time t_N conditional to the information available at present time t_i.

$$E\big[C_H(t_N)|\mathcal{F}_{t_i}\big] \tag{5.26}$$

Recalling equation 5.23, which may be written in terms of future times t_h with $h \in [i, N-1]$,

$$C_H(t_N) = C_H(t_i)G(t_i, t_N) - \sum_{h=i}^{N-2} \big[\Delta q_H(t_{h+1})H(t_{h+1})G(t_{h+1}, t_N)\big] - \Delta q_H(t_N)H(t_N)$$

we now look into its conditional expected value 5.26. In absence of specific information on interest rates, we may assume that for each future time the expected value of the risk free rate is equal to the present one and that all are independent one to another.

$$E\big[r(t_h)|\mathcal{F}_{t_i}\big] = r(t_i) = r$$

Furthermore, it appears reasonable to assume that the risk free rate is independent from Δq_H and H, so that we may write

$$E\big[C_H(t_N)|\mathcal{F}_{t_i}\big] = C_H(t_i)(1+r)^n - E\big[\sum_{h=i}^{N-1} \Delta q_H(t_{h+1})H(t_{h+1})(1+r)^{N-h-1}|\mathcal{F}_{t_i}\big]$$

where $n := N-1$ identifies the projection period

$$E\big[C_H(t_N)|\mathcal{F}_{t_i}\big] = C_H(t_i)(1+r)^n - \sum_{h=i}^{N-1} E\big[\Delta q_H(t_{h+1})H(t_{h+1})|\mathcal{F}_{t_i}\big](1+r)^{N-h-1} \tag{5.27}$$

We are now interested in the expression

$$E\big[\Delta q_H(t_{h+1})H(t_{h+1})|\mathcal{F}_{t_i}\big] \tag{5.28}$$

Recalling equation 5.25,

$$\Delta q_H(t_{h+1}) = -\beta(t_{h+1})\frac{\lambda S(t_{h+1})}{kH(t_{h+1})} - \beta(t_h)\frac{\lambda S(t_h)}{kH(t_h)}$$

$$\Delta q_H(t_{h+1})H(t_{h+1}) = -\frac{\lambda}{k}\left[\beta(t_{h+1})S(t_{h+1}) - \beta(t_h)S(t_h)\frac{H(t_{h+1})}{H(t_h)}\right]$$

$$\Delta q_H(t_{h+1})H(t_{h+1}) = -\frac{\lambda}{k}S(t_h)\left[\beta(t_{h+1})\frac{S(t_{h+1})}{S(t_h)} - \beta(t_h)\frac{H(t_{h+1})}{H(t_h)}\right]$$

$$\Delta q_H(t_{h+1})H(t_{h+1}) = -\frac{\lambda}{k}S(t_h)\Big[\beta(t_{h+1})\big(r_S(t_{h+1})+1\big) - \beta(t_h)\big(r_H(t_{h+1})+1\big)\Big]$$

To the extent of our purposes, we assume that the variation in parameter β from one time step to another is negligible

$$\beta(t_{h+1}) = \beta(t_h) + \Delta\beta(t_{h+1}) \approx \beta(t_h)$$

which enables us to write

$$\Delta q_H(t_{h+1})H(t_{h+1}) = -\frac{\lambda\beta(t_h)}{k}S(t_h)\Big[r_S(t_{h+1}) - r_H(t_{h+1})\Big]$$

$$\Delta q_H(t_{h+1})H(t_{h+1}) = -\frac{\lambda\beta(t_h)}{k}S(t_h)\Big[r_\alpha(t_{h+1}) + \big(\beta(t_h) - k - \delta k(t_{h+1})\big)r_X(t_{h+1})\Big]$$

Passing from this last expression to its expected value 5.28, it is convenient to asses whether some of the random variables featuring in it are independent one to another, in which case the expected value of their product may be written as the product of the expected values. In particular, $S(t_h)$, which is evaluated at time t_h, can be considered independent from the returns r_α and r_X evaluated at time t_{h+1}. We may thus write the expectation as follows

$$E\big[\Delta q_H(t_{h+1})H(t_{h+1})|\mathcal{F}_{t_i}\big] = -\frac{\lambda\beta(t_i)}{k}\Bigg[E\big[S(t_h)|\mathcal{F}_{t_i}\big]\big[E\big[r_\alpha(t_{h+1})|\mathcal{F}_{t_i}\big]+$$

$$\big(\beta(t_h)-k\big)E[r_X(t_{h+1})|\mathcal{F}_{t_i}\big] - E[\delta k(t_{h+1})r_X(t_{h+1})|\mathcal{F}_{t_i}\big]\Bigg] \tag{5.29}$$

We know that

$$E\big[r_\alpha(t_{h+1})|\mathcal{F}_{t_i}\big] = \alpha(t_i)$$

$$E\big[r_X(t_{h+1})|\mathcal{F}_{t_i}\big] = \mu_X(t_i)$$

$$E[\delta k(t_{h+1})r_X(t_{h+1})|\mathcal{F}_{t_i}\big] = 0$$

whereas as to $E\big[S(t_h)|\mathcal{F}_{t_i}\big]$, we can express it in terms of the growth process

$$S(t_h) = S(t_i)\prod_{g=i}^{h-1}\big[1+r_S(t_{g+1})\big]$$

$$E\big[S(t_h)|\mathcal{F}_{t_i}\big] = S(t_i)\big[1+\mu_S(t_i)\big]^{h-i}$$

$$E\big[S(t_h)|\mathcal{F}_{t_i}\big] = S(t_i)\big[1+\alpha(t_i)+\beta(t_i)\mu_X(t_i)\big]^{h-i}$$

and substituting these last results in equation 5.29, we obtain

$$E\big[\Delta q_H(t_{h+1})H(t_{h+1})|\mathcal{F}_{t_i}\big] = -\frac{\lambda\beta(t_i)}{k}S(t_i)\big[1+\alpha(t_i)+\beta\mu_X(t_i)\big]^{h-i}\big[\alpha(t_i)+\big(\beta(t_i)-k\big)\mu_X(t_i)\big]$$

Since

$$\frac{\lambda\beta(t_i)}{k}S(t_i) = -q_H(t_i)H(t_i)$$

$$E\big[\Delta q_H(t_{h+1})H(t_{h+1})|\mathcal{F}_{t_i}\big] = q_H(t_i)H(t_i)\big[1+\alpha(t_i)+\beta(t_i)\mu_X(t_i)\big]^{h-i}\big[\alpha(t_i)+\big(\beta(t_i)-k\big)\mu_X(t_i)\big]$$

We are now able to express the expected value of cash given in equation 5.27.

$$E\big[C_H(t_N)|\mathcal{F}_{t_i}\big] = C_H(t_i)(1+r)^n+$$

$$-q_H(t_i)H(t_i)\big[\alpha(t_i)+\big(\beta(t_i)-k\big)\mu_X(t_i)\big]\sum_{h=i}^{N-1}\big[1+\alpha(t_i)+\beta(t_i)\mu_X(t_i)\big]^{h-i}(1+r)^{N-h-1} \tag{5.30}$$

To better interpret this last result, let's assume that the risk free rate is null and for notation's sake we indicate the dependance on time by use of the underscore $[]_i = [](t_i)$

$$E\big[C_N|\mathcal{F}_i\big] = C_i - q_iH_i\big[\alpha_i+(\beta_i-k)\mu_{Xi}\big]\sum_{h=i}^{N-1}\big[1+\alpha_i+\beta_i\mu_{Xi}\big]^{h-i} \tag{5.31}$$

This equation tells us that the expected value of cash at time t_N conditional to the information available at time t_i, will be equal to the starting amount of cash minus a quantity which is function of the market drift μ_X and α given at starting time. This quantity also takes into account the length of the time span between t_N and t_i through the power terms in the sum. Notice that, in accordance to what observed in the previous section, $E\big[C_N|\mathcal{F}_i\big]$

is a decreasing function of μ_X, given that the term $(\beta - k)$ for negative k and positive β is positive. This implies that, apart from the impact of α, cash is consumed when market drift is positive, and cash is cumulated when market drift is negative.
If we impose $E\big[C_N|\mathcal{F}_i\big]$ to be null, we obtain the expression

$$C_i = q_i H_i \big[\alpha_i + (\beta_i - k)\mu_{X_i}\big] \sum_{h=i}^{N-1} \big[1 + \alpha_i + \beta_i \mu_{X_i}\big]^{h-i}$$

which, in case of positive market drift, determines the amount of cash necessary at starting time t_i to have an *autonomy* of $n = N - i$ time steps before all cash in the portfolio is consumed, in function of the current market drift and of the alpha mean return. Equivalently, we can also solve numerically the equation to determine the autonomy period in function of the starting cash and the market drift. Both these approaches are important from an operational point of view.

5.2 Portfolio hedged with Futures

5.2.1 Expected portfolio return

Now we apply a similar hedging technique by use of Futures rather than Short ETFs. We consider at times t_i with $i \in [1, N-1]$ a portfolio $\{\Pi_F\}$ composed of cash $C_F(t_i)$ and of the same quantity λ of the core-asset $\{S\}$ priced $S(t_i)$ as in the other portfolio. In place of the Short ETFs we now adopt as hedging tool a deterministically variable position $q_F(t_i)$ entered into a Futures contract $\{F\}$ priced $F(t_i)$.

Differently than in the previous case for which the ETF was part of the portfolio, now we only hold the core-asset and cash in the portfolio. The Futures position does not appear within the assets comprising the portfolio. It may be considered as a *contractual* asset, which does not appear directly in the list of holdings, but it appears indirectly through the obligations and the effects it produces in terms of cash flow. As mentioned in the previous chapter, when entering into either a long or a short position, each party needs to deposit a certain amount of cash defined as *margin*. At the end of each day, the positions are cleared by adding or taking away a quantity of cash equal to the variation of the futures contract value with respect to the previous day.

At time t_i, the portfolio value is

$$\Pi_F(t_i) = \lambda S(t_i) + C_F(t_i) \tag{5.32}$$

At this same time, we perform an estimation of the portfolio variation between the next time step and the current time. Similarly to the previous case, we will determine the quantity $q_F(t_i)$ of positions to enter into the Futures contract in function of this evaluation with the scope of eliminating from such variation any dependance to the benchmark index return.

$$E[\Pi_H(t_{i+1})|\mathcal{F}_{t_i}] = \lambda E[S(t_{i+1})|\mathcal{F}_{t_i}] + E[C_F(t_{i+1})|\mathcal{F}_{t_i}]$$

Before proceeding, let's consider the value of cash at each time t_{i+1}. We write it as the sum of two terms, one indicating the cash available in the portfolio and one equal to the cash deposited at the Futures exchange institution as margin.

$$C_F(t_{i+1}) = C_A(t_{i+1}) + M(t_{i+1})$$

where

$$C_A(t_{i+1}) := C_A(t_i)\big[1 + r(t_i)\big] + q_F(t_i)\big[F(t_{i+1}) - F(t_i)\big] + \big(|q_F(t_i)| - |q_F(t_{i+1})|\big)mF(t_{i+1})$$

is defined as *available cash*, and

$$M(t_{i+1}) = M(t_i)\big[1 + r_M(t_i)\big] - \big(|q_F(t_i)| - |q_F(t_{i+1})|\big)mF(t_{i+1})$$

represents the *margin* in deposit. Globally,

$$C_F(t_{i+1}) = C_A(t_i)\big[1 + r(t_i)\big] + M(t_i)\big[1 + r_M(t_i)\big] + q_F(t_i)\big[F(t_{i+1}) - F(t_i)\big]$$

Let's analyze each of the terms appearing in these expressions.

- Starting with $C_A(t_i)\big[1+r(t_i)\big]$, this is equal to the available cash balance at the previous time step plus the interests matured on such balance in function of the risk-free interest rate $r(t_i)$ at that time.
- Similarly, $M(t_i)\big[1+r_M(t_i)\big]$ is equal to the margin deposited at the previous time step increased by the interests, if any, matured at the rate given by $r_M(t_i)$.
- Different possibilities arise according to the combinations of values of $r(t_i)$ and $r_M(t_i)$. The simplest scenario corresponds to having these values equal one to another, so that the margin deposited at the exchange institution matures interest with the same rate of the available cash. In this case, we could write

$$C_F(t_{i+1}) = C_F(t_i)\big[1 + r(t_i)\big] + q_F(t_i)\big[F(t_{i+1}) - F(t_i)\big]$$

- The term $q_F(t_i)\big[F(t_{i+1}) - F(t_i)\big]$ represents the obligation that a Futures contract subscriber must comply at the end of each trading day. It is equal to the difference of the contract value with respect to the previous day, multiplied by the number of contracts subscribed. We allow the quantity $q_F(t_i)$ to be either positive or negative. In the first case, it represents a *long* position on the Futures, as opposed to the second case which indicates a *short* position on the contract. We will see that in order to apply the hedge against market return, quantity $q_F(t_i)$ needs to be negative. In this case, when the underlying index increases its value, and so does the Futures, a negative cash flow is produced on the portfolio, which means that money needs to be transferred to the other party in order to settle the position for that trading day. On the contrary, when the index return is negative, the contract price decreases and with a short position we are entitled to receive the corresponding amount of money.
- Finally, the third term $\big(|q_F(t_i)|-|q_F(t_{i+1})|\big)mF(t_{i+1})$ represents the amount of cash we need to transfer from the available cash C_A to the clearing house in order to maintain a

given level $m \in [0,1]$ of margin M in proportion to the Futures current value $F(t_{i+1})$ and to the variation of the quantity of contracts subscribed, as determined by the hedging trading. If we were to maintain a constant quantity of contracts, there would be no such cash transfer even if the contract price changes in time. At time t_0, we assume this quantity to be equal to $-|q(t_0)|mF(t_0) < 0$, which from the available cash point of view is negative both if you enter into a long or in a short position, since it represents the money you need to deposit to the clearing house independently of the position you enter into. An increase in the number of contracts, either long or short, will lead to an outgoing cash flow. Vice-versa, if we reduce the quantity of contracts subscribed we receive back a corresponding amount of margin previously deposited. This mechanism is represented by the expression $|q_F(t_i)| - |q_F(t_{i+1})|$.
If, for all $i \in [0, N-1]$ we assume to hold only short positions $q_F(t_i) < 0$ we may write

$$|q_F(t_i)| - |q_F(t_{i+1})| = q_F(t_{i+1}) - q_F(t_i)$$

whereas in the opposite case of $q_F(t_i) > 0$ it results that

$$|q_F(t_i)| - |q_F(t_{i+1})| = q_F(t_i) - q_F(t_{i+1})$$

Getting back to the cash value at time t_{i+1}, we consider exclusively short positions $q_F(t_i) < 0$ for all $i \in [0, N-1]$ and assume that $r(t_i) = r_M(t_i)$ so that

$$C_F(t_{i+1}) = C_F(t_i)\big[1 + r(t_i)\big] + q_F(t_i)\big[F(t_{i+1}) - F(t_i)\big]$$

At this point, we try to express the second term $q_F(t_i)\big[F(t_{i+1}) - F(t_i)\big]$. We assume that the expiry date falls at time t_N, and we shall indicate the remaining time to expiry as $n := N - i$, so that the price is equal to (equation 4.21)

$$F(t_i) = X(t_i)\big[1 + r(t_i)\big]^n + dF(t_i)$$

$$F(t_{i+1}) = X(t_{i+1})\big[1 + r(t_{i+1})\big]^{n-1} + dF(t_{i+1})$$

but also

$$X(t_{i+1}) = X(t_i)\big[1 + r_X(t_{i+1})\big]$$

so that

$$F(t_{i+1}) = X(t_i)\big[1 + r_X(t_{i+1})\big]\big[1 + r(t_{i+1})\big]^{n-1} + dF(t_{i+1})$$

$$\Delta F(t_{i+1}) = X(t_i)\big[1 + r_X(t_{i+1})\big]\big[1 + r(t_{i+1})\big]^{n-1} - X(t_i)\big[1 + r(t_i)\big]^n + dF(t_{i+1}) - dF(t_i)$$

$$\Delta F(t_{i+1}) = X(t_i)\big[1 + r(t_{i+1})\big]^{n-1}\left[1 + r_X(t_{i+1}) - \frac{\big[1 + r(t_i)\big]^n}{\big[1 + r(t_{i+1})\big]^{n-1}}\right] + dF(t_{i+1}) - dF(t_i) \tag{5.33}$$

At this point it is convenient to introduce an error term, defined as *futures error* which takes into account both the stochastic difference between no-arbitrage price and real price and the discounting given by the expectations on the dividend distributions.

$$\zeta_f(t_{i+1}) := \frac{dF(t_{i+1}) - dF(t_i)}{X(t_i)\big[1 + r(t_{i+1})\big]^{n-1}} \tag{5.34}$$

Recalling equation 4.20,

$$dF(t_i) := \big[X(t_i) - D(t_i)\big]\sigma_F(t_i)\Psi_F(t_i) - D(t_i)[1 + r(t_i)]^n$$

$$dF(t_{i+1}) := \big[X(t_{i+1}) - D(t_{i+1})\big]\sigma_F(t_{i+1})\Psi_F(t_{i+1}) - D(t_{i+1})[1 + r(t_{i+1})]^{n-1}$$

the futures error is equal to

$$\zeta_f(t_{i+1}) = \left[\frac{X(t_{i+1}) - D(t_{i+1})}{X(t_i)\big[1 + r(t_{i+1})\big]^{n-1}}\sigma_F(t_{i+1})\Psi_F(t_{i+1}) - \frac{X(t_i) - D(t_i)}{X(t_i)\big[1 + r(t_{i+1})\big]^{n-1}}\sigma_F(t_i)\Psi_F(t_i)\right] +$$

$$\frac{D(t_i)}{X(t_i)}\frac{[1 + r(t_i)]^n}{\big[1 + r(t_{i+1})\big]^{n-1}} - \frac{D(t_{i+1})}{X(t_i)}$$

We can therefore write

$$\Delta F(t_{i+1}) = X(t_i)\big[1 + r(t_{i+1})\big]^{n-1}\left[1 + r_X(t_{i+1}) - \frac{\big[1 + r(t_i)\big]^n}{\big[1 + r(t_{i+1})\big]^{n-1}} + \zeta_f(t_{i+1})\right] \tag{5.35}$$

If we consider its conditional expected value, variables $r(t_{i+1})$, $r(t_i)$ and $r_X(t_{i+1})$ are treated as independent one to another, and

$$E\Big[\big[1 + r(t_{i+1})\big]^{n-1}|\mathcal{F}_{t_i}\Big] = \big[1 + r(t_i)\big]^{n-1}$$

which allows us to write

$$E\big[\Delta F(t_{i+1})|\mathcal{F}_{t_i}\big] = X(t_i)\big[1+r(t_i)\big]^{n-1}\left[1+\mu_X(t_i) - \frac{\big[1 + r(t_i)\big]^n}{\big[1 + r(t_i)\big]^{n-1}}\right] + E[dF(t_{i+1}) - dF(t_i)|\mathcal{F}_{t_i}]$$

Given that

$$E[dF(t_{i+1}) - dF(t_i)|\mathcal{F}_{t_i}] = 0$$

it results

$$E[\Delta F(t_{i+1})|\mathcal{F}_{t_i}] = X(t_i)\big[1 + r(t_i)\big]^{n-1}\Big[\mu_X(t_i) - r(t_i)\Big] \tag{5.36}$$

As an alternative, we may also express it in function of $F(t_i)$ as

$$E[\Delta F(t_{i+1})|\mathcal{F}_{t_i}] = F(t_i)\big[1 + r(t_i)\big]^{-1}\big[\mu_X(t_i) - r(t_i)\big] \tag{5.37}$$

We can now get back to the portfolio, and consider its expected variation as follows

$$E[\Delta\Pi_F(t_{i+1})|\mathcal{F}_{t_i}] = \lambda E[\Delta S(t_{i+1})|\mathcal{F}_{t_i}] + q_F(t_i)E[\Delta F(t_{i+1})|\mathcal{F}_{t_i}]$$

We have already seen

$$\lambda E[\Delta S(t_{i+1})|\mathcal{F}_{t_i}] = \lambda S(t_i)\big[\alpha(t_i) + \beta(t_i)\mu_X(t_i)\big]$$

and we have just found that

$$q_F(t_i)E[\Delta F(t_{i+1})|\mathcal{F}_{t_i}] = q(t_i)F(t_i)\big[1+r(t_i)\big]^{-1}\big[\mu_X(t_i)-r(t_i)\big]$$

Putting together, we can write

$$E[\Delta\Pi_F(t_{i+1})|\mathcal{F}_{t_i}] = \lambda S(t_i)\big[\alpha(t_i)+$$

$$\beta(t_i)\mu_X(t_i)\big] + q_F(t_i)F(t_i)\big[1+r(t_i)\big]^{-1}\big[\mu_X(t_i)-r(t_i)\big] + C_F(t_i)r(t_i)$$

which can be rearranged into

$$E[\Delta\Pi_F(t_{i+1})|\mathcal{F}_{t_i}] = \lambda S(t_i)\alpha(t_i) + C_F(t_i)r(t_i)+$$

$$\Big[\lambda S(t_i)\beta(t_i) + q_F(t_i)F(t_i)\big[1+r(t_i)\big]^{-1}\Big]\mu_X(t_i) - q_F(t_i)F(t_i)\big[1+r(t_i)\big]^{-1}r(t_i) \tag{5.38}$$

At this point, if we impose to enter at time t_i into the following quantity of contracts, where the negative sign stands for short positions

$$q_F(t_i) = -\beta(t_i)\frac{\lambda S(t_i)}{F(t_i)}\big[1+r(t_i)\big] \tag{5.39}$$

which is equivalent to

$$q_F(t_i) = -\beta(t_i)\frac{\lambda S(t_i)}{X(t_i)}\big[1+r(t_i)\big]^{-(n-1)} \tag{5.40}$$

then the term proportional to the market drift $\mu_X(t_i)$ will vanish and the expected portfolio variation will be

$$E[\Delta\Pi_F(t_{i+1})|\mathcal{F}_{t_i}] = \lambda S(t_i)\alpha(t_i) + C_F(t_i)r(t_i) - q_F(t_i)F(t_i)\big[1+r(t_i)\big]^{-1}r(t_i)$$

Substituting

$$q_F(t_i)F(t_i)\big[1+r(t_i)\big]^{-1} = -\lambda S(t_i)\beta(t_i)$$

we obtain

$$E[\Delta\Pi_F(t_{i+1})|\mathcal{F}_{t_i}] = \lambda S(t_i)\alpha(t_i) + C_F(t_i)r(t_i) + \lambda S(t_i)\beta(t_i)r(t_i)$$

which can be seen as either

$$E[\Delta\Pi_F(t_{i+1})|\mathcal{F}_{t_i}] = \Big[\alpha(t_i)+\beta(t_i)r(t_i)\Big]\lambda S(t_i) + \Big[r(t_i)\Big]C_F(t_i) \tag{5.41}$$

or

$$E[\Delta\Pi_F(t_{i+1})|\mathcal{F}_{t_i}] = \Big[\lambda S(t_i)\Big]\alpha(t_i) + \Big[\lambda S(t_i)\beta(t_i) + C_F(t_i)\Big]r(t_i) \tag{5.42}$$

If we introduce the following portfolio allocation ratios,

$$\theta_F(t_i) := \frac{\lambda S(t_i)}{\Pi_F(t_i)} \qquad 1-\theta_F(t_i) = \frac{C_F(t_i)}{\Pi_F(t_i)} \tag{5.43}$$

and we divide equation 5.41 by $\Pi_F(t_i)$ we can express the expected portfolio return as

$$E[r_{\Pi_F}(t_{i+1})|\mathcal{F}_{t_i}] = \Big[\alpha(t_i) + \beta(t_i)r(t_i)\Big]\theta_F(t_i) + \Big[r(t_i)\Big]\big(1 - \theta_F(t_i)\big) \tag{5.44}$$

which is a sum of two terms. The first is proportional to the core-asset allocated fraction $\theta_F(t_i)$ and drifts with the alpha return $\alpha(t_i)$ plus an extra return given by $\beta(t_i)r(t_i)$. The second, is instead proportional to the cash fraction $1 - \theta_F(t_i)$ and drifts with the risk-free return $r(t_i)$.
If we divide also equation 5.42 by $\Pi_F(t_i)$, the expected portfolio drift is equal to

$$E[r_{\Pi_F}(t_{i+1})|\mathcal{F}_{t_i}] = \Big[\theta_F(t_i)\Big]\alpha(t_i) + \Big[1 - \big(1 - \beta(t_i)\big)\theta_F(t_i)\Big]r(t_i) \tag{5.45}$$

which again is the sum of two terms. The first now indicates that the fraction $\theta_F(t_i)$ drifts with the alpha mean return $\alpha(t_i)$, whereas a fraction given by $1 - \big(1 - \beta(t_i)\big)\theta_F(t_i)$ drifts with the risk-free return.

