

# An optimal Markovian consumption-investment problem in a market with longevity bonds

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Longevity bonds are the first financial products to offer longevity protection by hedging the trend in longevity. Longevity bonds are needed because lifetime has been constantly increasing (medical improvements, better life standards, etc) and so there is a demand for instruments hedging the longevity risk, i.e., the risk that members of some reference population might live longer, on average, than anticipated, for example, in the life companies' mortality tables (assuming constant longevity can lead to a bankrupt of a pension plan or a life insurer). The uncertainty of longevity projections is illustrated by the fact that life expectancy for men aged 60 is more than 5 years' longer in 2005 than it was anticipated to be in mortality projections made in the 1980 (we refer to Hardy [15]).

To meet this demand, the Capital markets offer longevity bonds with coupons depending on the survival rate of a given population. They can be used to hedge a big portion of the longevity risk. Following Menoncin [19], we consider a simple market which, besides a riskless asset  $G(t)$ , a risk asset  $S(t)$  and a (zero coupon) Bond  $B(t, T)$ , contains a (zero coupon) longevity bond  $L(t, T)$ . The riskless asset and the (zero coupon) bond are characterized as

$$dG(t) = r(t)G(t) dt,$$

$$B(t, T) = E_t^Q \left[ e^{-\int_t^T r(s) ds} \right],$$

where  $r(t)$  is the interest rate and  $Q$  is the risk neutral measure: we assume that  $Q$  is uniquely determined, and that the (vector) market price of risk  $\xi(t)$  is known. The interest rate and the risk asset are modeled as Itô process

$$dr(t) = \hat{\mu}^r(t, r(t)) dt + \hat{\sigma}^r(t, r(t)) dW^r(t),$$

$$\frac{dS(t)}{S(t)} = \hat{\mu}^S(t, r(t), S(t)) dt + \hat{\sigma}_r^S(t, r(t), S(t)) dW^r(t) + \hat{\sigma}_S^S(t, r(t), S(t)) dW^S(t),$$

where  $W^r(t)$  and  $W^S(t)$  are independent one-dimensional standard Wiener processes (in [19] the functions  $\hat{\mu}^S$ ,  $\hat{\sigma}_r^S$  and  $\hat{\sigma}_S^S$  are constant, i.e.,  $S(t)$  is a Geometric Brownian motion).

The hedging problem in such a market is faced as a control problem (the controls being consumption-portfolio strategies) with random horizon  $\tau$ , the death time of the investor. In modeling a market with longevity bonds life time modeling plays an important role.

As in Menoncin [19], (see also, e.g., Biffis [1], Luciano and Vigna [10]) we describe such random times by means of the first jump time of a doubly stochastic Poisson process, with an intensity process  $\lambda(t)$ , i.e., we follow the classical approach of Brémaud [6] (see also Duffie[8]).

The process  $\lambda(t)$  is described by the stochastic differential equation

$$d\lambda(t) = \hat{\mu}^\lambda(t, r(t), \lambda(t)) dt + \hat{\sigma}^\lambda(t, r(t), \lambda(t)) dW^\lambda(t),$$

where the *driving noise*  $W^\lambda(t)$  is a Wiener process, independent of  $W^r(t)$ , the *driving noise* of the interest rate. Note that the processes  $\lambda(t)$  and  $r(t)$  are not independent, though the driving noises are independent. An intensity process  $\lambda(t)$  has to satisfy two essential properties:

$$(i) \lambda(t) \geq 0 \quad \text{and} \quad (ii) \int_0^\infty \lambda(s) ds = \infty, \text{ a.s.}$$

(in particular property (ii) implies that  $\tau$  is finite).

A zero coupon longevity bond is a financial security that pays, at time  $T$ , the value of the survival rate (given by the number of survived people in  $T$  with respect to the number of survived people in  $t$ ) of the given population of  $m$  individuals, all with the same age. The model considered by Menoncin [19]

$$L(t, T) = E^Q \left[ e^{-\int_t^T r(s)ds} e^{-\int_t^T \lambda(u)du} \middle| \mathcal{F}_t^r \vee \mathcal{F}_t^\lambda \right]$$

can be obtained (cfr. Sezione 4.3) as a limit for  $m$  go to  $\infty$ , under the condition that all  $m$ -individuals of the cohort have the same death rate  $\lambda(t)$ .

Similarly to [19] we study the optimization problem

$$\sup E_{t_0}^P \left[ \int_{t_0}^{\tau} e^{-\rho t} U(C(t)) dt \right],$$

where  $P$  is the real probability measure,  $E_{t_0}^P$  is the conditional expectation given the information at time  $t_0$  ( $t_0 < \tau$ ),  $\rho > 0$  measures the subjective discount rate,  $U$  is a utility function satisfying Inada's conditions,  $C(t)$  is the consumption process.

Instead of the usual strategies, we consider the relative ones, denoted by  $u^G(t)$ ,  $u^B(t)$ ,  $u^L(t)$ ,  $u^S(t)$ , where the superscript indicates the corresponding asset. Since  $u^G(t)$  is automatically determined by the equality  $u^G(t) + u^B(t) + u^L(t) + u^S(t) = 1$ , we consider only the vector  $u^A(t) = (u^B(t), u^L(t), u^S(t))$ .