5.2.2 Actual portfolio return

At this point we move the present time from t_i to t_{i+1} and consider the *actual* portfolio variation and return (Figure 5.1).

$$\Delta\Pi_F(t_{i+1}) = \lambda\Delta S(t_{i+1}) + C_F(t_i)r(t_i) + q_F(t_i)\Delta F(t_{i+1}) \tag{5.46}$$

We have seen that the asset variation is equal to

$$\Delta S(t_{i+1}) = S(t_i)r_S(t_{i+1}) = S(t_i)\Big[\alpha(t_{i+1}) + \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1}) + \beta(t_{i+1})r_X(t_{i+1})\Big]$$

whereas the Futures contract variation is given by equation 5.35

$$\Delta F(t_{i+1}) = X(t_i)\big[1 + r(t_{i+1})\big]^{n-1}\Bigg[1 + r_X(t_{i+1}) - \frac{\big[1 + r(t_i)\big]^n}{\big[1 + r(t_{i+1})\big]^{n-1}} + \zeta_f(t_{i+1})\Bigg]$$

This equation may be approximated with the method examined in detail in Chapter 2 as follows. We adopt the simplified notation where $[]_i = [](t_i)$

$$\big(1 + r(t_i)\big)^n = (1 + r_i)^n \qquad \big(1 + r(t_{i+1})\big)^{n-1} = (1 + r_{i+1})^{n-1}$$

$$(1 + r_{i+1})^{n-1} = (1 + r_i + \Delta r_{i+1})^{n-1} = \sum_{g=0}^{n-1}\binom{n-1}{g}(1 + r_i)^{n-1-g}(\Delta r_{i+1})^g$$

If we look into the order of magnitude of the single terms, and consider that the order of magnitude of Δr_{i+1} does not exceed 10^{-5},

$$w(g) := \binom{n-1}{g}(1 + r_i)^{n-1-g}(\Delta r_{i+1})^g$$

$$w(0) = (1 + r_i)^{n-1} = o(10^0)$$

$$w(1) = (n-1)(1+r_i)^{n-2}(\Delta r_{i+1}) = o(10^2)o(10^0)o(10{-}5) = o(10^{-3})$$
$$w(2) = \binom{n-1}{2}(1+r_i)^{n-3}(\Delta r_{i+1})^2 = o(10^3)o(10^0)o(10{-}10) = o(10^{-7})$$

then we can truncate the binomial sum to the first term. This shows that

$$(1+r_{i+1})^{n-1} \approx (1+r_i)^{n-1}$$

which results in

$$1 - \frac{(1+r_i)^n}{(1+r_{i+1})^{n-1}} \approx 1 - \frac{(1+r_i)^n}{(1+r_i)^{n-1}} = 1 - (1+r_i) = -r_i$$

We may also write this expression in function of an approximation term, which is not stochastic and differently from the others may at any time be ignored by adopting the full analytical expression rather than the approximated one. This term we define as *approximation error*

$$\zeta_r(t_{i+1}) := 1 - \frac{\big[1+r(t_i)\big]^n}{\big[1+r(t_{i+1})\big]^{n-1}} + r(t_i) \tag{5.47}$$

so that

$$1 - \frac{\big[1+r(t_i)\big]^n}{\big[1+r(t_{i+1})\big]^{n-1}} = -r(t_i) + \zeta_r(t_{i+1})$$

Re-adopting full notation, we may write equation 5.35 as

$$\Delta F(t_{i+1}) = X(t_i)\big[1+r(t_{i+1})\big]^{n-1}\Big[r_X(t_{i+1}) - r(t_i) + \zeta_r(t_{i+1}) + \zeta_f(t_{i+1})\Big]$$

We point out that, as with most approximations presented throughout the thesis, also this one is performed with the sole intent to achieve a readable expression for each of the quantities appearing in the dynamics. On the other side, all numerical simulations shown throughout the work are computed with the exact analytical expressions rather than with the approximated ones.

At this point we can get back to equation 5.46 and write

$$\Delta\Pi_F(t_{i+1}) = \lambda S(t_i)\Big[r_\alpha(t_{i+1}) + \beta(t_{i+1})r_X(t_{i+1})\Big] + C_F(t_i)r(t_i)$$

$$+q_F(t_i)X(t_i)\big[1+r(t_{i+1})\big]^{n-1}\Big[r_X(t_{i+1}) - r(t_i) + \zeta_r(t_{i+1}) + \zeta_f(t_{i+1})\Big]$$

We rearrange now the expression in function of the returns

$$\Delta\Pi_F(t_{i+1}) = \lambda S(t_i)r_\alpha(t_{i+1}) + \Big[C_F(t_i) - q_F(t_i)X(t_i)\big[1+r(t_{i+1})\big]^{n-1}\Big]r(t_i) +$$

$$\Big[\lambda S(t_i)\beta(t_{i+1}) + q_F(t_i)X(t_i)\big[1+r(t_{i+1})\big]^{n-1}\Big]r_X(t_{i+1}) +$$

$$q_F(t_i)X(t_i)\big[1+r(t_{i+1})\big]^{n-1}\Big[\zeta_r(t_{i+1}) + \zeta_f(t_{i+1})\Big]$$

We can substitute the value $q(t_i)$ as previously determined

$$q(t_i) = -\beta(t_i)\frac{\lambda S(t_i)}{X(t_i)}\big[1 + r(t_i)\big]^{-(n-1)}$$

$$q(t_i)X(t_i)\big[1 + r(t_i)\big]^{n-1} = -\lambda S(t_i)\beta(t_i)$$

and by exploiting the approximation previously justified

$$\big[1 + r(t_{i+1})\big]^{n-1} \approx \big[1 + r(t_i)\big]^{n-1}$$

we can write

$$q(t_i)X(t_i)\big[1 + r(t_{i+1})\big]^{n-1} \approx -\lambda S(t_i)\beta(t_i)$$

At this point by applying this relation to the portfolio variation, we obtain

$$\Delta\Pi_F(t_{i+1}) = \lambda S(t_i)r_\alpha(t_{i+1}) + \Bigg[C_F(t_i) + \lambda S(t_i)\beta(t_i)\Bigg]r(t_i)+$$

$$\Bigg[\lambda S(t_i)\beta(t_{i+1}) - \lambda S(t_i)\beta(t_i)\Bigg]r_X(t_{i+1}) - \lambda S(t_i)\beta(t_i)\Bigg[\zeta_r(t_{i+1}) + \zeta_f(t_{i+1})\Bigg]$$

or

$$\Delta\Pi_F(t_{i+1}) = \Bigg[C_F(t_i) + \lambda S(t_i)\beta(t_i)\Bigg]r(t_i)+$$

$$\lambda S(t_i)\Bigg[\alpha(t_{i+1})+\sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})+\big(\beta(t_{i+1})-\beta(t_i)\big)r_X(t_{i+1})-\beta(t_i)\big(\zeta_r(t_{i+1})+\zeta_f(t_{i+1})\big)\Bigg]$$

Now, if we add and subtract the term $\alpha(t_i)$ from the portfolio variation, we can write

$$\Delta\Pi_F(t_{i+1}) = \Bigg[C_F(t_i) + \lambda S(t_i)\beta(t_i)\Bigg]r(t_i)+$$

$$\lambda S(t_i)\Bigg[\alpha(t_i) + \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1}) + \alpha(t_{i+1}) - \alpha(t_i) + \big(\beta(t_{i+1}) - \beta(t_i)\big)r_X(t_{i+1})+$$

$$-\beta(t_i)\big(\zeta_r(t_{i+1}) + \zeta_f(t_{i+1})\big)\Bigg]$$

and, recalling the definitions of alpha error and of estimation error

$$\zeta_\alpha(t_{i+1}) := \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})$$

$$\zeta_x(t_{i+1}) := \alpha(t_{i+1}) - \alpha(t_i) + \big(\beta(t_{i+1}) - \beta(t_i)\big)r_X(t_{i+1})$$

we finally obtain

$$\Delta\Pi_F(t_{i+1}) = C_F(t_i)r(t_i)+$$
$$\lambda S(t_i)\Bigg[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\Bigg(r(t_i) - \zeta_r(t_{i+1}) - \zeta_f(t_{i+1})\Bigg)\Bigg] \tag{5.48}$$

which, in terms of return is equal to

$$r_{\Pi_F}(t_{i+1}) = \Bigg[r(t_i)\Bigg]\big(1 - \theta(t_i)\big)+$$

Table 5.2: **Estimation error**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
From date	22.07.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
Mean	-1.205E-06	1.173E-08	-1.877E-07
Std. dev.	1.865E-05	1.557E-05	2.066E-05

$$\Big[\alpha(t_i)+\zeta_\alpha(t_{i+1})+\zeta_x(t_{i+1})+\beta(t_i)\Big(r(t_i)-\zeta_r(t_{i+1})-\zeta_f(t_{i+1})\Big)\Big]\theta(t_i) \tag{5.49}$$

5.2.3 Portfolio performance

Errors

If we compare this last equation with the expected portfolio return given by equation 5.44

$$E[r_{\Pi_F}(t_{i+1})|\mathcal{F}_{t_i}] = \Big[\alpha(t_i)+\beta(t_i)r(t_i)\Big]\theta(t_i)+\Big[r(t_i)\Big]\big(1-\theta(t_i)\big)$$

$$r_{\Pi_F}(t_{i+1})-E[r_{\Pi_F}(t_{i+1})|\mathcal{F}_{t_i}] = \Big[\zeta_\alpha(t_{i+1})+\zeta_x(t_{i+1})-\beta(t_i)\Big(\zeta_r(t_{i+1})+\zeta_f(t_{i+1})\Big)\Big]\theta(t_i) \tag{5.50}$$

As with the portfolio hedged with ETFs (equation 5.18), in proportion to $\theta(t_i)$ we have both the alpha error $\zeta_\alpha(t_{i+1})$ and the estimation error $\zeta_x(t_{i+1})$. There is also the futures error $\zeta_f(t_{i+1})$, equivalent to the etf error $\zeta_h(t_{i+1})$. Note that both these terms are proportional to $\beta(t_i)$ given that they represent either an under or over exposure to the market return given by the non ideal pricing of the respective securities. Finally, there is the approximation error $\zeta_r(t_{i+1})$, which, as we have mentioned may be disregarded by adopting the exact analytical solution in equation 5.47.
In Table 5.2 we show again the values of the estimation error for the three Sample Cases, which appear to be negligible on a daily time scale. In Figure 5.3 we show the evolution over time of the estimation error for the core-asset of Sample Case 1 together with its mean and standard deviation.

Leverage

For the ETF portfolio we have introduced a *leverage* parameter, defined as the ratio between the cash necessary to hedge the portfolio and the overall portfolio value. For the Futures, the cash required to implement the hedge is equal to the *margin* that needs to be deposited at the clearing house, equal to

$$mq_F(t_i)F(t_i) = -m\beta(t_i)\lambda S(t_i)\big[1+r(t_i)\big]$$

which determines the parameter

$$l_F(t_i) := \frac{mq_F(t_i)F(t_i)}{\Pi_F(t_i)} = -m\beta(t_i)\big[1+r(t_i)\big]\theta_F(t_i) \tag{5.51}$$

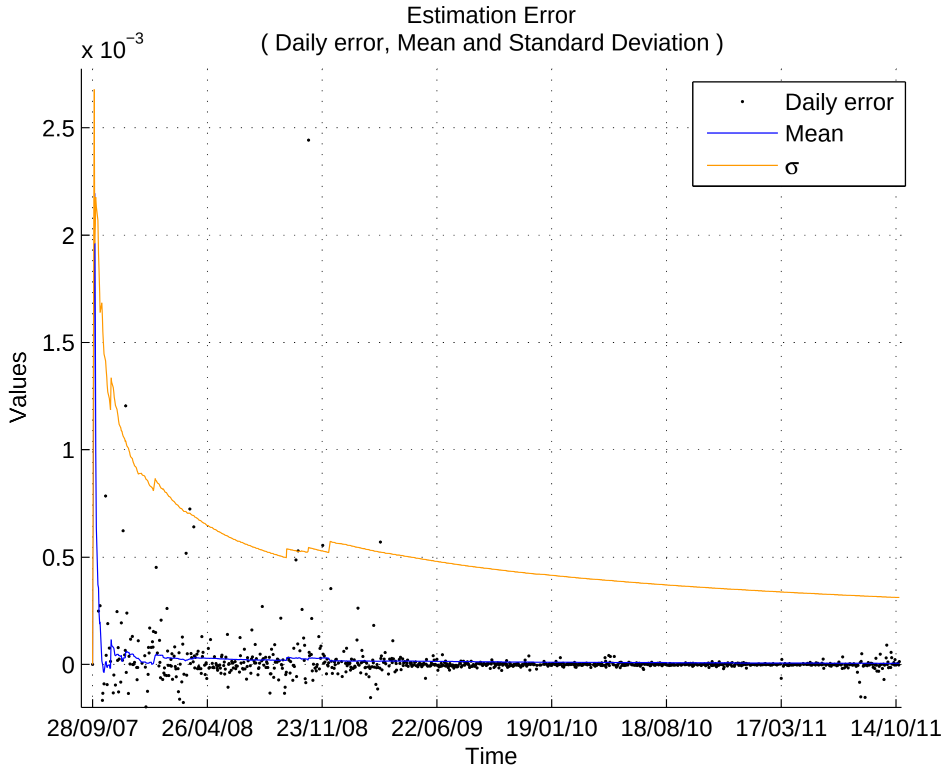


Figure 5.3: Estimation error for Sample Case 1 asset.

the inverse of which is a measure of the portfolio *leverage*.

Portfolio process

As per the ETF portfolio, also in this case we look for an approximated expression of the Futures portfolio return process.

$$r_{\Pi_F}(t_{i+1}) = \mu_{\Pi_F}(t_{i+1}) + \sigma_{\Pi_F}(t_{i+1})\Psi_{\Pi_F}(t_{i+1}) \tag{5.52}$$

To this scope we neglect the estimation, futures and approximation errors

$$\zeta_x(t_{i+1}) \approx 0$$

$$\zeta_f(t_{i+1}) \approx 0$$

$$\zeta_r(t_{i+1}) \approx 0$$

and write

$$\zeta_\alpha(t_{i+1}) \approx \sigma_\alpha(t_i)\Delta W_\alpha(t_{i+1})$$

so that the portfolio return of equation 5.49 may be approximated by

$$r_{\Pi_F}(t_{i+1}) \approx \Big[\alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{i+1}) + \beta(t_i)r(t_i)\Big]\theta_F(t_i) + \Big[r(t_i)\Big]\big(1 - \theta_F(t_i)\big)$$

which can be rearranged as

$$r_{\Pi_F}(t_{i+1}) \approx \Big[\alpha(t_i)\theta_F(t_i) + \beta(t_i)r(t_i)\theta_F(t_i) + \big(1-\theta_F(t_i)\big)r(t_i)\Big] + \Big[\sigma_\alpha(t_i)\theta_F(t_i)\Big]\Delta W_\alpha(t_{i+1})$$

Looking into the drift term, this can be written as

$$\alpha(t_i)\theta_F(t_i) + \beta(t_i)r(t_i)\theta_F(t_i) + \big(1-\theta_F(t_i)\big)r(t_i) = \Big[\alpha(t_i) + \big(\beta(t_i)-1\big)r(t_i)\Big]\theta_F(t_i) + r(t_i)$$

so that

$$\mu_{\Pi_F}(t_{i+1}) \approx \Big[\alpha(t_i) + \big(\beta(t_i)-1\big)r(t_i)\Big]\theta_F(t_i) + r(t_i)$$

and

$$\sigma_{\Pi_F}(t_{i+1}) \approx \sigma_\alpha(t_i)\theta_F(t_i)$$

The Sharpe ratio is equal to

$$\text{Sharpe}_F(t_{i+1}) := \frac{\mu_{\Pi_F}(t_{i+1}) - r(t_i)}{\sigma_{\Pi_F}(t_{i+1})} \approx \frac{\alpha(t_i) + \big(\beta(t_i)-1\big)r(t_i)}{\sigma_\alpha(t_i)}$$

And if $r(t_i) = 0$, it is equal to the ratio of the alpha mean drift and its volatility

$$\text{Sharpe}_F(t_{i+1}) \approx \frac{\alpha(t_i)}{\sigma_\alpha(t_i)}$$

Volatility reduction

Finally we show that the portfolio hedged with Futures is characterized by a lower volatility of the core-asset alone. We can in fact verify that the ratio of the portfolio standard deviation

$$\sigma_{\Pi_F}(t_i) \approx \sigma_\alpha(t_i)\theta_F(t_i)$$

over the asset one

$$\sigma_S(t_i) = \sqrt{\sigma_\alpha(t_i)^2 + \beta(t_i)^2\sigma_X(t_i)^2}$$

is always inferior or equal to one. In fact, if we write

$$\frac{\sigma_\alpha(t_i)\theta_F(t_i)}{\sqrt{\sigma_\alpha(t_i)^2 + \beta(t_i)^2\sigma_X(t_i)^2}} \leq 1$$

this implies

$$\sigma_\alpha(t_i)\theta_F(t_i) \leq \sqrt{\sigma_\alpha(t_i)^2 + \beta(t_i)^2\sigma_X(t_i)^2}$$

$$\sigma_\alpha(t_i)^2\theta_F(t_i)^2 \leq \sigma_\alpha(t_i)^2 + \beta(t_i)^2\sigma_X(t_i)^2$$

and gives the following condition which is always verified.

$$\Big[1-\theta_F(t_i)^2\Big]\sigma_\alpha(t_i)^2 + \beta(t_i)^2\sigma_X(t_i)^2 \geq 0$$

In Table 5.3 we summarize the main performance data relevant to the portfolio and compare it with an equivalent investment in the sole core-asset.

Table 5.3: **Performance of portfolio with Futures daily hedge**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
From date	22.07.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
Core-asset			
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
μ_S	1.486E-04	8.142E-04	6.061E-04
σ_S	1.422E-02	1.326E-02	9.767E-03
R_S	4.575E-02	5.158E-01	2.838E-01
Annualized R_S	2.014E-02	1.999E-01	1.552E-01
Sharpe ratio	8.956E-03	5.981E-02	6.015E-02
Sortino ratio	5.247E-04	3.551E-03	4.184E-03
RAP (benchmark $r\alpha$*)*	-1.503E-04	2.185E-05	-6.685E-06
RAP (benchmark r_X*)*	2.213E-04	1.217E-04	1.405E-04
Portfolio with Futures daily hedge			
μ_{Π_F}	1.508E-04	1.956E-04	2.154E-04
σ_{Π_F}	5.037E-03	6.035E-03	3.212E-03
R_Π	8.549E-02	1.038E-01	9.846E-02
Annualized R_{Π_F}	3.728E-02	4.428E-02	5.582E-02
Sharpe ratio	2.574E-02	2.893E-02	6.130E-02
Sortino ratio	1.527E-03	1.727E-03	4.225E-03
RAP (benchmark $r\alpha$*)*	-8.346E-05	-9.021E-05	-1.270E-06
RAP (benchmark r_X*)*	4.539E-04	-2.719E-04	1.653E-04

5.2.4 Ex-post evolution of cash

We now consider from an ex-post point of view the cash cumulated at present time t_i assuming to having started the strategy at time t_0. To this scope, we adopt a simplified notation for which the dependance on time is represented as $[]_i = [](t_i)$ and

$$C_i := C_F(t_i) \qquad q_i := q_F(t_i)$$

At each time step, we will have to either compensate or receive money in function of the contract price change.

$$C_1 = C_0(1 + r_0) + q_0(F_1 - F_0)$$

$$C_2 = C_1(1 + r_1) + q_0(F_2 - F_1)$$

$$C_2 = C_0(1 + r_0)(1 + r_1) + q_0(F_1 - F_0)(1 + r_1) + q_0(F_2 - F_1)$$

$$C_3 = C_2(1 + r_2) + q_0(F_3 - F_2)$$

$$C_3 = C_0(1 + r_0)(1 + r_1)(1 + r_2) + q_0(F_1 - F_0)(1 + r_1)(1 + r_2)$$

$$+q_0(F_2 - F_1)(1 + r_2) + q_0(F_3 - F_2)$$

and

$$C_4 = C_3(1 + r_3) + q_0(F_4 - F_3)$$

$$C_4 = C_0(1 + r_0)(1 + r_1)(1 + r_2)(1 + r_3) + q_0(F_1 - F_0)(1 + r_1)(1 + r_2)(1 + r_3)$$

$$+q_0(F_2 - F_1)(1 + r_2)(1 + r_3) + q_0(F_3 - F_2)(1 + r_3) + q_0(F_4 - F_3)$$

By induction,

$$C_i = C_0 \prod_{j=0}^{i-1}(1 + r_j) + q_0 \sum_{j=0}^{i-2} \left[(F_{j+1} - F_j) \prod_{g=j+1}^{i-1} (1 + r_g)\right] + q_0 \left[F_i - F_{i-1}\right]$$

Note that, with the sole exception of F_0 and F_i, each F_j appears twice in the sum above. Getting back to C_3, if we isolate the first and last term then we can write:

$$C_3 = C_0(1 + r_0)(1 + r_1)(1 + r_2) + q_0 \left[F_3 - F_0(1 + r_1)(1 + r_2)\right] +$$

$$q_0 \left[F_1\big((1 + r_1)(1 + r_2) - (1 + r_2)\big) + F_2\big((1 + r_2) - 1\big)\right]$$

or

$$C_3 = C_0(1 + r_0)(1 + r_1)(1 + r_2) + q_0 \left[F_3 - F_0(1 + r_1)(1 + r_2)\right]$$

$$q_0 \left[F_1 r_1 (1 + r_2) + F_2 r_2\right]$$

which we can generalize in the form

$$C_i = C_0 \prod_{j=0}^{i-1}(1+r_j) + q_0 \left[F_i - F_0 \prod_{j=1}^{i-1}(1+r_j)\right] + q_0 \left[\sum_{j=1}^{i-2} F_j r_j \prod_{g=j+1}^{i-1} (1+r_g) + F_{i-1} r_{i-1}\right] \quad (5.53)$$

If $r(t_j) = r$ it simplifies into

$$C_i = C_0(1 + r)^n + q_0 \left[F_i - F_0(1 + r)^{i-1}\right] + q_0 r \left[\sum_{j=1}^{i-2} F_j (1 + r)^{i-j-1} + F_{i-1}\right]$$

If $r = 0$

$$C_i = C_0 + q_0(F_i - F_0)$$

We can simplify the expression above by introducing the following compound growth parameters:

$$G(t_0, t_i) := \prod_{j=0}^{i-1}(1 + r_j)$$

$$G(t_1, t_i) := \prod_{j=1}^{i-1}(1 + r_j)$$

$$G(t_{j+1}, t_i) := \prod_{g=j+1}^{i-1} (1 + r_g)$$

so that

$$C_i = C_0 G(t_0, t_i) + q_0 \left[F_i - F_0 G(t_1, t_i)\right] + q_0 \left[\sum_{j=1}^{i-2} F_j r_j G(t_{j+1}, t_i) + F_{i-1} r_{i-1}\right] \quad (5.54)$$

The reason for adopting this form is that in this way we have highlighted the proportionality to r_j of the last term, which as a consequence may be neglected with respect to other terms. Since $r = o(10^{-4})$

$$q_0\left[\sum_{j=1}^{i-2} F_j r_j G(t_{j+1}, t_i) + F_{i-1} r_{i-1}\right] = o(q_0 F_j) o(10^{-4}) \approx 0$$

so that

$$C_i \approx C_0 G(t_0, t_i) + q_0\left[F_i - F_0 G(t_1, t_i)\right] \tag{5.55}$$

5.2.5 Ex-ante evolution of cash

Equation 5.55 may be written also in terms of the cumulated cash at time t_N starting to trade from time t_i

$$C_N \approx C_i G(t_i, t_N) + q_i\left[F_N - F_i G(t_{i+1}, t_N)\right]$$

of which we now take the expectation

$$E[C_N|\mathcal{F}_i] \approx C_i E[G(t_i, t_N)|\mathcal{F}_i] + q_i E[F_N|\mathcal{F}_i] - q_i F_i E[G(t_{i+1}, t_N)|\mathcal{F}_i]$$

Since

$$E[G(t_i, t_N)|\mathcal{F}_i] = (1 + r_i)^n = E[R(t_i, t_N)|\mathcal{F}_i] + 1 = E[R|\mathcal{F}_i] + 1$$

$$E[G(t_{i+1}, t_N)|\mathcal{F}_i] = (1 + r_i)^{n-1}$$

As to the contract price, we introduce the quantity

$$G_X := G_X(t_i, t_N) := \prod_{h=0}^{N-1} (1 + r_{X_{h+1}}) = R_X + 1$$

to write

$$F_i = X_i (1 + r_i)^n + dF_i$$

$$F_N = X_N = X_i G_X = X_i (1 + R_X) + dF_N$$

so that

$$E[F_N|\mathcal{F}_i] = X_i\big(1 + E[R_X|\mathcal{F}_i]\big)$$

Putting together, we can write

$$E[C_N|\mathcal{F}_i] \approx C_i\left[E[R|\mathcal{F}_i] + 1\right] + q_i X_i\left[1 + E[R_X|\mathcal{F}_i]\right] - q_i X_i (1 + r_i)^n E[G(t_{i+1}, t_N)|\mathcal{F}_i]$$

We now introduce a compound growth quantity defined as

$$\tilde{G} := \tilde{G}(t_i, t_N) := (1 + r_i)^n G(t_{i+1}, t_N) = (1 + r_i)^n \prod_{h=i+1}^{N-1} (1 + r_{h+1}) \tag{5.56}$$

which has an expected value of

$$E[\tilde{G}|\mathcal{F}_i] = (1 + r_i)^{2n-1} = E[G(t_i, t_{i+2n-1}|\mathcal{F}_i]$$

equivalent to the expected compound growth over $2n-1$ periods of the risk-free interest rate. This expected value is also equal to

$$E[\tilde{G}|\mathcal{F}_i] = (1+r_i)^n E[G(t_{i+1}, t_N)|\mathcal{F}_i]$$

so that

$$E[C_N|\mathcal{F}_i] \approx C_i \Big[E[R|\mathcal{F}_i] + 1\Big] + q_i X_i \Big[1 + E[R_X|\mathcal{F}_i] - E[\tilde{G}|\mathcal{F}_i]\Big] \tag{5.57}$$

5.3 Comparison between portfolio hedged with ETFs and with Futures

5.3.1 Returns

We are now able to compare the results obtained for the portfolio hedged on a daily basis with Short ETFs and with Futures. Looking into the portfolio return, we recall equation 5.15 for the ETF,

$$r_{\Pi_H}(t_{i+1}) = \Big[r(t_i)\Big]\big(1-\theta_H(t_i)\big) + \Big[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big]\gamma(t_i)\theta_H(t_i)$$

and equation 5.49 for the Futures

$$r_{\Pi_F}(t_{i+1}) = \Big[r(t_i)\Big]\big(1-\theta_F(t_i)\big) +$$

$$\Big[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\Big(r(t_i) - \zeta_r(t_{i+1}) - \zeta_f(t_{i+1})\Big)\Big]\theta_F(t_i)$$

In the first place, there is a difference in the *path dependent* allocation quantities θ_H and θ_F. The latter is generally higher $\theta_F > \theta_H$ because only a portion of cash equal to the margin needs to be put as guarantee in order to enter the Futures contract, whereas for the ETF an amount of cash equal to the whole price needs to allocated in order to hold the security in the portfolio.
Let's put aside for a moment this main difference and assume that a given time these allocation fractions are equal $\theta_F = \theta_H = \theta$ and that the error terms may be neglected. In this case, the difference between the two would be

$$r_{\Pi_F}(t_{i+1}) - r_{\Pi_H}(t_{i+1}) = \big(1-\gamma(t_i)\big)\alpha(t_i)\theta(t_i) + \beta(t_i)r(t_i)\theta(t_i)$$

which is a positive quantity. The Futures portfolio out-performs the ETF one because it achieves the whole alpha return as opposed to the ETF which realizes only a fraction γ of it, and because it has also the additional return term given by $\beta(t_i)r(t_i)\theta(t_i)$ proportional to the risk-free rate.

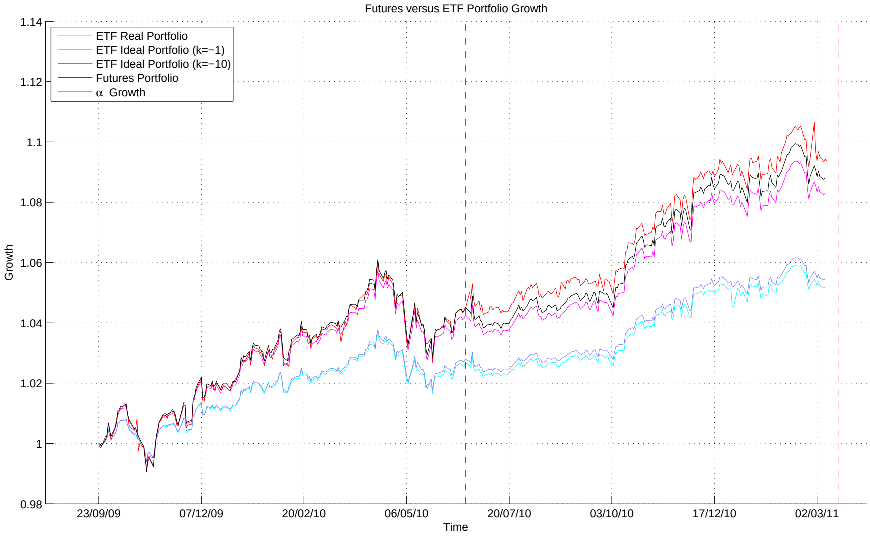


Figure 5.4: Growth comparison between Futures portfolio, alpha growth, and ETF portfolios with $k = -1$ and $k = -10$. Note that the $k = -10$ and the Futures portfolio deliver a comparable trajectory.

5.3.2 Leverage

Let's compare the different level of *leverage* between ETF and Futures portfolio. For the first (equation 5.19),

$$l_H(t_i) := \frac{q_H(t_i)H(t_i)}{\Pi_H(t_i)} = -\frac{\beta(t_i)}{k}\gamma(t_i)\theta_H(t_i)$$

whereas for the second (equation 5.51),

$$l_F(t_i) := -\frac{mq_F(t_i)F(t_i)}{\Pi_F(t_i)} = m\beta(t_i)\big[1 + r(t_i)\big]\theta_F(t_i)$$

Their ratio is equal to

$$\frac{l_F(t_i)}{l_H(t_i)} = -mk\big[1 + r(t_i)\big]\frac{\theta_F(t_i)}{\gamma(t_i)\theta_H(t_i)} \geq -mk$$

Not considering the other terms, the Futures leverage is only a fraction mk of the ETF one. Parameter m indicates the percentage of contract price to be deposited as margin to enter into a Futures contract, and its inverse $\frac{1}{m}$ may be compared to k. The leverage parameters are of the same order of magnitude when $k = -\frac{1}{m}$, which in the case of $m = 0.10$ is equal to $k = -10$.

In other words, a Futures portfolio with $m = 0.10$ is comparable to an ETF one with $k = -10$, as shown in Figure 5.4.

5.3.3 Performance

Let's consider the portfolio return processes given in equations 5.20 and 5.52. The ETF portfolio has a drift of

$$\mu_{\Pi_H}(t_{i+1}) \approx \alpha(t_i)\gamma(t_i)\theta_H(t_i) + r(t_i)\big(1 - \theta_H(t_i)\big)$$

whereas the Futures one of

$$\mu_{\Pi_F}(t_{i+1}) \approx \Big[\alpha(t_i) + \big(\beta(t_i) - 1\big)r(t_i)\Big]\theta_F(t_i) + r(t_i)$$

If we take the difference, and impose it to be positive

$$\mu_{\Pi_F}(t_{i+1}) - \mu_{\Pi_H}(t_{i+1}) \approx \Big[\alpha(t_i) + \big(\beta(t_i) - 1\big)r(t_i)\Big]\theta_F(t_i) - \alpha(t_i)\gamma(t_i)\theta_H(t_i) + r(t_i)\theta_H(t_i) =$$

$$\Big[\alpha(t_i) + \big(\beta(t_i) - 1\big)r(t_i)\Big]\theta_F(t_i) - \Big[\alpha(t_i)\gamma(t_i) - r(t_i)\Big]\theta_H(t_i) \geq 0$$

we can express it in terms of the allocation fractions as follows

$$\mu_{\Pi_F}(t_{i+1}) - \mu_{\Pi_H}(t_{i+1}) \geq 0 \qquad \rightarrow \qquad \theta_F(t_i) \geq \frac{\alpha(t_i)\gamma(t_i) - r(t_i)}{\alpha(t_i) + \big(\beta(t_i) - 1\big)r(t_i)}\theta_H(t_i)$$

which, if $r(t_i) = 0$, becomes

$$\theta_F(t_i) \geq \gamma(t_i)\theta_H(t_i)$$

a condition that is in general satisfied. This increase in the portfolio drift is made at a price of an increasing volatility. In fact if we take the ratio of

$$\sigma_{\Pi_F}(t_{i+1}) \approx \sigma_\alpha(t_i)\theta_F(t_i)$$

over

$$\sigma_{\Pi_H}(t_{i+1}) \approx \gamma(t_i)\theta_H(t_i)\sigma_\alpha(t_i)$$

it results

$$\frac{\sigma_{\Pi_F}(t_{i+1})}{\sigma_{\Pi_H}(t_{i+1})} \approx \frac{\theta_F(t_i)}{\gamma(t_i)\theta_H(t_i)} \geq 1$$

However, by comparing the Sharpe ratios, we can see that the Futures portfolio delivers a better risk-return performance compared to the ETF one. In fact, if we take the ratio of

$$\text{Sharpe}_F(t_{i+1}) \approx \frac{\alpha(t_i) + \big(\beta(t_i) - 1\big)r(t_i)}{\sigma_\alpha(t_i)}$$

over

$$\text{Sharpe}_H(t_{i+1}) \approx \frac{\alpha(t_i)\gamma(t_i) - r(t_i)}{\gamma(t_i)\sigma_\alpha(t_i)}$$

and impose

$$\frac{\text{Sharpe}_F(t_{i+1})}{\text{Sharpe}_H(t_{i+1})} \approx \frac{\alpha(t_i)\gamma(t_i) + \big(\beta(t_i) - 1\big)\gamma(t_i)r(t_i)}{\alpha(t_i)\gamma(t_i) - r(t_i)} \geq 1$$

it results that

$$\alpha(t_i)\gamma(t_i) + \big(\beta(t_i) - 1\big)\gamma(t_i)r(t_i) \geq \alpha(t_i)\gamma(t_i) - r(t_i)$$

or

$$\big(\beta(t_i) - 1\big)\gamma(t_i) \geq -1$$

$$\big(1 - \beta(t_i)\big)\gamma(t_i) \leq 1$$

which is in general true.

5.3.4 Implementation

Let's look into the numerical simulations performed on the three Sample Cases.

Risk-free interest rate

We have adopted as reference interest rate the *Euribor one month 360 days basis.* For positive cash balance we apply a negative spread s_P and for negative cash balance a positive spread s_N. Defining for $i \in [0, N-1]$ the following:

- $r(t_i)$ interest rate at time t_i;
- t_u for $u \leq i$ reference time falling at the first day of the month of t_i;
- $r_E(t_u)$ reference interest rate (Euribor 1M 360) at time t_u;
- Spread for positive interest rate $s_P = -10$ bps;
- Spread for negative interest rate $s_N = 200$ bps.

$$C(t_i) \geq 0 \qquad r(t_i) = r_E(t_u) + s_P$$

$$C(t_i) < 0 \qquad r(t_i) = r_E(t_u) + s_N$$

The reference interest rate to which apply the negative or positive spread is taken at the first day if the month in which the rate is considered. Figure 5.5 shows the value used for the Sample Cases.

Transaction costs

As to transaction costs, we assume that the investment strategy is implemented by a retail investor through a *trading online* platform, which often also provide the possibility to insert transaction orders on an automatic basis. In Italy for example, there are services for which fees may be applied on a fixed amount in the order of 10 to 20 Euros together with a pre-defined annual or monthly fee for the service activation. Other offers feature flat commissions on a certain period allowing for a pre-defined number of transactions. If the client exceeds the given allowance, more expensive fees are applied. Such commercial offers may change in time and depend also on a possible negotiation with the service provider. In general however, this type of platform allows for a considerable reduction in the fee cost if compared to traditional services where commissions are charged as a percentage of the transaction amount.
To our purposes, we assume that a flat commission of 10 Euros is applied for each daily transaction. The impact of such number obviously depends on the portfolio amount. Introducing the following quantities,

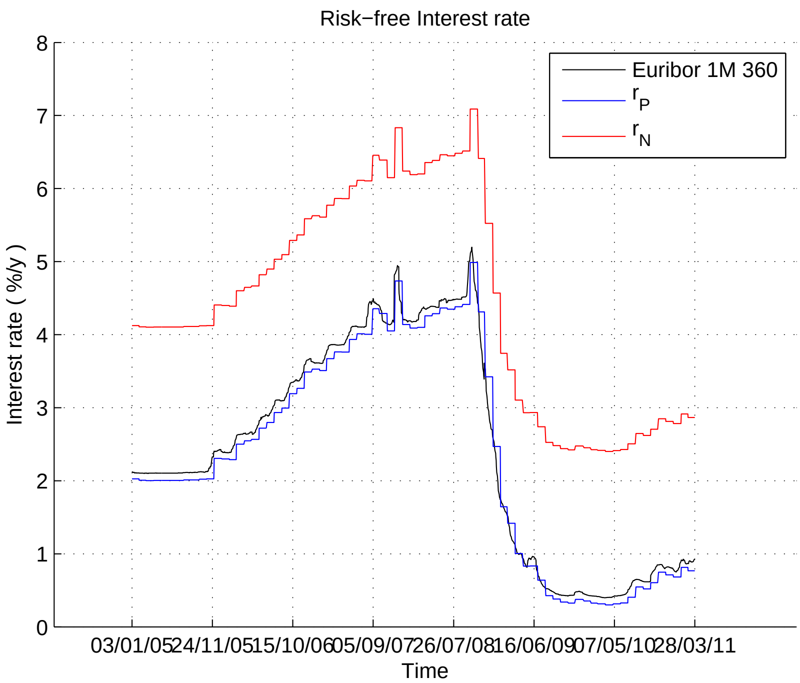


Figure 5.5: Risk-free rate over time

- c fixed commission per transaction (Euro);
- Π portfolio amount (Euro);
- α reference daily return;
- ϵ maximum desired ratio of commission to daily profit

then by writing

$$\frac{c}{\Pi\alpha} \leq \epsilon$$

we determine the minimum amount of portfolio such that the impact of the daily commission to the daily profit does not exceed a pre-defined ratio.

$$\Pi \geq \frac{c}{\epsilon\alpha}$$

Typically, such quantities have the following orders of magnitude:

$$\alpha \approx 10^{-3} \div 10^{-4} \qquad c \approx 10^{1} \qquad \epsilon \approx 10^{-1} \qquad \rightarrow \qquad \Pi \approx 10^{5} \div 10^{6}$$

Putting in some realistic values,

$$\alpha = 4 \cdot 10^{-4} \qquad c = 10 \qquad \epsilon = 0.1 \qquad \rightarrow \qquad \Pi = 250.000$$

Portfolio initial allocation

For both portfolios we have assumed to start with an overall value of 10^6 Euros of which 25% allocated into cash for the ETF portfolio and 25% is left as available cash for the Futures portfolio:

$$\Pi_H(t_0) = \Pi_H(t_0) = 1.00 \cdot 10^6 \text{ Euro}$$

$$C_H(t_0) = A_F(t_0) = 0.25 \cdot 10^6 \text{ Euro}$$

$$\theta_H(t_0) = \theta_F(t_0) = 0.75$$

$$1 - \theta_H(t_0) = 1 - \theta_F(t_0) = 0.25$$

Results

Finally, let's look into the results of the strategy implementation.

- In Table 5.4 we summarize the main performance data relevant to the portfolio hedged on a daily basis with ETFs and with Futures and compare them to the core-asset and its alpha dynamics.
  - Looking into the mean return, the Futures portfolio has a drift which is very close in value to alpha and in the order of 10^{-4}, which is inferior to the core-asset mean return, given that the implementation period covers mostly a phase of growing markets. The ETF portfolio instead performs very weakly and only achieves a small fraction of the alpha return. This poor result is determined by the negative impact of the tracking errors and by the lower leverage with respect to the Futures portfolio.
  - In terms of standard deviation, we point out that the core-asset has a very volatile path in the order of 10^{-2}. As already mentioned, the alpha dynamics has instead a volatility which is one order of magnitude inferior, and this more stable path is transferred to the hedged portfolios. Of these, the Futures pays the price of its high return in terms of volatility, whereas the ETF path is the most stable of all. These differences however are much smaller than the ones in the returns, since each of the alpha, the Futures and ETF volatilities are in the order of 10^{-3}.
  - As a result, the performance indexes clearly show that the Futures portfolio has the best risk-return profile of all, followed by the ETF portfolio, as also shown in the following figures.
  - Over the implementation period, the ETF portfolios achieves a compound return form 1% to 4% whereas the Futures portfolio ranges from 10% to 15%. On an annual basis, the ETF portfolio return ranges from 0% to 2.5% and the Futures portfolio from 3% to 6%.
- In Figures 5.6, 5.7 and 5.8, we show the growth trajectory of the Futures and the ETF portfolios hedged on a daily basis and compare them to the growth of the core-asset and the benchmark index. It can be noted that in all the Sample Cases the hedged portfolios present a stable increase in value with low volatility and that the Futures portfolio out-performs the ETF one.
- In Figures 5.10, 5.12 and 5.14 we track the portfolio allocation over time for the three Sample Cases. Note that the allocation θ_F into the core-asset for the Futures portfolio (orange line) is much higher the correspondent $\gamma\theta_H$ for the ETF portfolio (red line).

Both portfolios start with the same cash allowance but more cash is consumed for the ETF portfolio because over the implementation period markets have been mainly bullish (blue lines). It is to be noted the anti-correlation between core-asset and cash for both portfolios. Cash is consumed with rising markets and cumulated during periods of downfall. For the ETF portfolio we also show the allocation into the same ETF (grey line) whereas for the Futures portfolio we show the evolution of the margin (black line).

- In Figures 5.11, 5.13 and 5.15 we show the evolution over time of four performance indexes, namely the Sharpe ratio, the Sortino Ratio and the Risk Adjusted Performance index with both the alpha return and the index return adopted as benchmark. In each figure we compare these indexes for the Futures portfolio, the ETF portfolio and the core-asset, and we do so for each of the three Sample Cases. In general, the Futures portfolio delivers a better risk-return profile than the other two, with the exception of Sample Case 2. Due to the impact of tracking errors, the ETF portfolio instead does not allow for an improvement of the profile with respect to the core-asset. Note also that for the RAP with benchmark the alpha return, we obtain an almost flat path for the Futures portfolio.

- In Figure 5.16, we show the same performance indexes for Sample Case 3 with reference to an *ideal* case rather than a *real* one, in which we do not consider the tracking errors for the ETF and the pricing errors on the Futures. Whereas for the latter there is no remarkable difference, for the ETF the impact of tracking errors is crucial and determines the passage from a risk-return profile improvement in the ideal case with respect to the core-asset (as shown in this figure) to a worsening of such profile in the real case (as shown in Figure 5.15).

- Finally, in Figure 5.9 we present the growth evolution for the Futures portfolio of Sample Case 1 over a longer period (from february 2008 to October 2011) than then the one considered elsewhere, to show how the strategy has delivered a fairly stable path and has remarkably performed also in periods of crisis. During the 2008 downfall of markets the portfolio has indeed experienced an increase in its volatility but has not lost value. From 2009 onwards it has delivered a very smooth and regular return, whereas with the new crisis of 2011 it has interrupted such trend with a new increase in volatility but limiting the losses. These results obviously depend on the alpha dynamics and the fund manager's capacity to front moments of crisis.

In Appendix A, we summarize the main formulas introduced throughout the thesis.

Table 5.4: **Comparison between ETF and Futures daily hedged portfolios**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
From date	22.07.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
Core-asset			
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
μ_S	1.486E-04	8.142E-04	6.061E-04
σ_S	1.422E-02	1.326E-02	9.767E-03
R_S	4.575E-02	5.158E-01	2.838E-01
Annualized R_S	2.014E-02	1.999E-01	1.552E-01
Sharpe ratio	8.956E-03	5.981E-02	6.015E-02
Sortino ratio	5.247E-04	3.551E-03	4.184E-03
RAP (benchmark $r\alpha$*)*	-1.503E-04	2.185E-05	-6.685E-06
RAP (benchmark r_X*)*	2.213E-04	1.217E-04	1.405E-04
Alpha dynamics			
α	2.042E-04	2.257E-04	2.086E-04
σ_α	3.653E-03	3.785E-03	3.048E-03
R_α	1.200E-01	1.348E-01	9.435E-02
Annualized R_α	5.182E-02	5.699E-02	5.345E-02
Portfolio with Short ETF daily hedge			
μ_{Π_H}	9.882E-06	6.439E-05	9.768E-05
σ_{Π_H}	1.978E-03	3.166E-03	2.485E-03
R_Π	6.672E-03	3.814E-02	4.420E-02
Annualized R_{Π_H}	2.968E-03	1.654E-02	2.533E-02
Sharpe ratio	-5.670E-03	1.368E-02	3.179E-02
Sortino ratio	-3.242E-04	7.695E-04	2.117E-03
RAP (benchmark $r\alpha$*)*	-1.830E-04	-1.646E-04	-1.043E-04
RAP (benchmark r_X*)*	-4.734E-05	-4.192E-04	-1.836E-04
Portfolio with Futures daily hedge			
μ_{Π_F}	1.508E-04	1.956E-04	2.154E-04
σ_{Π_F}	5.037E-03	6.035E-03	3.212E-03
R_Π	8.549E-02	1.038E-01	9.846E-02
Annualized R_{Π_F}	3.728E-02	4.428E-02	5.582E-02
Sharpe ratio	2.574E-02	2.893E-02	6.130E-02
Sortino ratio	1.527E-03	1.727E-03	4.225E-03
RAP (benchmark $r\alpha$*)*	-8.346E-05	-9.021E-05	-1.270E-06
RAP (benchmark r_X*)*	4.539E-04	-2.719E-04	1.653E-04

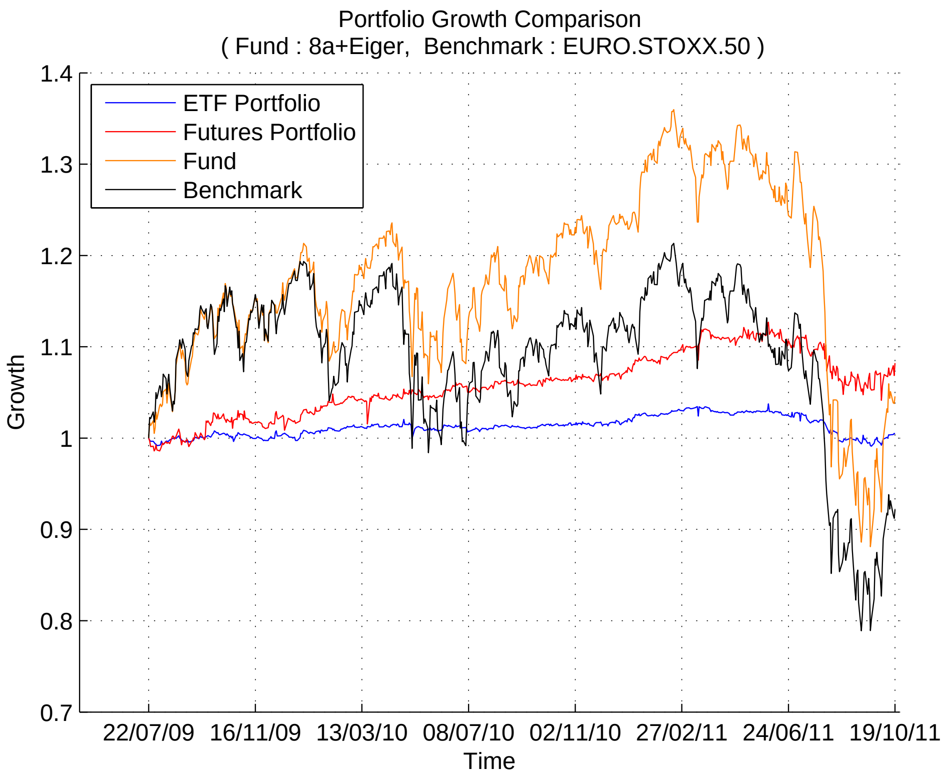


Figure 5.6: Growth comparison between Futures portfolio and ETF portfolio with daily hedging together with core-asset and benchmark index for Sample Case 1.

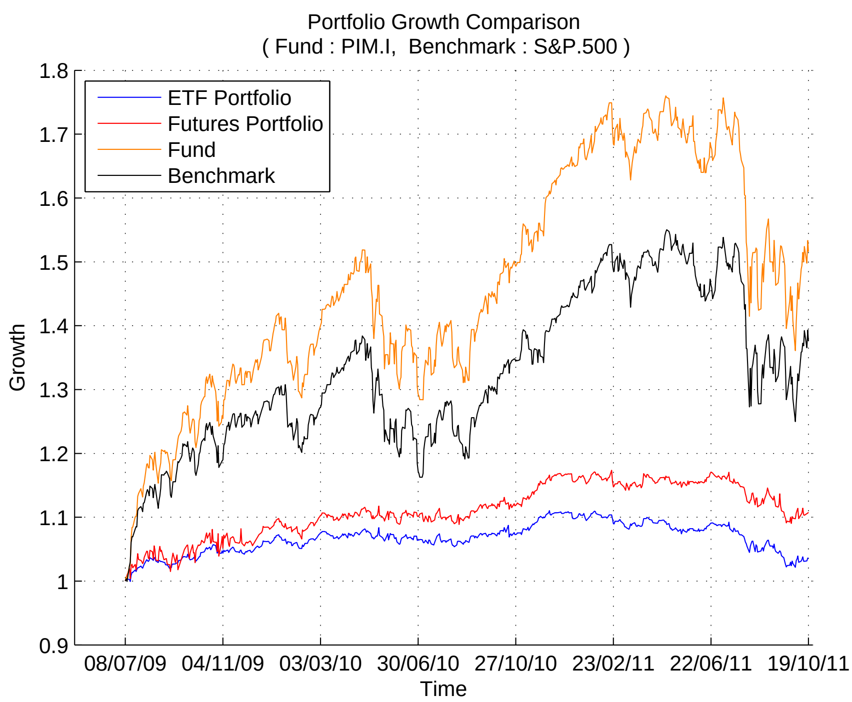


Figure 5.7: Growth comparison between Futures portfolio and ETF portfolio with daily hedging together with core-asset and benchmark index for Sample Case 2.

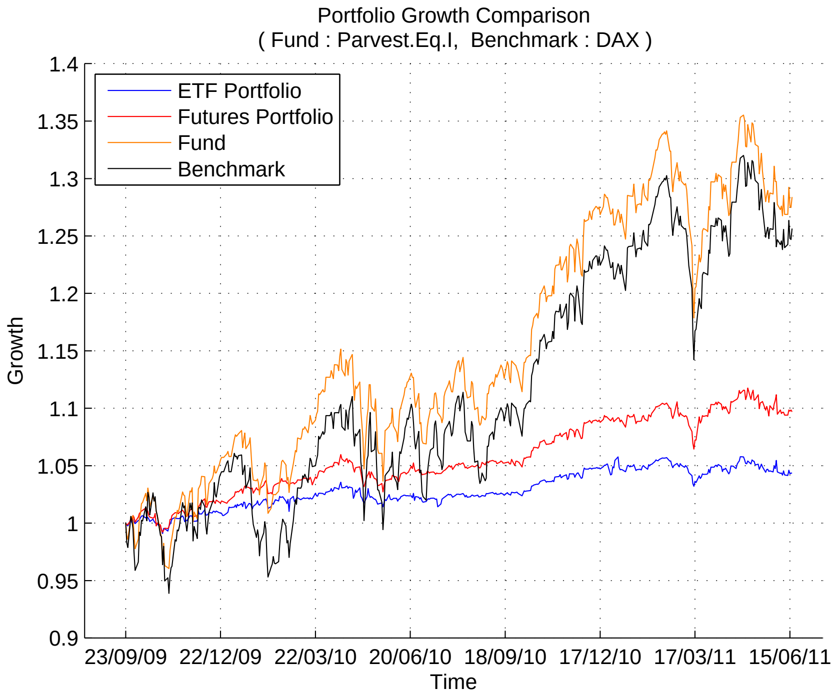


Figure 5.8: Growth comparison between Futures portfolio and ETF portfolio with daily hedging together with core-asset and benchmark index for Sample Case 3.

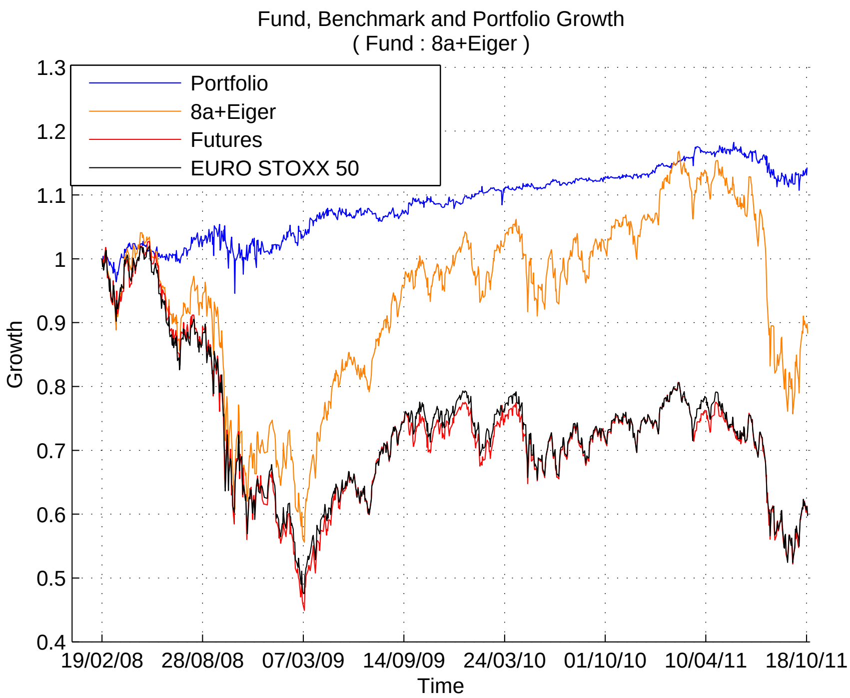


Figure 5.9: Long period growth evolution for the Futures portfolio of Sample Case 1.

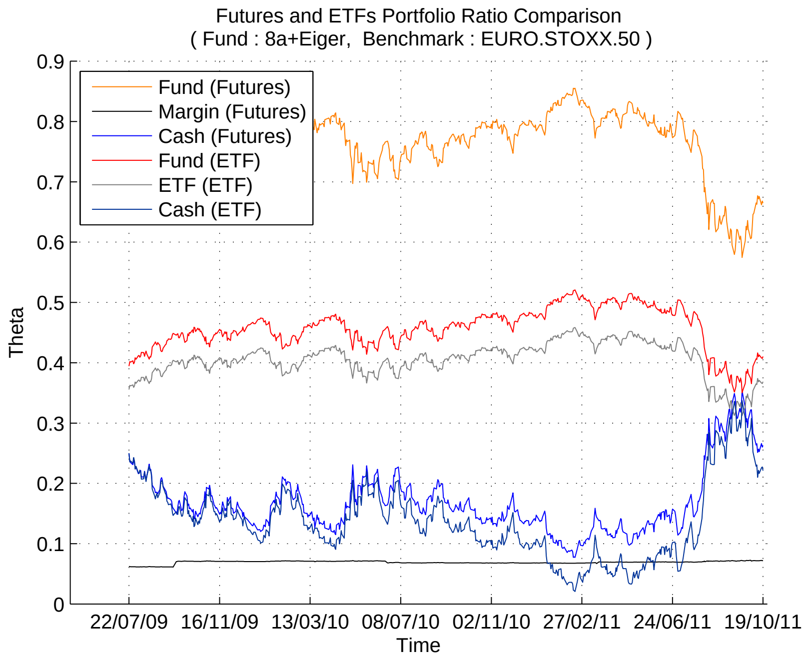
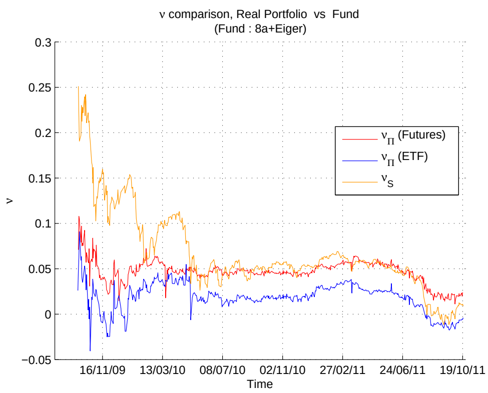
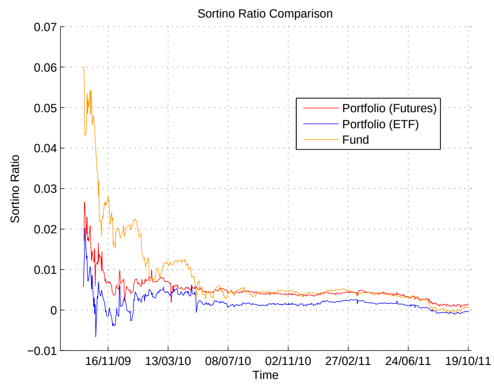


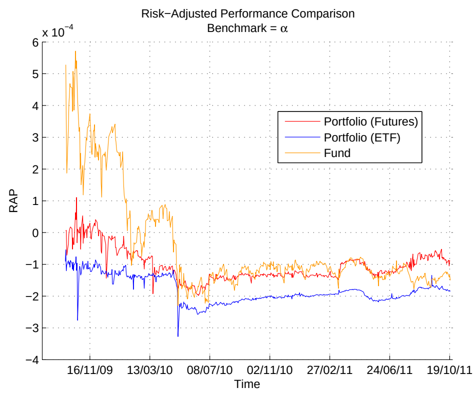
Figure 5.10: Allocation over time for the Futures and ETF portfolio for Sample Case 1.



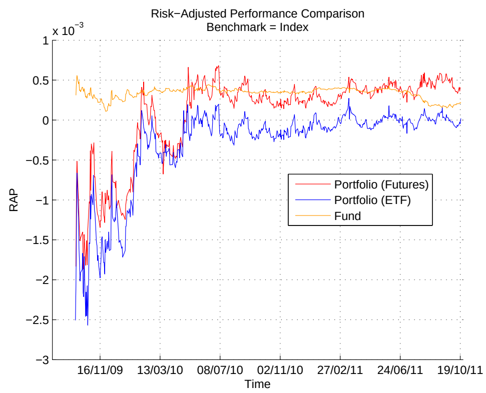
(a) *Sharpe ratio*



(b) *Sortino ratio*



(c) *RAP with benchmark index the alpha return*



(d) *RAP with benchmark index the index return*

Figure 5.11: A comparison of risk-return performance indexes between the Futures portfolio, the ETF portfolio and the core-asset for Sample Case 1.

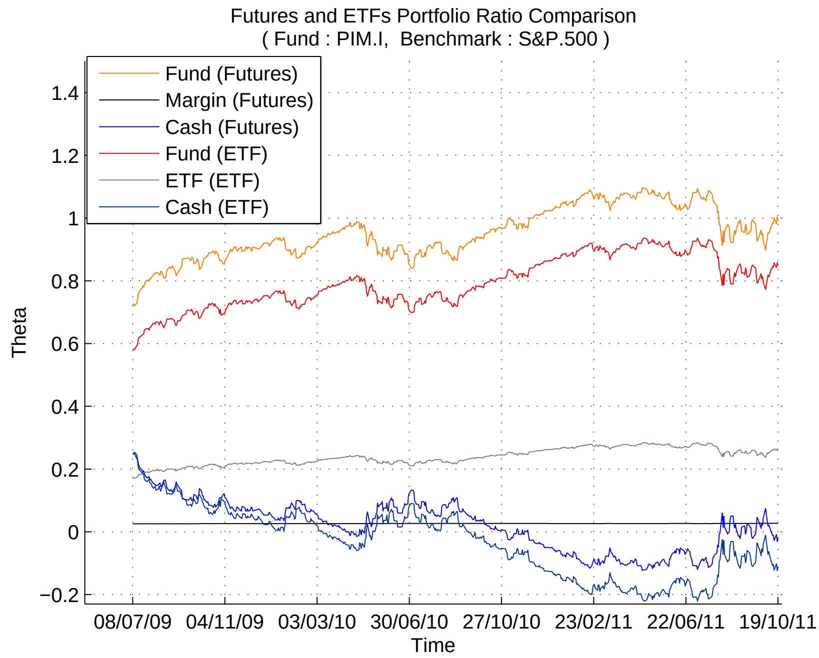
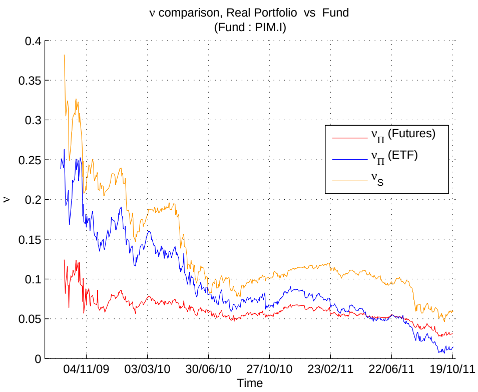
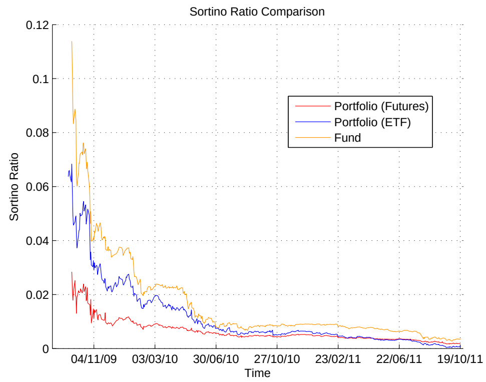


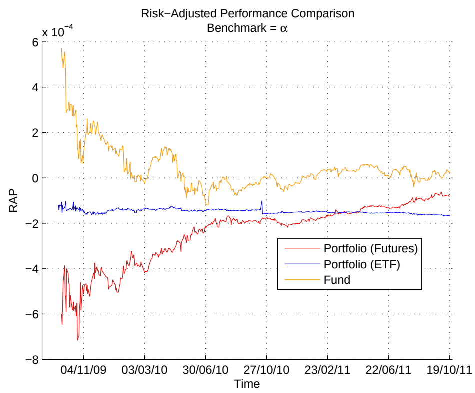
Figure 5.12: Allocation over time for the Futures and ETF portfolio for Sample Case 2.



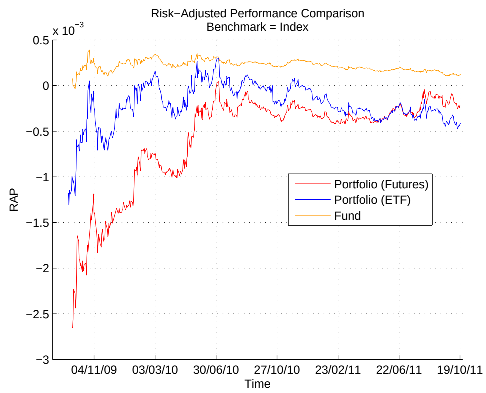
(a) *Sharpe ratio*



(b) *Sortino ratio*



(c) *RAP with benchmark index the alpha return*



(d) *RAP with benchmark index the index return*

Figure 5.13: A comparison of risk-return performance indexes between the Futures portfolio, the ETF portfolio and the core-asset for Sample Case 2.

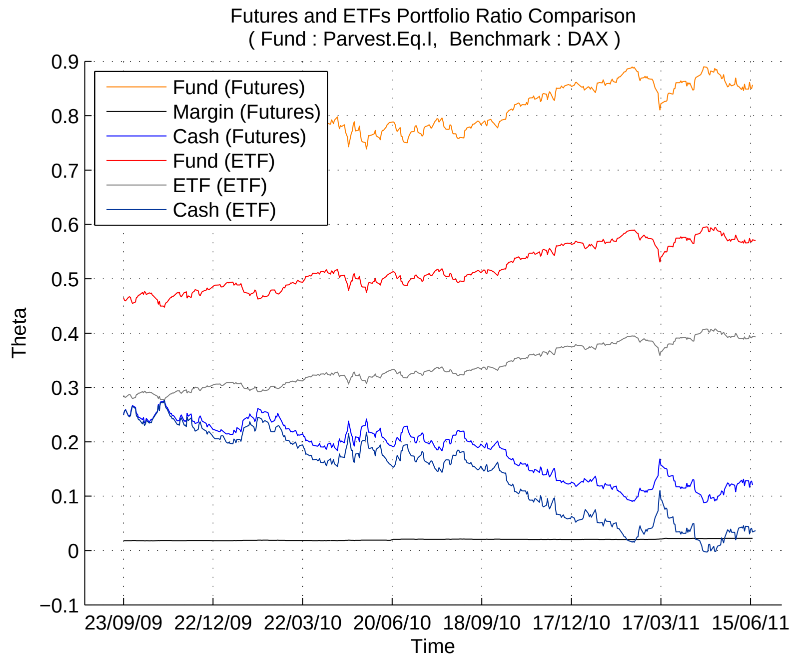
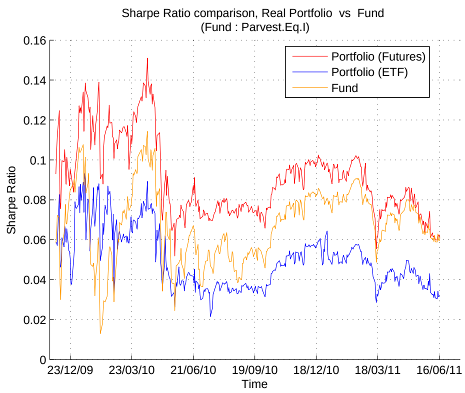
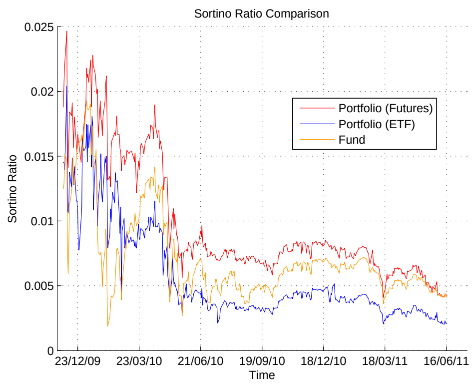


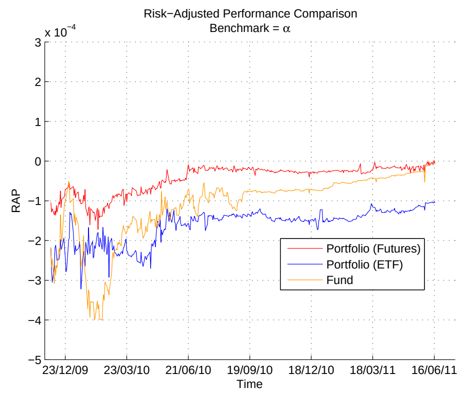
Figure 5.14: Allocation over time for the Futures and ETF portfolio for Sample Case 3.



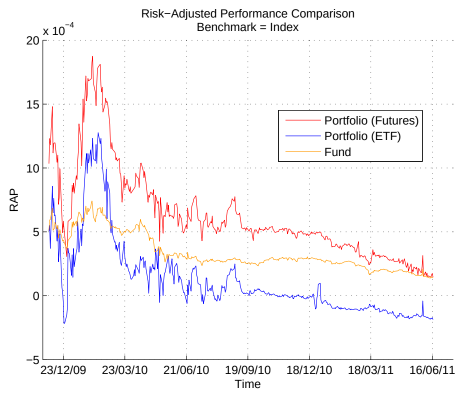
(a) *Sharpe ratio*



(b) *Sortino ratio*



(c) *RAP with benchmark index the alpha return*



(d) *RAP with benchmark index the index return*

Figure 5.15: A comparison of risk-return performance indexes between the Futures portfolio, the ETF portfolio and the core-asset for Sample Case 3.

(a) *Sharpe ratio*

(b) *Sortino ratio*

(c) *RAP with benchmark index the alpha return*

(d) *RAP with benchmark index the index return*

Figure 5.16: A comparison of risk-return performance indexes between the Futures portfolio, the ETF portfolio and the core-asset for Sample Case 3 in the *ideal* case.

Chapter 6

Portfolio with non-daily hedging

This last chapter is dedicated to the analysis of the third and last portfolio indicated in Section 2.1.2, for which the hedging is implemented on a longer time scale than the daily one by use of the Futures contracts.

6.1 Constraint on Short ETFs trading frequency

We have seen in section 4.1.1 that Short ETFs, when not affected by tracking errors, guarantee the nominal pre-defined proportionality with the index return exclusively on a given time scale, which typically is the daily one. It is not possible to hold a Short ETF for a longer period an be assured that in this time it will produce a *compound* return which delivers the nominal proportionality given by k with respect to the compound index return over the same period. The reason for this lies in the fact that equation 4.4 written as

$$R_H(t_i, t_N) = kR_X(t_i, t_N)$$

or

$$\prod_{i=0}^{N-1} \big(1 + kr_X(t_{i+1})\big) - 1 = k\big[\prod_{i=0}^{N-1} \big(1 + r_X(t_{i+1})\big) - 1\big]$$

is equivalent to a polynomial equation of order $N-1$ in the variable k. As seen, there are $N-1$ complex solutions which depend on the combinations of the index returns during the time period. This implies that solutions k are valid only for a particular set of index returns, and thus once fixed a certain value, it is not guaranteed that this will be a solution for any combinations of returns. For this reason, in general we can state that

$$R_H(t_i, t_N) \neq kR_X(t_i, t_N)$$

which implies that when using Short ETFs as hedging tools, we are obliged to trade them on a daily basis in order to be effective.

6.2 Non-daily trading of Futures

This is not the case with Futures, for which we may implement a hedging strategy which trades with a frequency longer than the daily one. To the extreme case, we can consider entering into the contract when it is issued and not perform any operation until its expiry. Obviously, we may also reduce this time span and consider intermediate trading frequencies in order to find an optimal periodicity. In this section, we will however consider the first case, which defines the maximum boundary of such period.

We recall the work set out in section 3.2, and consider the case for which we do not have sufficient data in order to perform a linear regression on sample compound returns and thus we perform the ex-ante estimation based on the linear regression of daily returns and transform this into compound returns to assess the future evolution.

We consider present time fixed at t_i with $i \in [0, N-1]$ and assume that t_N falls at the expiry date of the Futures, indicating with $n := N - i$ the residual time up to expiration. At present time we also exploit the data available from past times t_j with $j \in [0, i-1]$ which give us sufficient information to perform a linear regression on a *daily* time scale but not on a time scale of n periods.

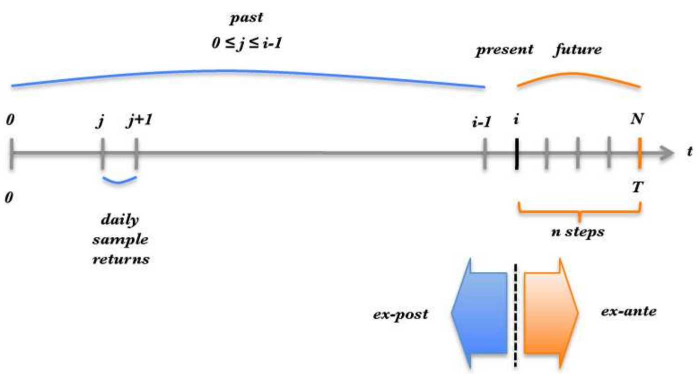


Figure 6.1: At present time t_i we look, with an ex-ante perspective, into the following n time steps up to terminal time t_N at which the Futures contract expires, and consider *compound* returns in such future period. These returns are computed from the *daily* returns known at time t_i for which we own data relevant to past times t_j.

6.2.1 Expected compound return

With a similar approach to the one put forward in the previous sections of this chapter, we consider in the first place the portfolio expected compound return and then compare it to the actual one. At time t_i, the portfolio value is

$$\Pi_F(t_i) = \lambda S(t_i) + C_F(t_i)$$

Now we estimate the portfolio variation between present time t_i and final time t_N, based on the expected compound return of its components. In function of this expectation we will determine the quantity $q_F(t_i)$ of contracts to subscribe in order to hedge the portfolio from the index compound return on the same time scale.

$$E[\Pi_F(t_N)|\mathcal{F}_{t_i}] = \lambda E[S(t_N)|\mathcal{F}_{t_i}] + E[C_F(t_N)|\mathcal{F}_{t_i}]$$

Asset

The asset expected value at terminal time can be expressed in function of its compound returns as in equation 6.1

$$E[R_S(t_i,t_N)|\mathcal{F}_{t_i}] = E[A(t_N)|\mathcal{F}_{t_i}] + E[B(t_N)|\mathcal{F}_{t_i}]E[R_X(t_i,t_N)|\mathcal{F}_{t_i}]$$

where from equations 3.38 and 3.37

$$E\big[A(t_N)|\mathcal{F}_{t_i}\big] \approx E\big[R_\alpha(t_i,t_N)|\mathcal{F}_i\big] = \big[1+\alpha(t_i)\big]^n - 1$$

$$E\big[B(t_N)|\mathcal{F}_{t_i}\big] \approx \beta(t_i)\big[1+\alpha(t_i)\big]^{n-1}$$

and

$$E\big[R_X(t_i,t_N)|\mathcal{F}_i\big] = \big[1+\mu_X(t_i)\big]^n - 1 \approx n\mu_X(t_i)$$

If we adopt a simplified notation for which the dependance on time is expressed as $[]_i = [](t_i)$ and omitt the explicit dependance on time for the compound returns

$$R_S := R_S(t_i,t_N) \qquad R_X := R_X(t_i,t_N) \qquad R_\alpha := R_\alpha(t_i,t_N)$$

$$A := A(t_N) \qquad B := B(t_N)$$

$$E[R_S|\mathcal{F}_i] \approx E[R_\alpha|\mathcal{F}_i] + E[B|\mathcal{F}_i]E[R_X|\mathcal{F}_i]] \tag{6.1}$$

Cash

We now perform the same analysis put forward in sections 5.2.4 and 5.2.5 to describe the cash evolution. We define

$$C_i := C_F(t_i) \qquad q_i := q_F(t_i)$$

Starting from t_i,

$$C_{i+1} = C_i(1+r_i) + q_i(F_{i+1} - F_i)$$

$$C_{i+2} = C_{i+1}(1+r_{i+1}) + q_i(F_{i+2} - F_{i+1})$$

$$C_{i+2} = C_i(1+r_i)(1+r_{i+1}) + q_i(F_{i+1} - F_i)(1+r_{i+1}) + q_i(F_{i+2} - F_{i+1})$$

$$C_{i+3} = C_{i+2}(1+r_{i+2}) + q_i(F_{i+3} - F_{i+2})$$

$$C_{i+3} = C_i(1+r_i)(1+r_{i+1})(1+r_{i+2}) + q_i(F_{i+1} - F_i)(1+r_{i+1})(1+r_{i+2})$$

$$+q_i(F_{i+2} - F_{i+1})(1+r_{i+2}) + q_i(F_{i+3} - F_{i+2})$$

and

$$C_{i+4} = C_{i+3}(1+r_{i+3}) + q_i(F_{i+4} - F_{i+3})$$

$$C_{i+4} = C_i(1+r_i)(1+r_{i+1})(1+r_{i+2})(1+r_{i+3}) + q_i(F_{i+1} - F_i)(1+r_{i+1})(1+r_{i+2})(1+r_{i+3})$$

$$+q_i(F_{i+2}-F_{i+1})(1+r_{i+2})(1+r_{i+3})+q_i(F_{i+3}-F_{i+2})(1+r_{i+3})+q_i(F_{i+4}-F_{i+3})$$

By induction,

$$C_N=C_i\prod_{h=i}^{N-1}(1+r_h)+q_i\sum_{h=i}^{N-2}\left[(F_{h+1}-F_h)\prod_{g=h+1}^{N-1}(1+r_g)\right]+q_i\Big[F_N-F_{N-1}\Big]$$

Again, note that with the sole exception of F_i and F_N, each F_h appears twice in the sum above. Getting back to C_{i+3}, if we isolate the first and last term then we can write:

$$C_{i+3}=C_i(1+r_i)(1+r_{i+1})(1+r_{i+2})+q_i\Big[F_{i+3}-F_i(1+r_{i+1})(1+r_{i+2})\Big]+$$

$$q_i\Big[F_{i+1}\big((1+r_{i+1})(1+r_{i+2})-(1+r_{i+2})\big)+F_{i+2}\big((1+r_{i+2})-1\big)\Big]$$

or

$$C_{i+3}=C_i(1+r_i)(1+r_{i+1})(1+r_{i+2})+q_i\Big[F_{i+3}-F_i(1+r_{i+1})(1+r_{i+2})\Big]$$

$$q_i\Big[F_{i+1}r_{i+1}(1+r_{i+2})+F_{i+2}r_{i+2}\Big]$$

which we can generalize in the form

$$C_N=C_i\prod_{h=i}^{N-1}(1+r_h)+q_i\left[F_N-F_i\prod_{h=i+1}^{N-1}(1+r_h)\right]+q_i\left[\sum_{h=i+1}^{N-2}F_hr_h\prod_{g=h+1}^{N-1}(1+r_g)+F_{N-1}r_{N-1}\right] \tag{6.2}$$

If $r(t_h)=r$ it simplifies into

$$C_N=C_i(1+r)^n+q_i\Big[F_N-F_i(1+r)^{n-1}\Big]+q_ir\left[\sum_{h=i+1}^{N-2}F_h(1+r)^{N-h-1}+F_{N-1}\right]$$

If $r=0$

$$C_N=C_i+q_i(F_N-F_i)$$

We can simplify the expression above by introducing the following compound growth parameters:

$$G(t_i,t_N):=\prod_{h=i}^{N-1}(1+r_h)$$

$$G(t_{i+1},t_N):=\prod_{h=i+1}^{N-1}(1+r_h)$$

$$G(t_{h+1},t_N):=\prod_{g=h+1}^{N-1}(1+r_g)$$

so that

$$C_N=C_iG(t_i,t_N)+q_i\Big[F_N-F_iG(t_{i+1},t_N)\Big]+q_i\left[\sum_{h=i+1}^{N-2}F_hr_hG(t_{h+1},t_N)+F_{N-1}r_{N-1}\right] \tag{6.3}$$

The reason for adopting this form is that in this way we have highlighted the proportionality to r_h of the last term, which as a consequence may be neglected with respect to other terms. Since $r = o(10^{-4})$

$$q_i\left[\sum_{h=i+1}^{N-2} F_h r_h G(t_{h+1}, t_N) + F_{N-1} r_{N-1}\right] = o(q_i F_h) o(10^{-4}) \approx 0$$

so that

$$C_N \approx C_i G(t_i, t_N) + q_i \left[F_N - F_i G(t_{i+1}, t_N)\right] \tag{6.4}$$

As to the conditional expected value of future cash, we recall equation 5.57

$$E[C_N|\mathcal{F}_i] \approx C_i \left[E[R|\mathcal{F}_i] + 1\right] + q_i X_i \left[1 + E[R_X|\mathcal{F}_i] - E[\tilde{G}|\mathcal{F}_i]\right] \tag{6.5}$$

Portfolio

Getting back to the expected compound return of the portfolio, with simplified notation for which

$$\Pi_{F_i} := \Pi_F(t_i) \qquad \Pi_{F_N} := \Pi_F(t_N)$$

it results that

$$E[\Pi_{F_N} - \Pi_{F_i}|\mathcal{F}_i] = E[\Pi_{F_N}|\mathcal{F}_i] - \Pi_{F_i} = \lambda E[S_N|\mathcal{F}_i] - \lambda S_i + E[C_N|\mathcal{F}_i] - C_i =$$

$$\lambda S_i E[R_S|\mathcal{F}_i] + E[C_N|\mathcal{F}_i] - C_i$$

Using equations 6.1 and 6.5, we obtain

$$E[\Pi_{F_N} - \Pi_{F_i}|\mathcal{F}_i] \approx \lambda S_i E[R_\alpha|\mathcal{F}_i] + \lambda S_i E[B|\mathcal{F}_i] E[R_X|\mathcal{F}_i] +$$

$$C_i \left[E[R|\mathcal{F}_i] + 1\right] + q_i X_i \left[1 + E[R_X|\mathcal{F}_i] - E[\tilde{G}|\mathcal{F}_i]\right] - C_i$$

which we can rearrange into

$$E[\Pi_{F_N} - \Pi_{F_i}|\mathcal{F}_i] \approx \lambda S_i E[R_\alpha|\mathcal{F}_i] + \left[\lambda S_i E[B|\mathcal{F}_i] + q_i X_i\right] E[R_X|\mathcal{F}_i] + C_i E[R|\mathcal{F}_i] + q_i X_i \left[1 - E[\tilde{G}|\mathcal{F}_i]\right]$$

If we impose that the term multiplying $E[R_X|\mathcal{F}_i]$ to be null, so that the portfolio variation in the given period be independent from the index compound return, then we find the value of $q_i = q_F(t_i)$ which determines this expected hedge

$$\lambda S_i E[B|\mathcal{F}_i] + q_i X_i = 0$$

so that

$$q_i = -E[B|\mathcal{F}_i] \frac{\lambda S_i}{X_i}$$

which is equivalent to

$$\lambda S_i \beta_i (1 + \alpha_i)^{n-1} + q_i X_i = 0$$

so that

$$q_i = -\beta_i \frac{\lambda S_i}{X_i}(1+\alpha_i)^{n-1}$$

or, readopting full notation,

$$q_F(t_i) = -\beta(t_i)\frac{\lambda S(t_i)}{X(t_i)}\big[1+\alpha(t_i)\big]^{n-1} \tag{6.6}$$

This way it results that

$$E[\Pi_{F_N} - \Pi_{F_i}|\mathcal{F}_i] \approx \lambda S_i E[R_\alpha|\mathcal{F}_i] + C_i E[R|\mathcal{F}_i] - q_i X_i E[\tilde{R}|\mathcal{F}_i]$$

where we have introduced the compound return

$$\tilde{R} := \tilde{R}(t_i, t_N) := (1+r_i)^n \prod_{h=i+1}^{N-1}(1+r_{h+1}) - 1 = \tilde{G} - 1 \tag{6.7}$$

If we now substitute the value q_i as just determined, we obtain

$$E[\Pi_{F_N} - \Pi_{F_i}|\mathcal{F}_i] \approx \lambda S_i E[R_\alpha|\mathcal{F}_i] + C_i E[R|\mathcal{F}_i] + \lambda S_i E[B|\mathcal{F}_i]E[\tilde{R}|\mathcal{F}_i]$$

which can be either seen as

$$E[\Pi_{F_N} - \Pi_{F_i}|\mathcal{F}_i] \approx \Big[E[R_\alpha|\mathcal{F}_i] + E[B|\mathcal{F}_i]E[\tilde{R}|\mathcal{F}_i]\Big]\lambda S_i + \Big[E[R|\mathcal{F}_i]\Big]C_i \tag{6.8}$$

or

$$E[\Pi_{F_N} - \Pi_{F_i}|\mathcal{F}_i] \approx \Big[\lambda S_i\Big]E[R_\alpha|\mathcal{F}_i] + \Big[\lambda S_i E[B|\mathcal{F}_i]\Big]E[\tilde{R}|\mathcal{F}_i] + \Big[C_i\Big]E[R|\mathcal{F}_i] \tag{6.9}$$

As previously done, we can divide these equations by the initial value of the portfolio and thus obtain two alternative expressions for the expected compound return of the portfolio.

$$E[R_{\Pi_F}(t_i,t_N)|\mathcal{F}_i] \approx \Big[E[R_\alpha|\mathcal{F}_i] + E[B|\mathcal{F}_i]E[\tilde{R}|\mathcal{F}_i]\Big]\theta_i + \Big[E[R|\mathcal{F}_i]\Big](1-\theta_i) \tag{6.10}$$

The expected return is given by two components. The first is proportional to the fraction θ_i of portfolio allocated in the core-asset and has a return equal to the sum of the expected alpha return $E[R_\alpha|\mathcal{F}_i]$ plus a proportion $E[B|\mathcal{F}_i]$ of a risk-free compound return $E[\tilde{R}|\mathcal{F}_i]$ for $2n-1$ periods. The second, is proportional to the cash allocation $(1-\theta_i)$ and has a return equal to the risk-free expected compound return $E[R|\mathcal{F}_i]$.
Alternatively,

$$E[R_{\Pi_F}(t_i,t_N)|\mathcal{F}_i] \approx \Big[\theta_i\Big]E[R_\alpha|\mathcal{F}_i] + \Big[\theta_i E[B|\mathcal{F}_i]\Big]E[\tilde{R}|\mathcal{F}_i] + \Big[1-\theta_i\Big]E[R|\mathcal{F}_i] \tag{6.11}$$

we can express the expected compound return as given by following three components. The first is the fraction θ_i of portfolio which has the compound expected alpha return $E[R_\alpha|\mathcal{F}_i]$; then a part of the portfolio equal to $\theta_i E[B|\mathcal{F}_i]$ has an expected compound return of $E[\tilde{R}|\mathcal{F}_i]$ and finally, the cash fraction $(1-\theta_i)$ has the risk-free compound return $E[R|\mathcal{F}_i]$.

6.2.2 Actual compound return

Now we move the present time from t_i to t_N and consider from an ex-post point of view which has effectively been the portfolio compound return over this period.

$$\Pi_{F_i} = \lambda S_i + C_i$$

$$\Pi_{F_N} = \lambda S_N + C_N$$

$$\Pi_{F_N} - \Pi_{F_i} = \lambda(S_N - S_i) + C_N - C_i = \lambda S_i R_S + C_N + C_i$$

As to the asset,

$$R_S(t_i, t_n) = A(t_N) + B(t_N)R_X(t_i, t_N) + \zeta_A(t_i, t_N)$$

where $\zeta_A := \zeta_A(t_i, t_N)$ indicates the compound alpha error. If we adopt the simplified notation

$$R_S = A_N + B_N R_X + \zeta_A$$

Recalling equation 6.3, the final cash may be written as

$$C_N = C_i G(t_i, t_N) + q_i \Big[F_N - F_i G(t_{i+1}, t_N) + X_i \zeta_R(t_i, t_N) \Big]$$

where we have defined as *compound approximation error* the previously approximated quantity

$$\zeta_R := \zeta_R(t_i, t_N) := \frac{1}{X_i} \sum_{h=i+1}^{N-2} F_h r_h G(t_{h+1}, t_N) + F_{N-1} r_{N-1} \tag{6.12}$$

The overall portfolio variation at time t_N will have been

$$\Pi_{F_N} - \Pi_{F_i} = \lambda S_i R_S + C_i R + q_i \Big[F_N - F_i G(t_{i+1}, t_N) + X_i \zeta_R(t_i, t_N) \Big]$$

If we substitute

$$F_i = X_i(1 + r_i)^n + dF_i$$

$$F_N = X_i(1 + R_X) + dF_N$$

then

$$\Pi_{F_N} - \Pi_{F_i} = \lambda S_i \Big[A_N + B_N R_X + \zeta_A \Big] + C_i R +$$

$$q_i \Big[X_i(1 + R_X) - X_i(1 + r_i)^n G(t_{i+1}, t_N) + X_i \zeta_R + dF_N - dF_i \Big]$$

Now we define as *futures compound error* the term

$$\zeta_F(t_i, t_N) := \frac{q(t_i)}{X(t_i)} \big[dF(t_n) - dF(t_i) \big] \tag{6.13}$$

or with simplified notation

$$\zeta_F := \frac{q_i}{X_i} \big[dF_N - dF_i \big]$$

to write

$$\Pi_{F_N} - \Pi_{F_i} = \lambda S_i \Big[A_N + B_N R_X + \zeta_A \Big] + C_i R + q_i X_i \Big[1 + R_X - (1+r_i)^n G(t_{i+1}, t_N) + \zeta_R + \zeta_F \Big]$$

We can also use the definition introduced in equation 6.7

$$\tilde{R} = (1+r_i)^n G(t_{i+1}, t_N) - 1$$

so that

$$\Pi_{F_N} - \Pi_{F_i} = \lambda S_i \Big[A_N + B_N R_X + \zeta_A \Big] + C_i R + q_i X_i \Big[R_X - \tilde{R} + \zeta_R + \zeta_F \Big]$$

which we can rearrange into

$$\Pi_{F_N} - \Pi_{F_i} = \lambda S_i \Big[A_N + \zeta_A \Big] + \Big[\lambda S_i B_N + q_i X_i \Big] R_X + C_i R - q_i X_i \Big[\tilde{R} - \zeta_R - \zeta_F \Big]$$

Now, if we substitute the previously determined value of q_i

$$q_i = -E[B|\mathcal{F}_i] \frac{\lambda S_i}{X_i}$$

equivalent to

$$q_i X_i = -E[B|\mathcal{F}_i] \lambda S_i$$

it results that

$$\Pi_{F_N} - \Pi_{F_i} = \lambda S_i \Big[A_N + \zeta_A + \big(B_N - E[B|\mathcal{F}_i] \big) R_X + E[B|\mathcal{F}_i] \big(\tilde{R} - \zeta_R - \zeta_F \big) \Big] + C_i R$$

In order to highlight the *compound estimation error* defined as

$$\zeta_X = \zeta_X(t_i, t_N) := A(t_N) - E[A(t_N)|\mathcal{F}_i] + \big(B(t_N) - E[B(t_N)|\mathcal{F}_i] \big) R_X$$

we add and subtract the term $E[A|\mathcal{F}_i]$ from the portfolio variation and obtain

$$\Pi_{F_N} - \Pi_{F_i} = \lambda S_i \Big[E[A|\mathcal{F}_i] + \zeta_A + \Big(A_N - E[A|\mathcal{F}_i] + \big(B_N - E[B|\mathcal{F}_i] \big) R_X \Big) + E[B|\mathcal{F}_i] \big(\tilde{R} - \zeta_R - \zeta_F \big) \Big] + C_i R$$

Since

$$E[A(t_N)|\mathcal{F}_i] \approx E[R_\alpha(t_N)|\mathcal{F}_i]$$

we can also write

$$\Pi_{F_N} - \Pi_{F_i} = \Big[E[R_\alpha|\mathcal{F}_i] + \zeta_A + \zeta_X + E[B|\mathcal{F}_i] \big(\tilde{R} - \zeta_R - \zeta_F \big) \Big] \lambda S_i + R C_i \tag{6.14}$$

We can now divide each term by the portfolio initial value Π_{F_i} to obtain an expression for the compound portfolio return

$$R_{\Pi_F}(t_i, t_N) = \Big[E[R_\alpha \mathcal{F}_i] + \zeta_A + \zeta_X + E[B|\mathcal{F}_i] \big(\tilde{R} - \zeta_R - \zeta_F \big) \Big] \theta_i + \Big[R \Big] (1 - \theta_i) \tag{6.15}$$

The portfolio return is composed of two terms. The first, is proportional to the fraction θ_i of portfolio allocated into the core-asset and has a return which is given by the alpha compound expected drift $E[R_\alpha|\mathcal{F}_i]$ modified by the compound alpha error ζ_A and increased by a risk-free return term $E[B|\mathcal{F}_i]\tilde{R}$. Then there are also three error terms, given by the compound estimation ζ_X, the compound approximation ζ_R and the compound futures ζ_F errors, the last two of which proportional to $E[B|\mathcal{F}_i]$. The second term refers to the fraction $(1-\theta_i)$ of allocated cash and has a compound return equal to the risk-free one R.
As usual, we may also express the return in a different form

$$R_{\Pi_F}(t_i,t_N) = \Big[\theta_i\Big]E[R_\alpha|\mathcal{F}_i]+\Big[\theta_i\Big]\Big(\zeta_A+\zeta_X-E[B|\mathcal{F}_i]\big(\zeta_R+\zeta_F\big)\Big)+\Big[1-\theta_i\Big]R+\Big[E[B|\mathcal{F}_i]\theta_i\Big]\tilde{R} \tag{6.16}$$

where we have highlighted the different compound returns appearing in equation 6.15. Return $E[R_\alpha|\mathcal{F}_i]$ is achieved by the fraction θ_i which also produces a return equal to the sum of the compound errors $\zeta_A+\zeta_X-E[B|\mathcal{F}_i]\big(\zeta_R+\zeta_F\big)$. A fraction $E[B|\mathcal{F}_i]\theta_i$ also produces the risk-free return $\tilde{R}$ as does the cash part $1-\theta_i$ which grows with R.

6.2.3 Portfolio performance

Error

Recalling the expected portfolio variation given by equation 6.8

$$E[\Pi_{F_N}-\Pi_{F_i}|\mathcal{F}_i] \approx \Big[E[R_\alpha|\mathcal{F}_i]+E[B|\mathcal{F}_i]E[\tilde{R}|\mathcal{F}_i]\Big]\lambda S_i + \Big[E[R|\mathcal{F}_i]\Big]C_i$$

we may compute the difference between the actual variation and the estimated one

$$\Pi_{F_N}-\Pi_{F_i}-E[\Pi_{F_N}-\Pi_{F_i}|\mathcal{F}_i] =$$

$$\lambda S_i\Big[\zeta_A+\zeta_X-E[B|\mathcal{F}_i]\big(\zeta_R+\zeta_F\big)+E[B|\mathcal{F}_i]\Big(\tilde{R}-E[\tilde{R}|\mathcal{F}_i]\Big)\Big]+\Big[R-E[R|\mathcal{F}_i]\Big]C_i \tag{6.17}$$

This expression indicates the error we fall into when implementing the hedge at time t_i with an ex-ante approach to the portfolio dynamics up to time t_N, and thus compute the *expected* portfolio variation, error which we commit with respect to the *actual* variation known at time t_N and computed then with an ex-post perspective.
As with the daily trading, we have same four types of errors as the ones of equation 5.50, but in a *compound* form rather than a single one. These terms are the *compound alpha error* ζ_A, the *compound estimation error* ζ_X, the *compound approximation error* ζ_R and the *compound futures error* ζ_F. Differently than in the daily case there are however two extra terms given by the differences $\tilde{R}-E[\tilde{R}|\mathcal{F}_i]$ and $R-E[R|\mathcal{F}_i]$ between expected compound risk-free returns and the actual ones.

The reason for which these terms do not appear in the daily trading case, is because we assume that at any given time t_i the interests we will receive on a sum $C(t_i)$ at the next time step t_{i+1} are known and equal to $C(t_i)r(t_i)$. In other words, interest rates are treated as *anticipated* rates, which means that the interest over a certain period is determined by the interest rate at the beginning of that period rather than at the end of it. On the contrary,

when referring to returns of securities, we have expressed the variation in value at time t_{i+1} as a function of the return at that same time. This variation is therefore an unknown quantity at time t_i, which is the beginning of the period of reference, and it becomes known only at the end of it. In this sense, such returns could be seen as *posticipated* rates.
Getting back to the new errors appearing in equation 6.17, although it is true that for the first time step the interest rate is known, this is not the case for the subsequent rates which remain unknown. As a consequence, there will be a difference between the estimated compound return and the actual one, a difference which as mentioned does not exist if we consider only a single period of time. This is based on the assumption that interest rates mature on a daily basis.

Implementation

For the non-daily trading case, we consider only the Futures portfolio. Numerical simulations are implemented considering that trading of Futures contracts occur exclusively at delivery times, when a contract ceases to exist and a new one is issued in its substitution. This long-period represents the upper limit in terms of trading frequency, whereas the daily one can be considered its inferior limit. Obviously, the strategy could be implemented also on an intermediate time frame.
We only present the results relevant to Sample Case 1 and 3 because we were not able to identify with enough accuracy the delivery times of the Futures contracts for the S&P Futures of Sample Case 3.
All other conditions, such as the adopted risk-free rate, the transaction costs and the initial portfolio allocation are the same as those given in the previous Chapter.

Results

- In Figures 6.2 and 6.3, we show the growth trajectory of the Futures portfolio with hedging at delivery time and compare it to the growth of the core-asset, the benchmark index and the ETF portfolio with daily hedge. The enhancement provided by the Futures portfolio also with non-daily hedging is remarkable with respect to the ETF one. This implies that the compound approximation error does not affect importantly the global return.
- In Figures 6.4 and 6.6 we present the portfolio allocation over time for Sample Cases 1 and 3. As with the daily hedging case, the allocation θ_F into the core-asset for the Futures portfolio with trading at delivery (orange line) is much higher the correspondent $\gamma\theta_H$ for the ETF portfolio with daily hedge (red line). Again, both portfolios start with the same cash allowance but more cash is consumed by the ETF portfolio in an anti-correlated path with respect to the core-asset. For the ETF portfolio we also show the allocation into the same ETF (grey line) whereas for the Futures portfolio we show the evolution of the margin (black line).
- In Figures 6.5 and 6.7 we show the evolution over time of the four performance indexes (Sharpe ratio, Sortino Ratio, Risk Adjusted Performance with benchmark the index and RAP with benchmark the alpha return). Once again, the Futures portfolio with trading at delivery is characterized by the best risk-return profile.

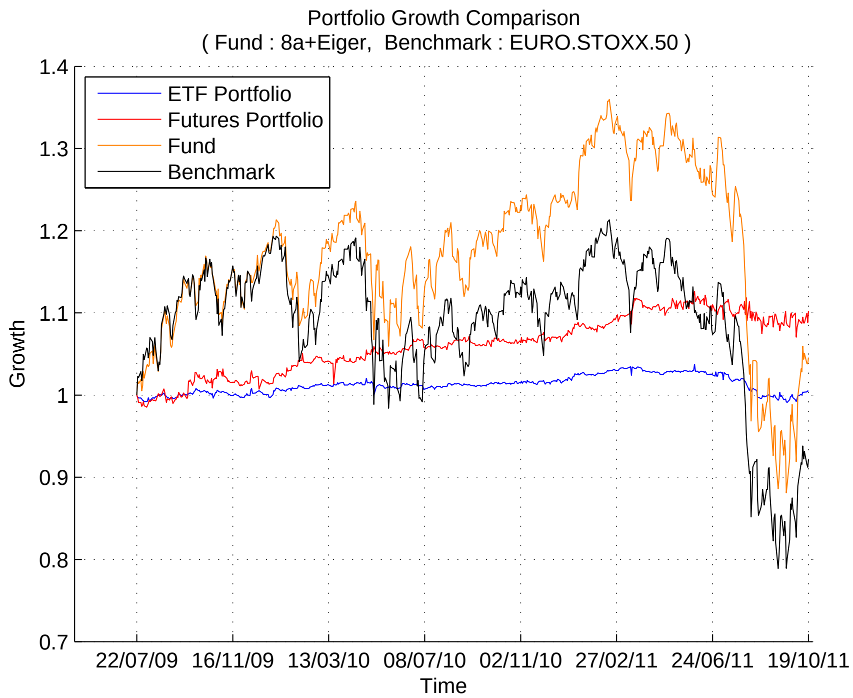


Figure 6.2: Growth comparison between Futures portfolio, ETF portfolio, core-asset and benchmark index for Sample Case 1.

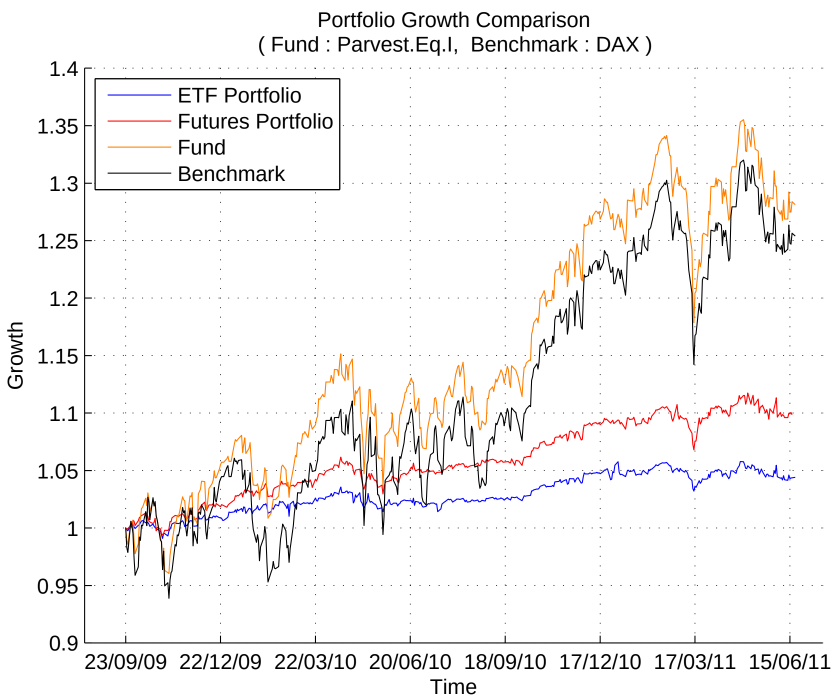


Figure 6.3: Growth comparison between Futures portfolio, ETF portfolio, core-asset and benchmark index for Sample Case 3.

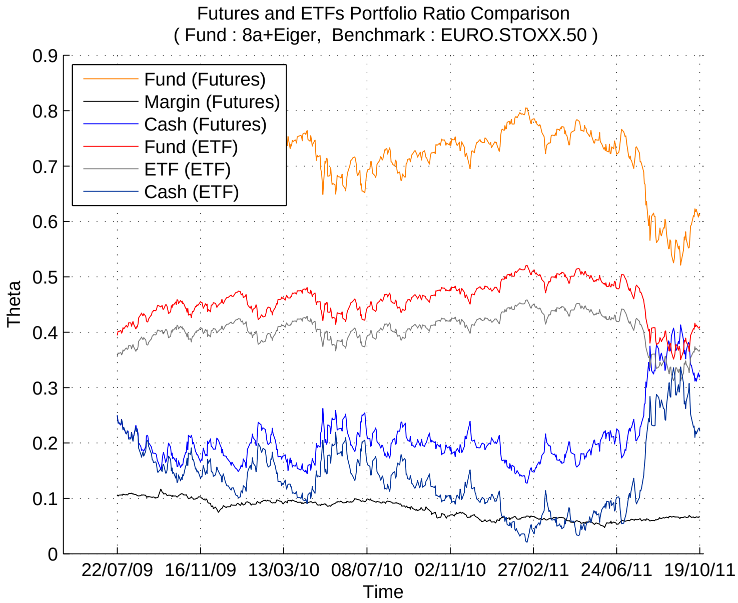
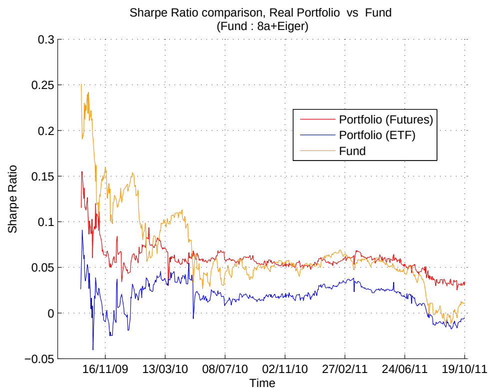
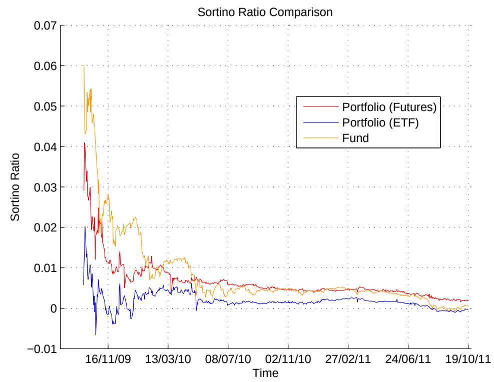


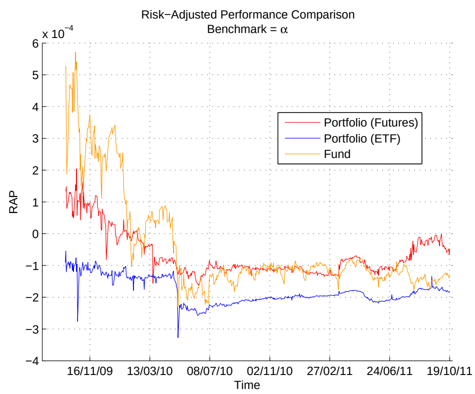
Figure 6.4: Allocation over time for the Futures portfolio with hedging at delivery compared to the ETF portfolio with daily hedging for Sample Case 1.



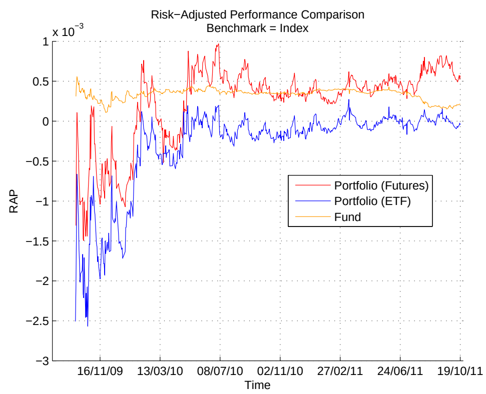
(a) *Sharpe ratio*



(b) *Sortino ratio*



(c) *RAP with benchmark index the alpha return*



(d) *RAP with benchmark index the index return*

Figure 6.5: A comparison of risk-return performance indexes between the Futures portfolio with trading at delivery date, the ETF portfolio with daily trading and the core-asset for Sample Case 1.

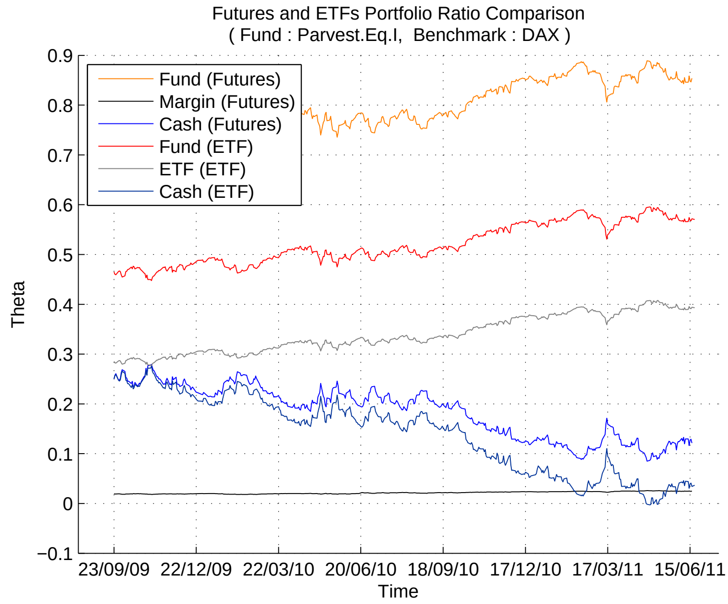
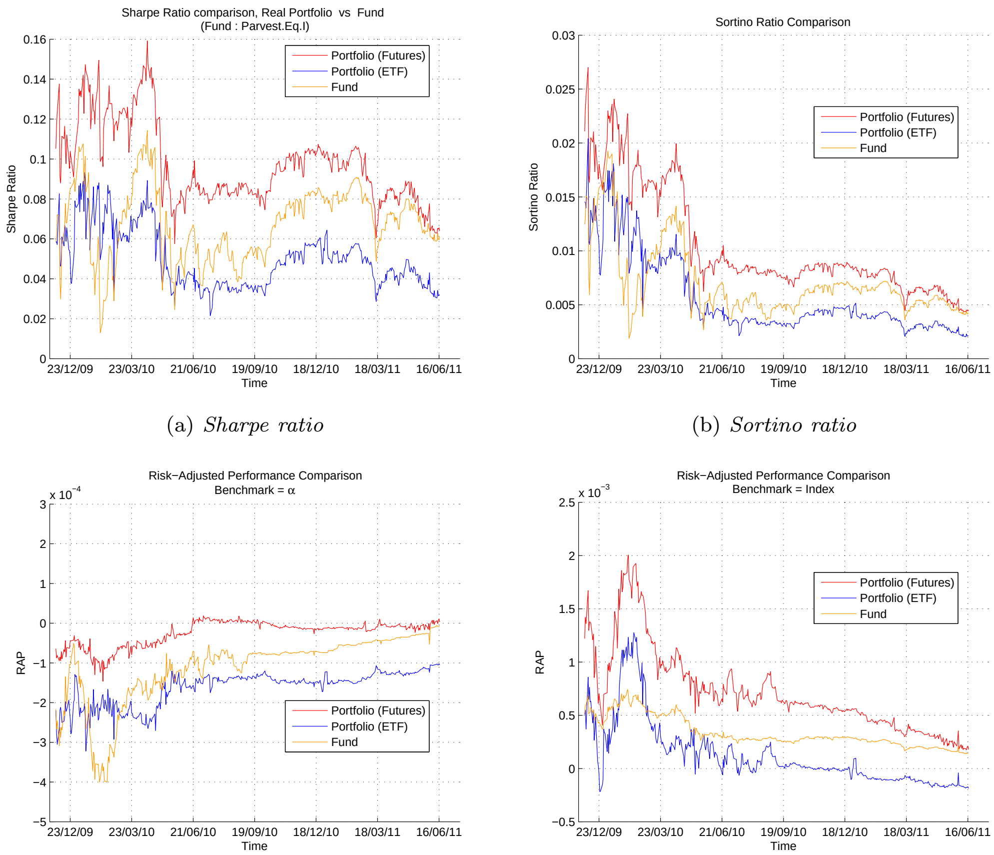


Figure 6.6: Allocation over time for the Futures portfolio with hedging at delivery compared to the ETF portfolio with daily hedging for Sample Case 3.



(a) *Sharpe ratio* (b) *Sortino ratio*

(c) *RAP with benchmark index the alpha return* (d) *RAP with benchmark index the index return*

Figure 6.7: A comparison of risk-return performance indexes between the Futures portfolio with trading at delivery date, the ETF portfolio with daily trading and the core-asset for Sample Case 3.

Table 6.1: **Daily and compound quantities**

Quantity	*Daily*	*Compound*
Portfolio return	$r_{\Pi_F}(t_{i+1})$	$R_{\Pi_F}(t_i, t_N)$
Risk-free return	$r(t_i)$	$R(t_i, t_N)$
Alpha	$\alpha(t_i)$	$E[R_\alpha(t_i, t_N)\mathcal{F}_i]$
Beta	$\beta(t_i)$	$E[B(t_i, t_N)\vert\mathcal{F}_i]$
Extra risk-free return	$r(t_i)$	$\tilde{R}(t_i, t_N)$
Alpha error	$\zeta_\alpha(t_{i+1})$	$\zeta_A(t_i, t_N)$
Estimation error	$\zeta_x(t_{i+1})$	$\zeta_X(t_i, t_N)$
Approximation error	$\zeta_r(t_{i+1})$	$\zeta_R(t_i, t_N)$
Futures error	$\zeta_f(t_{i+1})$	$\zeta_F(t_i, t_N)$

6.3 Comparison between Futures portfolio with daily and non-daily hedging

Finally, we now compare the results for the Futures portfolio with hedging at delivery dates with the same portfolio hedged on a daily basis.

Returns

Recalling equation 5.49, the portfolio with daily hedging presents a return equal to

$$r_{\Pi_F}(t_{i+1}) = \Big[r(t_i)\Big]\big(1-\theta_F(t_i)\big)+$$

$$\Big[\alpha(t_i)+\zeta_\alpha(t_{i+1})+\zeta_x(t_{i+1})+\beta(t_i)\Big(r(t_i)-\zeta_r(t_{i+1})-\zeta_f(t_{i+1})\Big)\Big]\theta(t_i)$$

whereas the non-daily one has a return given by equation 6.15

$$R_{\Pi_F}(t_i, t_N) = \Big[R(t_i, t_N)\Big]\big(1-\theta_F(t_i)\big)+$$

$$\Big[E[R_\alpha(t_i,t_N)\mathcal{F}_i]+\zeta_A(t_i,t_N)+\zeta_X(t_i,t_N)+E[B(t_i,t_N)|\mathcal{F}_i]\big(\tilde{R}(t_i,t_N)-\zeta_R(t_i,t_N)-\zeta_F(t_i,t_N)\big)\Big]\theta_F(t_i)$$

These equations are very similar one to another. Each daily return and error in the first equation has its equivalent in compound form in the second, as shown in Table 6.1

Results

We finally look into the numerical results.

- In Table 6.2 we summarize the main performance data for the two Futures portfolios and compare them to the core-asset and its alpha dynamics.
  - Looking into the mean return, we note that the non-daily portfolio has a return which is very close to the daily portfolio one and the alpha dynamics.

  - The standard deviation is also very similar between the two portfolios and slightly higher than the alpha one, although still in the same order of magnitude of 10^{-3}, as opposed to the core-asset which has a volatility in the order of 10^{-2}.
  - The performance indexes are also very similar between the two portfolios and bot show an improvement on the risk-return profile with respect to the core-asset as also shown in the following figures.
  - Over the implementation period, the hedged portfolios gain a compound return between 10% and 15% equal to annualized returns in the range from 3% to 6%.

- Figures 6.8 and 6.10 compare the growth evolution of the two futures portfolios. It can be noted that the path is very similar, which implies that the compound estimation error, arising from the fact that we hedge the portfolio at the delivery dates making an estimation of its future correlation over the life time of the contract, does not affect the performance significantly.

- Finally, Figures 6.9 and 6.11 compare the performance indexes for the two portfolios and once again confirm that no major difference is notable between them. Both show an important improvement in the risk-return profile.

As a consequence, it is possible to effectively implement the strategy by trading the futures on a long time frame, which to limit can also be the time frame of delivery dates of the contracts. This possibility obviously simplifies remarkably the operations and reduces to minimum the transaction costs and the resources that need to be placed in order to execute the operations.

Once again, we mention that all main formulas are summarized in Appendix A.

Table 6.2: **Comparison between ETF and Futures daily hedged portfolios**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
From date	22.07.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
Core-asset			
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
μ_S	1.486E-04	8.142E-04	6.061E-04
σ_S	1.422E-02	1.326E-02	9.767E-03
R_S	4.575E-02	5.158E-01	2.838E-01
Annualized R_S	2.014E-02	1.999E-01	1.552E-01
Sharpe ratio	8.956E-03	5.981E-02	6.015E-02
Sortino ratio	5.247E-04	3.551E-03	4.184E-03
RAP (benchmark $r\alpha$*)*	-1.503E-04	2.185E-05	-6.685E-06
RAP (benchmark r_X*)*	2.213E-04	1.217E-04	1.405E-04
Alpha dynamics			
α	2.042E-04	2.257E-04	2.086E-04
σ_α	3.653E-03	3.785E-03	3.048E-03
R_α	1.200E-01	1.348E-01	9.435E-02
Annualized R_α	5.182E-02	5.699E-02	5.345E-02
Portfolio with Futures daily hedge			
μ_{Π_F}	1.508E-04	1.956E-04	2.154E-04
σ_{Π_F}	5.037E-03	6.035E-03	3.212E-03
R_Π	8.549E-02	1.038E-01	9.846E-02
Annualized R_{Π_F}	3.728E-02	4.428E-02	5.582E-02
Sharpe ratio	2.574E-02	2.893E-02	6.130E-02
Sortino ratio	1.527E-03	1.727E-03	4.225E-03
RAP (benchmark $r\alpha$*)*	-8.346E-05	-9.021E-05	-1.270E-06
RAP (benchmark r_X*)*	4.539E-04	-2.719E-04	1.653E-04
Portfolio with Futures single hedge at delivery date			
μ_{Π_F}	1.831E-04	n.a.	2.186E-04
σ_{Π_F}	5.121E-03	n.a.	3.121E-03
R_Π	1.057E-01	n.a.	9.957E-02
Annualized R_{Π_F}	4.585E-02;	n.a.	5.644E-02
Sharpe ratio	3.162E-02	n.a.	6.411E-02
Sortino ratio	1.914E-03	n.a.	4.416E-03
RAP (benchmark $r\alpha$*)*	-6.197E-05	n.a.	7.312E-06
RAP (benchmark r_X*)*	5.446-04	n.a.	1.980E-04

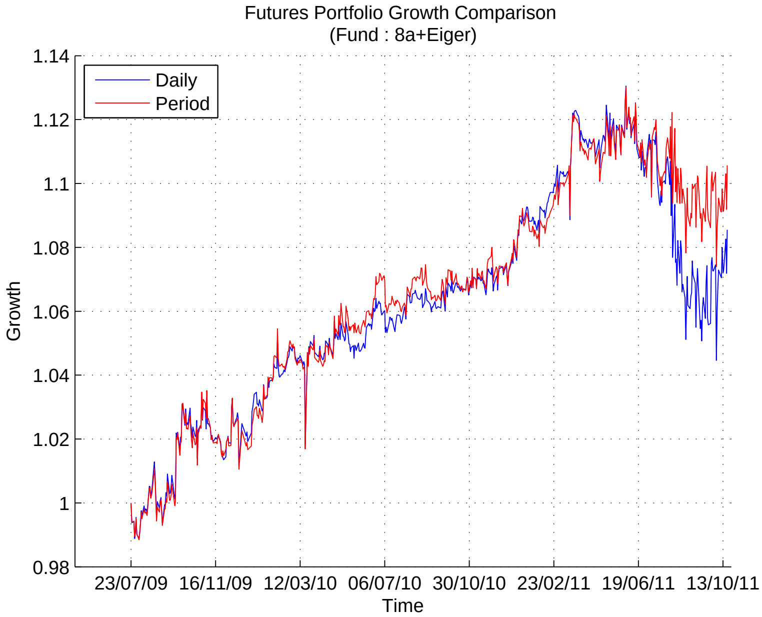
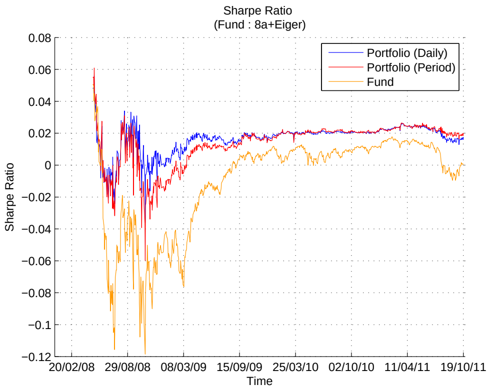
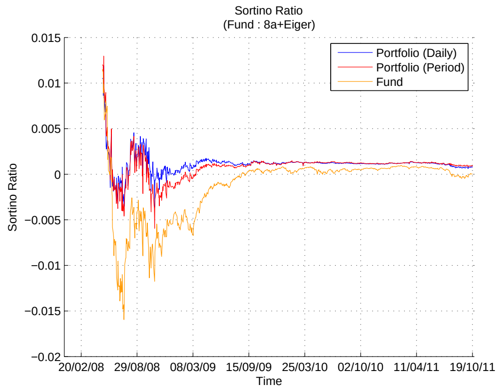


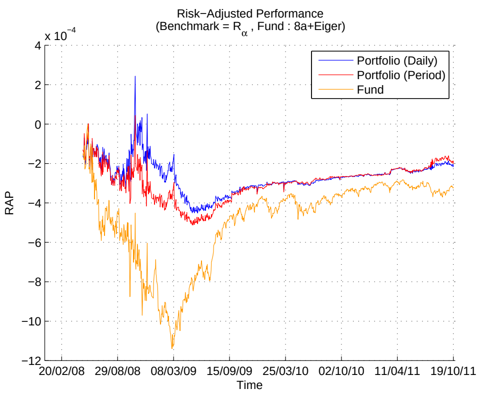
Figure 6.8: Growth comparison between the Futures portfolio with daily hedging and delivery date hedging for Sample Case 1.



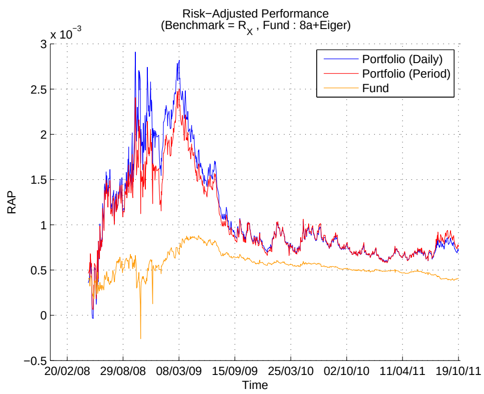
(a) *Sharpe ratio*



(b) *Sortino ratio*



(c) *RAP with benchmark index the alpha return*



(d) *RAP with benchmark index the index return*

Figure 6.9: A comparison of risk-return performance indexes between the Futures portfolio with daily hedging and delivery hedging for Sample Case 1.

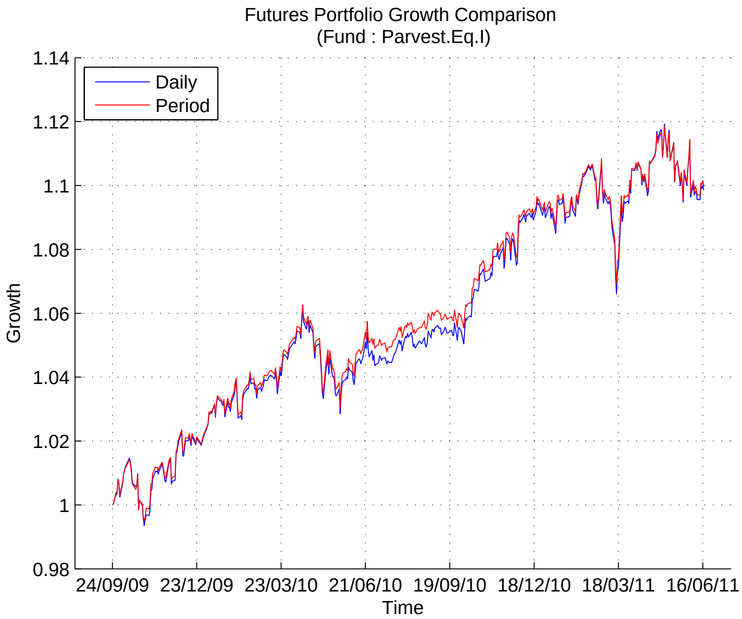
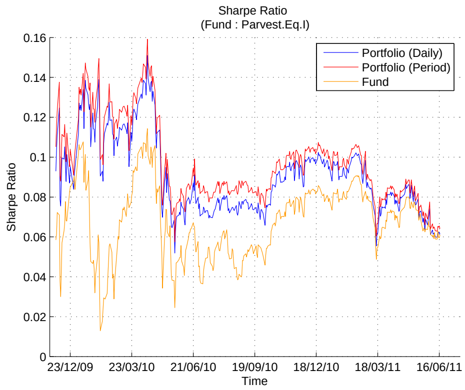
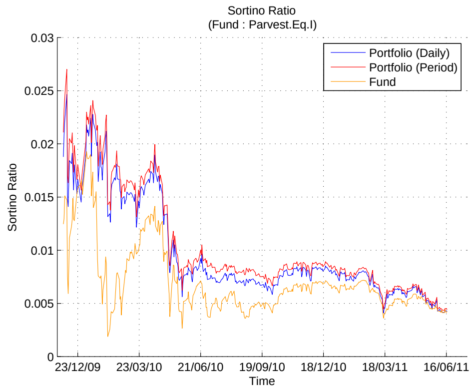


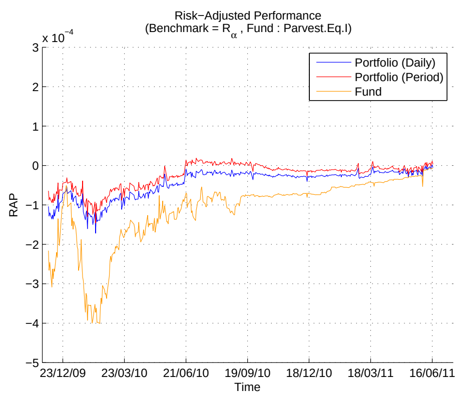
Figure 6.10: Growth comparison between the Futures portfolio with daily hedging and delivery date hedging for Sample Case 3.



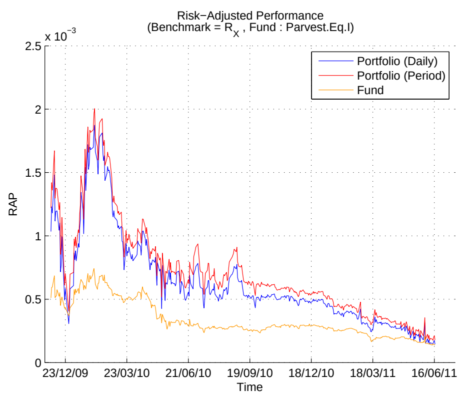
(a) *Sharpe ratio*



(b) *Sortino ratio*



(c) *RAP with benchmark index the alpha return*



(d) *RAP with benchmark index the index return*

Figure 6.11: A comparison of risk-return performance indexes between the Futures portfolio with daily hedging and delivery hedging for Sample Case 3.

Conclusions

We have introduced a defensive investment strategy that aims at the reduction of the market risk from a portfolio, and we have seen that such strategy may be implemented starting from an accurate selection of a core-asset and by appropriately hedging against the market risk by using either Short ETFs or Futures contracts.

The effectivity of the hedge depends on the accuracy of estimating the correlation between the core-asset and its benchmark index, both on a daily and on a non-daily basis. Once the hedge has been appropriately implemented, the *alpha dynamics* characteristics are transferred to the portfolio, which as a consequence benefits from a reduction in volatility and an increase in its risk-return profile if compared to an investment in the sole core-asset. Although the daily returns are of an order of magnitude inferior to their volatility, the portfolio will deliver with a certain level of confidence a positive return in the mean period if we allow the strategy to live for a sufficient amount of time, typically in the order of 10^2 days. As a consequence, the portfolio shows a stable and smooth growth path.

ETFs and Futures don't have the same efficiency and it results that the latter perform significantly better and are easier to adopt. Moreover, Futures may be traded on a longer time scale than the daily one, and still maintain a hedging effectivity, which from an operational level results in a major simplification and accessibility of the investment strategy.

For each of the three portfolios considered, we have presented analytical expressions of their expected and actual returns and also of the errors that arise when implementing the hedge together with the expected and actual cash evolution. Numerical simulations relevant to three different sample cases have also been presented.

There are a number of possible extensions of this work. In the first place, it could be of interest to investigate the origin of the alpha dynamics and the specific funds' characteristics. The strategy could also be considered as complementary to others focusing on market drift phases.

Appendix A

Formulae

Time frame

Time	Present		$t_i \quad i \in [0, N-1]$
	Past		$t_j \quad j \in [0, i-1]$
	Future		$t_h \quad h \in [i, N-1]$

Ex-post return process

Process	Past time		$t_j \quad j \in [0, i-1]$
	Value or price		$V(t_j)$
	Daily returns	1.1	$r_V(t_{j+1}) := \dfrac{V(t_{j+1}) - V(t_j)}{V(t_j)} = \dfrac{\Delta V(t_{j+1})}{V(t_j)}$
Sample daily return	Sample mean	1.2	$\mu_V(t_i) := \dfrac{1}{i}\sum_{j=0}^{i-1} r_V(t_{j+1})$
	Sample mean (memory)	1.3	$\mu_V(t_i) := \dfrac{1}{m}\sum_{j=i-m}^{i-1} r_V(t_{j+1})$
	Sample variance	1.4	$\sigma_V^2(t_i) := \dfrac{1}{i-1}\sum_{j=0}^{i-1}\Big(r_V(t_{j+1}) - \mu_V(t_i)\Big)^2$
	Sample variance (memory)	1.5	$\sigma_V^2(t_i) := \dfrac{1}{m-1}\sum_{j=i-m}^{i-1}\Big(r_V(t_{j+1}) - \mu_V(t_i)\Big)^2$
	Sample covariance	1.6	$\sigma_{VU}^2(t_i) := \dfrac{1}{i-1}\sum_{j=0}^{i-1}\Big[(r_V(t_{j+1}) - \mu_V(t_i))(r_U(t_{j+1}) - \mu_U(t_i))\Big]$
	Sample covariance (memory)	1.7	$\sigma_{VU}^2(t_i) := \dfrac{1}{m-1}\sum_{j=i-m}^{i-1}\Big[(r_V(t_{j+1}) - \mu_V(t_i))(r_U(t_{j+1}) - \mu_U(t_i)\Big]$
	Return process	1.8	$r_V(t_{j+1}) := \dfrac{\Delta V(t_{j+1})}{V(t_j)} = \mu_V(t_i) + \sigma_V(t_i)\Psi_V(t_{j+1})$
Sample compound return	Growth	1.11	$G_V(t_0, t_i) := \dfrac{V(t_i)}{V(t_0)} = \prod_{j=0}^{i-1}\Big[1 + r_V(t_{j+1})\Big]$
	Return	1.12	$R_V(t_0, t_i) := \dfrac{V(t_i)}{V(t_0)} - 1 = \prod_{j=0}^{i-1}\Big[1 + r_V(t_{j+1})\Big] - 1$
	Annual return	1.13	$R_V^{\text{year}}(t_0, t_i) := \Big(1 + R_V(t_0, t_i)\Big)^{\text{days}/365} - 1$

Ex-ante return process

Process	Future time		$t_h \quad h \in [i, N-1]$
	Evaluation period		$n := N - i$
	Value or price		$V(t_h)$
	Daily returns	1.14	$r_V(t_{h+1}) = E\big[r_V(t_{h+1})\vert\mathcal{F}_{t_i}\big] + \sqrt{E\Big[\Big(r_V(t_{h+1}) - E\big[r_V(t_{h+1})\vert\mathcal{F}_{t_i}\big]\Big)^2\Big\vert\mathcal{F}_{t_i}\Big]}\,\Psi_V(t_{h+1})$
Conditional daily return	Drift	1.15	$E\big[r_V(t_{h+1})\vert\mathcal{F}_{t_i}\big] = \mu_V(t_i)$
	Variance	1.16	$E\big[\big(r_V(t_{h+1}) - E\big[r_V(t_{h+1})\vert\mathcal{F}_{t_i}\big]\big)^2\vert\mathcal{F}_{t_i}\big] = \sigma_V^2(t_i)$
	Covariance	1.17	$E\Big[\big(r_V(t_{h+1}) - \mu_V(t_i)\big)\big(r_U(t_{h+1}) - \mu_U(t_i)\big)\vert\mathcal{F}_{t_i}\Big] = \sigma_{VU}^2(t_i)$
	Return process	1.18	$r_V(t_{h+1}) = \mu_V(t_i) + \sigma_V(t_i)\Psi_V(t_{h+1})$
Expected compound return	Growth		$E\big[G_V(t_i, t_N)\vert\mathcal{F}_{t_i}\big] = E\Big[\prod_{h=i}^{N-1}\Big[1 + r_V(t_{h+1})\Big]\Big\vert\mathcal{F}_{t_i}\Big]$
	Return		$E\big[R_V(t_i, t_N)\vert\mathcal{F}_{t_i}\big] = E\Big[\prod_{h=i}^{N-1}\Big[1 + r_V(t_{h+1})\Big]\Big\vert\mathcal{F}_{t_i}\Big] - 1$

Correlation between core-asset and benchmark index

Correlation	Sample covariance		$\sigma^2_{SX}(t_i) := \frac{1}{i-1}\sum_{j=0}^{i-1}\left[(r_S(t_{j+1}) - \mu_S(t_i))(r_X(t_{j+1}) - \mu_X(t_i)\right]$
	Correlation coefficient		$\rho(t_i) := \frac{\sigma^2_{SX}(t_i)}{\sigma_S(t_i)\sigma_X(t_i)}$
Linear regression	Asset return	2.10	$r_S(t_{j+1}) = \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1}) + \beta(t_i) r_X(t_{j+1})$
	Alpha	2.8	$\alpha(t_i) = \mu_S(t_i) - \beta(t_i)\mu_X(t_i)$
	Beta	2.9	$\beta(t_i) = \frac{\sigma^2_{SX}(t_i)}{\sigma^2_X(t_i)} = \rho(t_i)\frac{\sigma_S(t_i)}{\sigma_X(t_i)}$

Core-asset return decomposition

Components	Index return	4.1	$r_X(t_{j+1}) := \mu_X(t_i) + \sigma_X(t_i)\Psi_X(t_{j+1})$
	Alpha return	2.11	$r_\alpha(t_{j+1}) := \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1})$
Decomposition	Core-asset return	2.13	$r_S(t_{j+1}) = r_\alpha(t_{j+1}) + \beta(t_i) r_X(t_{j+1})$ $r_S(t_{j+1}) = \Big[\alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1})\Big] + \beta(t_i)\Big[\mu_X(t_i) + \sigma_X(t_i)\Psi_X(t_{j+1})\Big]$ $r_S(t_{j+1}) = \Big[\alpha(t_i) + \beta(t_i)\mu_X(t_i)\Big] + \Big[\sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1}) + \sigma_X(t_i)\Psi_X(t_{j+1})\Big]$
Sample daily return	Sample mean		$\mu_S(t_i) = \alpha(t_i) + \beta(t_i)\mu_X(t_i)$
	Sample variance		$\sigma^2_S(t_i) = \sigma^2_\alpha + \beta^2\sigma^2_X$

Daily return errors

Error terms	Overall error	2.20	$\zeta_S(t_{i+1}) = \zeta_{\text{hedge}}(t_{i+1}) + \zeta_X(t_{i+1}) + \zeta_\alpha(t_{i+1})$
	Hedgable error	2.21	$\zeta_{\text{hedge}}(t_{i+1}) := \Big[r_X(t_{i+1}) - \mu_X(t_i)\Big]\beta(t_i)$
	Estimation error	2.22	$\zeta_x(t_{i+1}) := \alpha(t_{i+1}) - \alpha(t_i) + \Big[\beta(t_{i+1}) - \beta(t_i)\Big]r_X(t_{i+1})$
	Alpha error	2.23	$\zeta_\alpha(t_{i+1}) := \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})$
	Etf error	5.10	$\zeta_h(t_{i+1}) := -\dfrac{\delta k(t_{i+1})}{k} r_X(t_{i+1})$
	Futures error	5.34	$\zeta_f(t_{i+1}) := \dfrac{dF(t_{i+1}) - dF(t_i)}{X(t_i)\big[1 + r(t_{i+1})\big]^{n-1}}$
	Approximation error	5.47	$\zeta_r(t_{i+1}) := 1 - \dfrac{\big[1 + r(t_i)\big]^n}{\big[1 + r(t_{i+1})\big]^{n-1}} + r(t_i)$

Compound return errors

Error terms	Overall error	3.41	$\zeta_S(t_i, t_N) = \zeta_{\text{hedge}}(t_i, t_N) + \zeta_X(t_i, t_N) + \zeta_A(t_i, t_N)$
	Hedgable error		$\zeta_{\text{hedge}}(t_i, t_N) := \Big[R_X - E[R_X(t_i, t_N)\vert\mathcal{F}_{t_i}]\Big]E[B(t_N)\vert\mathcal{F}_{t_i}]$
	Estimation error		$\zeta_X(t_i, t_N) := A(t_N) - E[A(t_N)\vert\mathcal{F}_{t_i}] + \Big(B(t_N) - E[B(t_N)\vert\mathcal{F}_{t_i})\Big)R_X(t_i, t_N)$
	Alpha error		$\zeta_A(t_i, t_N) := \sigma_A(t_N)\Delta W_A(t_i, t_N)$
	Futures error	6.13	$\zeta_F(t_i, t_N) := \dfrac{q(t_i)}{X(t_i)}\big[dF(t_n) - dF(t_i)\big]$
	Approximation error	6.12	$\zeta_R(t_i, t_N) := \dfrac{1}{X_i}\sum_{h=i+1}^{N-2} F_h r_h G(t_{h+1}, t_N) + F_{N-1}r_{N-1}$

Ex-post alpha return process

Sample daily return	Past time		$t_j \qquad j \in [0, i-1]$
	Daily returns	2.11	$r_\alpha(t_{j+1}) := \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{j+1})$
	Sample mean	2.8	$\alpha(t_i) = \mu_S(t_i) - \beta(t_i)\mu_X(t_i)$
	Sample variance	2.16	$\sigma_\alpha(t_i) = \sigma_S(t_i)\sqrt{1-\rho^2(t_i)}$
Sample compound return	Growth		$G_\alpha(t_0, t_i) := \prod_{j=0}^{i-1}\Big[1 + r_\alpha(t_{j+1})\Big]$
	Return		$R_\alpha(t_0, t_i) := \prod_{j=0}^{i-1}\Big[1 + r_\alpha(t_{j+1})\Big] - 1 = G_\alpha(t_0, t_i) - 1$
	Annual return		$R_\alpha^{\text{year}}(t_0, t_i) := \Big(1 + R_\alpha(t_0, t_i)\Big)^{\text{days}/365} - 1$

Ex-ante alpha return process

Expected daily return	Future time		$t_h \qquad h \in [i, N-1]$
	Evaluation period		$n := N - i$
	Daily returns	2.17	$r_\alpha(t_{h+1}) = \alpha(t_i) + \sigma_\alpha(t_i)\Delta W_\alpha(t_{h+1})$
	Conditional drift		$E\big[r_\alpha(t_{h+1})\vert\mathcal{F}_{t_i}\big] = \alpha(t_i)$
	Conditional variance		$\mathrm{Var}\big[r_\alpha(t_{h+1})\vert\mathcal{F}_{t_i}\big] = \sigma_\alpha^2(t_i)$
Expected growth	Exact	3.7	$E\big[G_\alpha(t_i, t_N)\vert\mathcal{F}_{t_i}\big] = \big(1 + \alpha(t_i)\big)^n$
	Approximated	3.10	$E\big[G_\alpha(t_i, t_N)\vert\mathcal{F}_{t_i}\big] \approx 1 + n\alpha(t_i)$
Expected compound return	Exact	3.8	$E\big[R_\alpha(t_i, t_N)\vert\mathcal{F}_{t_i}\big] = \big(1 + \alpha(t_i)\big)^n - 1$
	Approximated	3.11	$E\big[R_\alpha(t_i, t_N)\vert\mathcal{F}_{t_i}\big] \approx n\alpha(t_i)$
- begin simplified notation -			
Growth variance	Exact	3.13	$\mathrm{Var}\big[G_N\vert\mathcal{F}_i\big] = \big[(1+\alpha_i)^2 + \sigma_i^2\big]^n - (1+\alpha_i)^{2n}$
	Approx. 1	3.16	$\mathrm{Var}\big[G_N\vert\mathcal{F}_i\big] \approx n\sigma_i^2(1+\alpha_i)^{2(n-1)}$
	Approx. 2	3.18	$\mathrm{Var}\big[G_N\vert\mathcal{F}_i\big] \approx n\sigma_i^2\big(1 + 2n\alpha_i - 2\alpha_i\big)$
	Approx. 3	3.19	$\mathrm{Var}\big[G_N\vert\mathcal{F}_i\big] \approx n\sigma_i^2\big(1 + 2n\alpha_i\big)$
	Approx. 4	3.20	$\mathrm{Var}\big[G_N\vert\mathcal{F}_i\big] \approx n\sigma_i^2$
Growth volataility	Exact		$\mathrm{Vol}\big[G_N\vert\mathcal{F}_i\big] = \sqrt{\big[(1+\alpha_i)^2 + \sigma_i^2\big]^n - (1+\alpha_i)^{2n}}$
	Approx. 1		$\mathrm{Vol}\big[G_N\vert\mathcal{F}_i\big] \approx \sqrt{n}\sigma_i(1+\alpha_i)^{(n-1)}$
	Approx. 2		$\mathrm{Vol}\big[G_N\vert\mathcal{F}_i\big] \approx \sqrt{n}\sigma_i\sqrt{1 + 2n\alpha_i - 2\alpha_i}$
	Approx. 3		$\mathrm{Vol}\big[G_N\vert\mathcal{F}_i\big] \approx \sqrt{n}\sigma_i\sqrt{1 + 2n\alpha_i}$
	Approx. 4		$\mathrm{Vol}\big[G_N\vert\mathcal{F}_i\big] \approx \sqrt{n}\sigma_i$
Waiting time	Definition	3.21	$Z(\mathcal{F}_i, n, z) := E\big[G_\alpha(t_i, t_N)\vert\mathcal{F}_i\big] + z\sqrt{\mathrm{Var}\big[G_\alpha(t_i, t_N)\vert\mathcal{F}_i\big]}$
	Target	3.22	$Z(\mathcal{F}_i, n, z) = \big(1 + \mu^*\big)^n$
	Case		$z = -1 \qquad \mu^* = 0$
	Equation		$(1+\alpha_i)^n - \sqrt{\big[(1+\alpha_i)^2 + \sigma_i^2\big]^n - (1+\alpha_i)^{2n}} = 1$
	Approx. solution 3	3.23	$n \approx \dfrac{\sigma_i^2}{\alpha_i^2 - 2\alpha_i\sigma_i^2}$
	Approx. solution 4	3.24	$n \approx \dfrac{\sigma_i^2}{\alpha_i^2}$
- end simplified notation -			

Correlation on a non-daily period between core-asset and benchmark index

Case 1. Sufficient available data to perform linear regression

Ex-post linear regression

Perspective	Present time		t_N
	Perspective		ex-post
	Over period		$[t_i, t_N]$
Definition	Compound return	3.29	$R_S(t_i,t_N) = A(t_N) + B(t_N)R_X(t_i,t_N) + \sigma_A(t_N)\Delta W_A(t_i,t_N)$
	Parameter A	3.30	$A(t_N) = M_S(t_N) - B(t_N)M_X(t_N)$
	Parameter B	3.31	$B(t_N) = \dfrac{\mathrm{Cov}_{\mathrm{sample}}[R_S, R_X]}{\mathrm{Var}_{\mathrm{sample}}[R_X]}$

Ex-ante estimated linear regression

Perspective	Present time		t_i
	Perspective		ex-ante
	Over period		$[t_i, t_N]$
Definition	Compound return		$R_S(t_i,t_N) = A(t_N) + B(t_N)R_X(t_i,t_N) + \sigma_A(t_N)\Delta W_A(t_i,t_N)$
	Parameter A		$A(t_N) = E\big[R_S(t_N)\vert\mathcal{F}_{t_i}\big] - B(t_N)E\big[R_X(t_N)\vert\mathcal{F}_{t_i}\big]$
	Parameter B		$B(t_N) = \dfrac{\mathrm{Cov}\big[R_S, R_X\vert\mathcal{F}_{t_i}\big]}{\mathrm{Var}\big[R_X\vert\mathcal{F}_{t_i}\big]}$
Estimation	Estimated return		$R_S(t_i,t_N) = A(t_i) + B(t_i)R_X(t_i,t_N) + \sigma_A(t_i)\Delta W_A(t_i,t_N)$
	Estimated A		$A(t_N) = A(t_i) = M_S(t_i) - B(t_i)M_X(t_i)$
	Estimated B		$B(t_N) = B(t_i) = \dfrac{\mathrm{Cov}_{\mathrm{sample}}[R_S, R_X]}{\mathrm{Var}_{\mathrm{sample}}[R_X]}$

Case 2. Regression determined from daily returns. Non-sufficient sample data

Ex-post linear regression

Perspective	Present time		t_N
	Perspective		ex-post
	Over period		$[t_i, t_N]$
Definition	Compound return		$R_S(t_i,t_N) = A(t_N) + B(t_N)R_X(t_i,t_N) + \sigma_A(t_N)\Delta W_A(t_i,t_N)$

Ex-ante estimated linear regression

Perspective	Present time		t_i
	Perspective		ex-ante
	Over period		$[t_i, t_N]$
Definition	Estimated return		$R_S(t_i,t_N) = E\big[A(t_N)\vert\mathcal{F}_{t_i}\big] + E\big[B(t_N)\vert\mathcal{F}_{t_i}\big]R_X(t_i,t_N) +$
		3.36	$E\big[\sigma_A(t_N)\vert\mathcal{F}_{t_i}\big]\Delta W_A(t_i,t_N)$
	Estimated A	3.33	$E\big[A(t_N)\vert\mathcal{F}_{t_i}\big] := E\big[R_S(t_i,t_N)\vert\mathcal{F}_{t_i}\big] -$
			$E\big[B(t_N)\vert\mathcal{F}_{t_i}\big]E\big[R_X(t_i,t_N)\vert\mathcal{F}_{t_i}\big]$
	Estimated B	3.34	$E\big[B(t_N)\vert\mathcal{F}_{t_i}\big] := \dfrac{\mathrm{Cov}\big[R_S, R_X\vert\mathcal{F}_{t_i}\big]}{\mathrm{Var}\big[R_X\vert\mathcal{F}_{t_i}\big]}$
Estimation	Estimated A		$E\big[A(t_N)\vert\mathcal{F}_{t_i}\big] \approx \big[1+\alpha(t_i)\big]^n - 1$
		3.38	$E\big[A(t_N)\vert\mathcal{F}_{t_i}\big] \approx E\big[R_\alpha(t_i,t_N)\vert\mathcal{F}_i\big]$
	Estimated B	3.37	$E\big[B(t_N)\vert\mathcal{F}_{t_i}\big] \approx \beta(t_i)\big[1+\alpha(t_i)\big]^{n-1}$
		3.39	$E\big[B(t_N)\vert\mathcal{F}_{t_i}\big] \approx \dfrac{E\big[R_S(t_i,t_N)\vert\mathcal{F}_{t_i}\big] - E\big[R_\alpha(t_i,t_N)\vert\mathcal{F}_{t_i}\big]}{E\big[R_X(t_i,t_N)\vert\mathcal{F}_{t_i}\big]}$

Short ETFs

Daily return	Definition	4.2	$r_H(t_{i+1}) := \dfrac{H(t_{i+1}) - H(t_i)}{H(t_i)} = \dfrac{\Delta H(t_{i+1})}{H(t_i)}$
	Replication	4.3	$r_H(t_{i+1}) = kr_X(t_{i+1})$
Compound return	Definition		$R_H(t_N) := \prod_{i=0}^{N-1} \left[1 + kr_X(t_{i+1})\right] - 1 = G_H(t_N) - 1$
	Replication	4.4	$R_H(t_N) \neq kR_X(t_N)$
Tracking errors	Daily return	4.9	$r_H(t_{i+1}) = \left[k + \delta k(t_{i+1})\right] r_X(t_{i+1})$
	Replication	4.10	$\tilde{k}(t_{i+1}) := k + \delta k(t_{i+1}) = \dfrac{r_H(t_{i+1})}{r_X(t_{i+1})}$

Forward contract on a security

Variables	Forward contract		$F(t_i)$
	Security		$V(t_i)$
Interest rates	Compound interests		$I(t_N) := \prod_{h=i}^{N-1} [1 + r(t_h)]$
	Assumption	4.12	$E\left[I(t_n) \middle| \mathcal{F}_{t_i}\right] = [1 + r(t_i)]^n$
Pricing	No-arbitrage price	4.11	$F(t_i) = V(t_i) E\left[I(t_n) \middle| \mathcal{F}_{t_i}\right]$
	Ideal price	4.13	$F(t_i) = V(t_i)[1 + r(t_i)]^n$
Dividends	Security		$V(t_d^-) = V(t_d^+) + d(t_d)$
	Forward contract		$F(t_d^-) = F(t_d^+) + d(t_d)[1 + r(t_i)]^n$
	Dividends		$D(t_i) := E\Big[d(t_d)[1 + r(t_i)]^{-(d-i)} \vert \mathcal{F}_{t_i}\Big]$
Pricing	No-arbitrage price		$F(t_i) = \left[V(t_i) - D(t_i)\right] E\left[I(t_n) \middle| \mathcal{F}_{t_i}\right]$
	Ideal price		$F(t_i) = \left[V(t_i) - D(t_i)\right] [1 + r(t_i)]^n$

Futures contract on a benchmark index

Variables	Futures contract		$F(t_i)$
	Benchmark index		$X(t_i)$
Interest rates	Compound interests		$I(t_N) := \prod_{h=i}^{N-1} [1 + r(t_h)]$
	Assumption	4.12	$E\left[I(t_n) \middle| \mathcal{F}_{t_i}\right] = [1 + r(t_i)]^n$
Pricing	No-arbitrage price	4.14	$F(t_i) = X(t_i) E\left[I(t_n) \middle| \mathcal{F}_{t_i}\right]$
	Ideal price	4.15	$F(t_i) = X(t_i)[1 + r(t_i)]^n$
	Real price	4.17	$F(t_i) = X(t_i)\Big[[1 + r(t_i)]^n + \sigma_F(t_i)\Psi_F(t_i)\Big]$
Dividends	Index dividends		$D(t_i) := E\Big[\sum_{z=1}^{Z} d_z(t_{d_z})\left[1 + r(t_i)\right]^{-(d_z - i)} \vert \mathcal{F}_{t_i}\Big]$
Pricing	No-arbitrage price	4.18	$F(t_i) = \left[X(t_i) - D(t_i)\right]\Big[E\left[I(t_n) \middle| \mathcal{F}_{t_i}\right] + \sigma_F(t_i)\Psi_F(t_i)\Big]$
	Real price	4.19	$F(t_i) = \left[X(t_i) - D(t_i)\right]\Big[[1 + r(t_i)]^n + \sigma_F(t_i)\Psi_F(t_i)\Big]$
Pricing	Error	4.20	$dF(t_i) := \left[X(t_i) - D(t_i)\right]\sigma_F(t_i)\Psi_F(t_i) - D(t_i)[1 + r(t_i)]^n$
	Real price	4.21	$F(t_i) = X(t_i)[1 + r(t_i)]^n + dF(t_i)$

Portfolio hedged daily with ETFs

Portfolio	Value	5.1	$\Pi_H(t_i) = \lambda S(t_i) + q_H(t_i)H(t_i) + C_H(t_i)$
Hedging	ETF quantity	5.2	$q(t_i) = -\beta(t_i)\dfrac{\lambda S(t_i)}{kH(t_i)}$
Portfolio variation	Expected variation	5.3	$E[\Delta\Pi_H(t_{i+1})\vert\mathcal{F}_{t_i}] = \Big[\alpha(t_i)\Big]\lambda S(t_i) + \Big[r(t_i)\Big]C_H(t_i)$
	Actual variation	5.12	$\Delta\Pi_H(t_{i+1}) = \Big[r(t_i)\Big]C_H(t_i) + \Big[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big]\lambda S(t_i)$
Errors	Alpha error	2.23	$\zeta_\alpha(t_{i+1}) := \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})$
	Estimation error	2.22	$\zeta_x(t_{i+1}) := \alpha(t_{i+1}) - \alpha(t_i) + \Big[\beta(t_{i+1}) - \beta(t_i)\Big]r_X(t_{i+1})$
	Etf error	5.10	$\zeta_h(t_{i+1}) := -\dfrac{\delta k(t_{i+1})}{k}r_X(t_{i+1})$
Alloc. "A"	ETF fraction	5.4	$\phi_H(t_i) := \dfrac{q(t_i)H(t_i)}{\Pi_H(t_i)}$
	Asset fraction		$\phi_S(t_i) := \dfrac{\lambda S(t_i)}{\Pi_H(t_i)}$
	Cash fraction		$\phi_C(t_i) := \dfrac{C_H(t_i)}{\Pi_H(t_i)}$
	Total		$\phi_S(t_i) + \phi_H(t_i) + \phi_C(t_i) = 1$
Alloc. "B"	Invested fraction	5.6	$\theta_H(t_i) := \phi_S(t_i) + \phi_H(t_i)$
	Gamma	5.7	$\gamma(t_i) := \dfrac{\phi_S(t_i)}{\theta_H(t_i)}$
Portfolio return "A"	Expected return	5.5	$E[r_{\Pi_H}(t_{i+1})\vert\mathcal{F}_{t_i}] = \Big[\alpha(t_i)\Big]\phi_S(t_i) + \Big[r(t_i)\Big]\phi_C(t_i)$
	Actual return	5.13	$r_{\Pi_H}(t_{i+1}) = \Big[r(t_i)\Big]\phi_C(t_i) + \Big[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big]\phi_S(t_i)$
	Error	5.17	$r_{\Pi_H}(t_{i+1}) - E[r_{\Pi_H}(t_{i+1})\vert\mathcal{F}_{t_i}] = \Big[\zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big]\phi_S(t_i)$
Portfolio return "B"	Expected return	5.9	$E[r_{\Pi_H}(t_{i+1})\vert\mathcal{F}_{t_i}] = \Big[\gamma(t_i)\alpha(t_i)\Big]\theta_H(t_i) + \Big[r(t_i)\Big]\big(1 - \theta_H(t_i)\big)$
	Actual return	5.15	$r_{\Pi_H}(t_{i+1}) = \Big[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big]\gamma(t_i)\theta_H(t_i) + \Big[r(t_i)\Big]\big(1 - \theta_H(t_i)\big)$
	Error	5.18	$r_{\Pi_H}(t_{i+1}) - E[r_{\Pi_H}(t_{i+1})\vert\mathcal{F}_{t_i}] = \Big[\zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\zeta_h(t_{i+1})\Big]\gamma(t_i)\theta_H(t_i)$
Cash	Recursive expression	5.21	$C_H(t_{j+1}) = C_H(t_j)(1 + r(t_j)) - \Delta q_H(t_{j+1})H(t_{j+1})$
	Ex-post evolution	5.23	$C_H(t_i) = C_H(t_0)G(t_0,t_i) - \sum_{j=0}^{i-2}\Big[\Delta q_H(t_{j+1})H(t_{j+1})G(t_{j+1},t_i)\Big] - \Delta q_H(t_i)H(t_i)$
	Ex-ante evolution	5.30	$E\big[C_H(t_N)\vert\mathcal{F}_{t_i}\big] = C_H(t_i)(1+r)^n + -q_H(t_i)H(t_i)\big[\alpha(t_i) + (\beta(t_i) - k)\mu_X(t_i)\big]\sum_{h=i}^{N-1}\big[1 + \alpha(t_i) + \beta(t_i)\mu_X(t_i)\big]^{h-i}(1+r)^{N-h-1}$
Process	Return	5.20	$r_{\Pi_H}(t_{i+1}) = \mu_{\Pi_H}(t_{i+1}) + \sigma_{\Pi_H}(t_{i+1})\Psi_{\Pi_H}(t_{i+1})$
	Drift		$\mu_{\Pi_H}(t_{i+1}) \approx \alpha(t_i)\gamma(t_i)\theta_H(t_i) + r(t_i)(1 - \theta_H(t_i))$
	Volatility		$\sigma_{\Pi_H}(t_{i+1}) \approx \gamma(t_i)\theta_H(t_i)\sigma_\alpha(t_i)$
	Sharpe ratio		$\text{Sharpe}_H(t_{i+1}) \approx \dfrac{\alpha(t_i)\gamma(t_i) - r(t_i)}{\gamma(t_i)\sigma_\alpha(t_i)}$

Portfolio hedged daily with Futures

Portfolio	Value	5.32	$\Pi_F(t_i) = \lambda S(t_i) + C_F(t_i)$
Hedging	Futures quantity	5.39	$q_F(t_i) = -\beta(t_i)\dfrac{\lambda S(t_i)}{F(t_i)}\big[1 + r(t_i)\big]$
		5.40	$q_F(t_i) = -\beta(t_i)\dfrac{\lambda S(t_i)}{X(t_i)}\big[1 + r(t_i)\big]^{-(n-1)}$
Allocation	Invested fraction	5.43	$\theta_F(t_i) := \dfrac{\lambda S(t_i)}{\Pi_F(t_i)} := \phi_S(t_i)$
	Cash fraction		$1 - \theta_F(t_i) = \dfrac{C_F(t_i)}{\Pi_F(t_i)} := \phi_C(t_i)$
Portfolio variation	Expected variation	5.41	$E[\Delta\Pi_F(t_{i+1})\vert\mathcal{F}_{t_i}] = \Big[\alpha(t_i) + \beta(t_i)r(t_i)\Big]\lambda S(t_i) + \Big[r(t_i)\Big]C_F(t_i)$
	Actual variation	5.48	$\Delta\Pi_F(t_{i+1}) = \Big[r(t_i)\Big]C_F(t_i) + \lambda S(t_i)\Big[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\Big(r(t_i) - \zeta_r(t_{i+1}) - \zeta_f(t_{i+1})\Big)\Big]$
Errors	Alpha error	2.23	$\zeta_\alpha(t_{i+1}) := \sigma_\alpha(t_{i+1})\Delta W_\alpha(t_{i+1})$
	Estimation error	2.22	$\zeta_x(t_{i+1}) := \alpha(t_{i+1}) - \alpha(t_i) + \Big[\beta(t_{i+1}) - \beta(t_i)\Big]r_X(t_{i+1})$
	Futures error	5.34	$\zeta_f(t_{i+1}) := \dfrac{dF(t_{i+1}) - dF(t_i)}{X(t_i)\big[1 + r(t_{i+1})\big]^{n-1}}$
	Approximation error	5.47	$\zeta_r(t_{i+1}) := 1 - \dfrac{\big[1 + r(t_i)\big]^n}{\big[1 + r(t_{i+1})\big]^{n-1}} + r(t_i)$
Portfolio return	Expected return	5.44	$E[r_{\Pi_F}(t_{i+1})\vert\mathcal{F}_{t_i}] = \Big[\alpha(t_i) + \beta(t_i)r(t_i)\Big]\theta_F(t_i) + \Big[r(t_i)\Big](1 - \theta_F(t_i))$
	Actual return	5.49	$r_{\Pi_F}(t_{i+1}) = \Big[r(t_i)\Big](1 - \theta_F(t_i)) + \Big[\alpha(t_i) + \zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) + \beta(t_i)\Big(r(t_i) - \zeta_r(t_{i+1}) - \zeta_f(t_{i+1})\Big)\Big]\theta(t_i)$
	Error	5.50	$r_{\Pi_F}(t_{i+1}) - E[r_{\Pi_F}(t_{i+1})\vert\mathcal{F}_{t_i}] = \Big[\zeta_\alpha(t_{i+1}) + \zeta_x(t_{i+1}) - \beta(t_i)\Big(\zeta_r(t_{i+1}) + \zeta_f(t_{i+1})\Big)\Big]\theta(t_i)$
Alternative expressions	Expected variation	5.42	$E[\Delta\Pi_F(t_{i+1})\vert\mathcal{F}_{t_i}] = \Big[\lambda S(t_i)\Big]\alpha(t_i) + \Big[\lambda S(t_i)\beta(t_i) + C_F(t_i)\Big]r(t_i)$
	Expected return	5.45	$E[r_{\Pi_F}(t_{i+1})\vert\mathcal{F}_{t_i}] = \Big[\theta_F(t_i)\Big]\alpha(t_i) + \Big[1 - \big(1 - \beta(t_i)\big)\theta_F(t_i)\Big]r(t_i)$
Cash	Recursive expression		$C_F(t_{i+1}) = C_F(t_i)\big[1 + r(t_i)\big] + q_F(t_i)\big[F(t_{i+1}) - F(t_i)\big]$
	Ex-post evolution	5.54	$C_i = C_0G(t_0, t_i) + q_0\Big[F_i - F_0G(t_1, t_i)\Big] + q_0\Big[\sum_{j=1}^{i-2} F_j r_j G(t_{j+1}, t_i) + F_{i-1}r_{i-1}\Big]$
		5.55	$C_i \approx C_0G(t_0, t_i) + q_0\Big[F_i - F_0G(t_1, t_i)\Big]$
	Ex-ante evolution	5.57	$E[C_N\vert\mathcal{F}_i] \approx C_i\Big[E[R\vert\mathcal{F}_i] + 1\Big] + q_iX_i\Big[1 + E[R_X\vert\mathcal{F}_i] - E[\tilde{G}\vert\mathcal{F}_i]\Big]$
Process	Return	5.52	$r_{\Pi_F}(t_{i+1}) = \mu_{\Pi_F}(t_{i+1}) + \sigma_{\Pi_F}(t_{i+1})\Psi_{\Pi_F}(t_{i+1})$
	Drift		$\mu_{\Pi_F}(t_{i+1}) \approx \Big[\alpha(t_i) + \big(\beta(t_i) - 1\big)r(t_i)\Big]\theta_F(t_i) + r(t_i)$
	Volatility		$\sigma_{\Pi_F}(t_{i+1}) \approx \sigma_\alpha(t_i)\theta_F(t_i)$
	Sharpe ratio		$\text{Sharpe}_F(t_{i+1}) \approx \dfrac{\alpha(t_i) + \big(\beta(t_i) - 1\big)r(t_i)}{\sigma_\alpha(t_i)}$

Portfolio hedged with Futures on non-daily periods

Portfolio	Value	5.32	$\Pi_F(t_i) = \lambda S(t_i) + C_F(t_i)$
Hedging	Futures quantity		$q_F(t_i) = -E[B(t_N)\|\mathcal{F}_{t_i}]\dfrac{\lambda S(t_i)}{X(t_i)}$
		6.6	$q_F(t_i) = -\beta(t_i)\dfrac{\lambda S(t_i)}{X(t_i)}\big[1+\alpha(t_i)\big]^{n-1}$
Allocation	Invested fraction	5.43	$\theta_F(t_i) := \dfrac{\lambda S(t_i)}{\Pi_F(t_i)} := \phi_S(t_i)$
	Cash fraction		$1-\theta_F(t_i) = \dfrac{C_F(t_i)}{\Pi_F(t_i)} := \phi_C(t_i)$
- begin simplified notation -			
Portfolio variation	Expected variation	6.8	$E[\Pi_{F_N} - \Pi_{F_i}\|\mathcal{F}_i] \approx \Big[E[R_\alpha\|\mathcal{F}_i] + E[B\|\mathcal{F}_i]E[\tilde{R}\|\mathcal{F}_i]\Big]\lambda S_i + \Big[E[R\|\mathcal{F}_i]\Big]C_i$
	Actual variation	6.14	$\Pi_{F_N} - \Pi_{F_i} = \Big[E[R_\alpha\|\mathcal{F}_i] + \zeta_A + \zeta_X + E[B\|\mathcal{F}_i]\big(\tilde{R} - \zeta_R - \zeta_F\big)\Big]\lambda S_i + RC_i$
Compound errors	Alpha error		$\zeta_A(t_i,t_N) := \sigma_A(t_N)\Delta W_A(t_i,t_N)$
	Estimation error		$\zeta_X(t_i,t_N) := A(t_N) - E[A(t_N)\|\mathcal{F}_{t_i}] + \Big(B(t_N) - E[B(t_N)\|\mathcal{F}_{t_i})\Big)R_X(t_i,t_N)$
	Futures error	6.13	$\zeta_F(t_i,t_N) := \dfrac{q(t_i)}{X(t_i)}\big[dF(t_n) - dF(t_i)\big]$
	Approximation error	6.12	$\zeta_R(t_i,t_N) := \dfrac{1}{X_i}\sum_{h=i+1}^{N-2} F_h r_h G(t_{h+1},t_N) + F_{N-1}r_{N-1}$
Portfolio return	Expected return	6.10	$E[R_{\Pi_F}(t_i,t_N)\|\mathcal{F}_i] \approx \Big[E[R_\alpha\|\mathcal{F}_i] + E[B\|\mathcal{F}_i]E[\tilde{R}\|\mathcal{F}_i]\Big]\theta_i + \Big[E[R\|\mathcal{F}_i]\Big](1-\theta_i)$
	Actual return	6.15	$R_{\Pi_F}(t_i,t_N) = \Big[E[R_\alpha \mathcal{F}_i] + \zeta_A + \zeta_X + E[B\|\mathcal{F}_i]\big(\tilde{R} - \zeta_R - \zeta_F\big)\Big]\theta_i + \Big[R\Big](1-\theta_i)$
	Error	6.17	$\lambda S_i\Big[\zeta_A + \zeta_X - E[B\|\mathcal{F}_i]\big(\zeta_R + \zeta_F\big) + E[B\|\mathcal{F}_i]\Big(\tilde{R} - E[\tilde{R}\|\mathcal{F}_i]\Big)\Big] + \Big[R - E[R\|\mathcal{F}_i]\Big]C_i$
Alternative expressions	Expected variation	6.9	$E[\Pi_{F_N} - \Pi_{F_i}\|\mathcal{F}_i] \approx \Big[\lambda S_i\Big]E[R_\alpha\|\mathcal{F}_i] + \Big[\lambda S_i E[B\|\mathcal{F}_i]\Big]E[\tilde{R}\|\mathcal{F}_i] + \Big[C_i\Big]E[R\|\mathcal{F}_i]$
	Expected return	6.11	$E[R_{\Pi_F}(t_i,t_N)\|\mathcal{F}_i] \approx \Big[\theta_i\Big]E[R_\alpha\|\mathcal{F}_i] + \Big[\theta_i E[B\|\mathcal{F}_i]\Big]E[\tilde{R}\|\mathcal{F}_i] + \Big[1-\theta_i\Big]E[R\|\mathcal{F}_i]$
	Actual return	6.16	$R_{\Pi_F}(t_i,t_N) = \Big[\theta_i\Big]E[R_\alpha \mathcal{F}_i] + \Big[\theta_i\Big]\Big(\zeta_A + \zeta_X - E[B\|\mathcal{F}_i]\big(\zeta_R + \zeta_F\big)\Big) + \Big[1-\theta_i\Big]R + \Big[E[B\|\mathcal{F}_i]\theta_i\Big]\tilde{R}$
Cash	Recursive expression		$C_F(t_{i+1}) = C_F(t_i)\big[1+r(t_i)\big] + q_F(t_i)\big[F(t_{i+1}) - F(t_i)\big]$
	Ex-post evolution	6.3	$C_N = C_i G(t_i,t_N) + q_i\Big[F_N - F_i G(t_{i+1},t_N)\Big] + q_i\Big[\sum_{h=i+1}^{N-2} F_h r_h G(t_{h+1},t_N) + F_{N-1}r_{N-1}\Big]$
		6.4	$C_N \approx C_i G(t_i,t_N) + q_i\Big[F_N - F_i G(t_{i+1},t_N)\Big]$
	Ex-ante evolution	6.5	$E[C_N\|\mathcal{F}_i] \approx C_i\Big[E[R\|\mathcal{F}_i] + 1\Big] + q_i X_i\Big[1 + E[R_X\|\mathcal{F}_i] - E[\tilde{G}\|\mathcal{F}_i]\Big]$
- end simplified notation -			

Appendix B

Implementation data

Table B.1: **Core-asset and benchmark index performance**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
Implementation period			
From date	22.Jul.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
Core-asset			
Fund	**8a+ Eiger**	**PIM America FCP IC USD**	**Parvest Eq. Germany I**
Isin code	IT0004255219	FR0010612770	LU0325630076
Management	8a Investimenti	PIM Gestion France	BNP Paribas
Fund type	Large Cap European Equity	Large Cap U.S. Equity	Large Cap German Equity
μ_S	1.486E-04	8.142E-04	6.061E-04
σ_S	1.422E-02	1.326E-02	9.767E-03
R_S	4.575E-02	5.158E-01	2.838E-01
Annualized R_S	2.014E-02	1.999E-01	1.552E-01
Sharpe ratio	8.956E-03	5.981E-02	6.015E-02
Sortino ratio	5.247E-04	3.551E-03	4.184E-03
RAP (benchmark $r\alpha$*)*	-1.503E-04	2.185E-05	-6.685E-06
RAP (benchmark r_X*)*	2.213E-04	1.217E-04	1.405E-04
Benchmark index			
Index	**Euro Stoxx 50**	**S&P 500**	**DAX**
μ_X	-6.192E-05	6.307E-04	5.765E-04
σ_X	1.543E-02	1.223E-02	1.161E-02
R_X	-7.860E-02	3.756E-01	2.564E-01
Annualized R_X	-3.582E-02	1.499E-01	1.409E-01
Risk-free rate			
Mean	2.222E-04	2.075E-04	2.108E-04
Alpha dynamics			
α	2.042E-04	2.257E-04	2.086E-04
σ_α	3.653E-03	3.785E-03	3.048E-03
Skewness	-2.874E-01	-3.426E-02	-2.874E-01
Kurtosis	9.869E+00	5.120E+00	6.270E+00
β	8.984E-01	9.331E-01	6.894E-01
ρ	8.627E-01	9.069E-01	8.580E-01
R_α	1.200E-01	1.348E-01	9.435E-02
Annualized R_α	5.182E-02	5.699E-02	5.345E-02
Estimation error on core-asset daily returns			
Mean	-1.205E-06	1.173E-08	-1.877E-07
Std. dev.	1.865E-05	1.557E-05	2.066E-05

Table B.2: **Portfolio performance**

Item	*Sample Case 1*	*Sample Case 2*	*Sample Case 3*
Implementation period			
From date	22.Jul.2009	08.Apr.2009	23.Sep.2009
To date	20.Oct.2011	20.Oct.2011	15.Jun.2011
Days	821	926	631
Short ETFs			
Short ETF	**Amundi ETF Short Dow Jones Euro Stoxx 50**	**ProShares Ultra-ProShort S&P 500**	**db x-trackers Short DAX**
Isin code	FR0010757781	US74347X8561	LU0292106241
Management	Amundi Investment	ProShares	DB Platinum Advisor
Portfolio with Short ETF daily hedge			
μ_{Π_H}	9.882E-06	6.439E-05	9.768E-05
σ_{Π_H}	1.978E-03	3.166E-03	2.485E-03
R_Π	6.672E-03	3.814E-02	4.420E-02
Annualized R_{Π_H}	2.968E-03	1.654E-02	2.533E-02
Sharpe ratio	-5.670E-03	1.368E-02	3.179E-02
Sortino ratio	-3.242E-04	7.695E-04	2.117E-03
RAP (benchmark $r\alpha$*)*	-1.830E-04	-1.646E-04	-1.043E-04
RAP (benchmark r_X*)*	-4.734E-05	-4.192E-04	-1.836E-04
Futures			
Futures	**Euro Stoxx 50 Index Futures**	**S&P 500 Futures**	**DAX Futures**
Ticker	FESX	FSP	FDAX
Isin code	DE0009652388	-	DE0008469594
Management	Eurex	CME Group	Eurex
Portfolio with Futures daily hedge			
μ_{Π_F}	1.508E-04	1.956E-04	2.154E-04
σ_{Π_F}	5.037E-03	6.035E-03	3.212E-03
R_Π	8.549E-02	1.038E-01	9.846E-02
Annualized R_{Π_F}	3.728E-02	4.428E-02	5.582E-02
Sharpe ratio	2.574E-02	2.893E-02	6.130E-02
Sortino ratio	1.527E-03	1.727E-03	4.225E-03
RAP (benchmark $r\alpha$*)*	-8.346E-05	-9.021E-05	-1.270E-06
RAP (benchmark r_X*)*	4.539E-04	-2.719E-04	1.653E-04
Portfolio with Futures single hedge at delivery date			
μ_{Π_F}	1.831E-04	n.a.	2.186E-04
σ_{Π_F}	5.121E-03	n.a.	3.121E-03
R_Π	1.057E-01	n.a.	9.957E-02
Annualized R_{Π_F}	4.585E-02;	n.a.	5.644E-02
Sharpe ratio	3.162E-02	n.a.	6.411E-02
Sortino ratio	1.914E-03	n.a.	4.416E-03
RAP (benchmark $r\alpha$*)*	-6.197E-05	n.a.	7.312E-06
RAP (benchmark r_X*)*	5.446-04	n.a.	1.980E-04

Bibliography

[1] M. Mezzedimi
Alpha return on portfolios hedged with Short ETFs. Oxford University MSc dissertation, 2011.

[2] Steven E. Shreve.
Stochastic Calculus for Finance II. Springer Finance , 2004.

[3] P. Justice.
Warning: Leveraged and Inverse ETFs Kill Portfolios, Morningstar Library Service , 22 Jan 2009.

[4] Morningstar Methodology Paper.
Modern Portfolio Theory (MPT) Statistics, Morningstar Library Service, 31 July 2008.

[5] P. Baldi.
Caldolo delle probabilità e statistica, Mc Graw Hil, 1998.

[6] J. Hill and G. Foster.
Understanding Returns of Leveraged and INverse Funds. Journal of Indexes, Oct 2009.

[7] G. Grimmet, D. Stirzaker
Probability and Random Processes. Oxford University Press, Third Edition 2001.

[8] E. Zeidler
Oxford Users' Guide to Mathematics. Oxford University Press, 2004.

[9] -
Plus - Exchange Traded Funds, Il Sole 24 Ore, March 2011

[10] J. C. Hull
Options, Futures and Other Derivatives. Prentice Hall Finance Series, Fifth Edition 2001.

[11] -
*Euro Stoxx 50 Index fact sheet.*Stoxx, http://www.stoxx.com/indices/factsheets

[12] -
Stoxx DVP Guide. Stoxx, February 2011.

[13] -
Stoxx Equity Index Calculation Guide. Stoxx, April 2011.

[14] -
S&P 500 indices fact sheet. Standard & Poor's, McGraw Hill, http://www.standardandpoors.com/indices/sp-500

[15] -
S&P U.S. Indices Methodology. Standard & Poor's, McGraw Hill, September 2011

[16] -
DAX index fact sheet. Deutsche Borse Market Data, www.dax-indices.com

[17] -
Equity and Equity Index Derivatives. Trading Startegies. Eurex brochure, www.eurex.com.

[18] -
Equity and Equity Index Derivatives. Trading Startegies - Questions and Case Studies. Eurex brochure, www.eurex.com.

[19] W. F. Sharpe
Mutual Fund Performance. Journal of Business 39 (S1), pages 119-138, 1966

[20] W. F. Sharpe
The Sharpe Ratio. Journal of Portfolio Management, pages 49-58, 1994

[21] F. A. Sortino
The Sortino Ratio. http://www.sortino.com/htm/Sortino%20Ratio.htm

[22] F. Modigliani
Risk-Adjusted Performance. Journal of Portfolio Management, pages 45-54, 1997

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