In order to avoid arbitrage the supremum is taken over all consumption-investment (relative) strategies satisfying a budget constraint, which can be loosely written as

$$V(t_0) = E_{t_0}^Q \left[ \int_{t_0}^{\tau} \frac{c(t)V(t)}{G(t)} dt + \frac{V(\tau)}{G(\tau)} \right],$$

where the measure  $Q$  is the risk neutral measure,  $V(t)$  is the wealth process, and  $c(t)$  is the relative consumption, i.e.,  $C(t) = c(t)V(t)$ . Furthermore we assume that the wealth process  $V(t)$  is strictly positive. Then the budget constraint has the satisfying interpretation that the initial wealth  $V(t_0)$  equals the expectation of the total discounted consumption  $\int_{t_0}^{\tau} \frac{c(t)V(t)}{G(t)} dt$  plus the discounted heredity  $\frac{V(\tau)}{G(\tau)}$ . The latter being positive, the agent does not leave debts to her/his heirs. Setting

$$\mathcal{Z}^A(t) = \exp \left\{ -\frac{1}{2} \int_0^t |\xi^A(s)|^2 ds - \int_0^t \xi^A(s) dW_s \right\}$$

where  $W(t) = (W^r(t), W^\lambda(t), W^S(t))$ , and  $\xi^A(t) = \xi(t) - \Sigma^A(t)u^A(t)$  is a suitable linear combination of the strategy  $u^A(t)$  and the market price of risk  $\xi(t)$ , we prove that the budget constraints are satisfied whenever, the process  $\mathcal{Z}^A(t)$  is a martingale (see Theorem 5.3.1).

The main idea consists in reducing the above problem to a control problem with infinite time horizon

$$\sup E_{t_0}^P \left[ \int_{t_0}^{\infty} e^{-\int_{t_0}^t \lambda(u)du} e^{-\rho t} U(c(t)V(t)) dt \right].$$

If we restrict to Markovian consumption-investment strategies, the process  $(r(t), \lambda(t), S(t), V(t))$  is Markovian, and the above conditional expectation coincides with

$$E^P \left[ \int_{t_0}^{\infty} e^{-\int_{t_0}^t \lambda(u)du} e^{-\rho t} U(c(t)V(t)) dt \middle| r(t_0), \lambda(t_0), S(t_0), V(t_0) \right].$$

The corresponding value function is

$$J(t, r, \lambda, S, V) = \sup E^P \left[ \int_t^\infty e^{-\int_t^u \lambda(s) ds} e^{-\rho u} U(c(u)V(u)) du \middle| r(t) = r, \lambda(t) = \lambda, S(t) = S, V(t) = V \right],$$

where the supremum is taken over Markovian controls.

Nevertheless, due to the term  $e^{-\int_{t_0}^t \lambda(u) du}$ , the corresponding control problem is not Markovian. Introducing the process  $dz_0(t) = -\lambda(t)z_0(t)dt$ , which morally represents the conditional survival function of  $\tau$ , we can put the control problem in a Markovian framework: the aim becomes to maximize

$$E^P \left[ \int_{t_0}^\infty z_0(t) e^{-\rho t} U(c(t)V(t)) dt \middle| z_0(t_0), r(t_0), \lambda(t_0), S(t_0), V(t_0) \right]$$

over Markovian controls. We face the latter extended optimal problem, disregarding the budget constrains, following the (stochastic) dynamic programming approach via the so-called Hamilton-Jacobi-Bellman equation and the verification technique.

Denoting by  $\bar{J}(t, z_0, r, \lambda, S, V)$  the corresponding value function, then the value functions of the two unconstrained problems are linked by the following relation

$$J(t, r, \lambda, S, V) = \bar{J}(t, 1, r, \lambda, S, V).$$

Similarly, if  $\bar{c}_{sup}(t, z_0(t), r(t), \lambda(t), S(t), V(t); \bar{J})$  and  $\bar{u}_{sup}^A(t, z_0(t), r(t), \lambda(t), S(t), V(t); \bar{J})$  denote the optimal Markovian controls for the extended unconstrained optimization problem, then

$$\begin{aligned} c_{sup}(t) &= \bar{c}_{sup}(t, 1, r(t), \lambda(t), S(t), V(t); \bar{J}), \\ u_{sup}^A(t) &= \bar{u}_{sup}^A(t, 1, r(t), \lambda(t), S(t), V(t); \bar{J}), \end{aligned}$$

are the optimal Markovian controls for our optimization unconstrained problem.

The above controls solve the our optimization problem with the budget constraint if  $\mathcal{Z}_{sup}^A(t)$  is a martingale, where

$$\mathcal{Z}_{sup}^A(t) = \exp \left\{ -\frac{1}{2} \int_0^t |\xi_{sup}^A(s)|^2 ds - \int_0^t \xi_{sup}^A(s) dW_s \right\},$$

and  $\xi_{sup}^A = \xi(t) - \Sigma^A(t)u_{sup}^A(t)$ .

When  $U$  is the CRRA utility function

$$U(C) = \frac{1}{1-\delta} C^{1-\delta}, \quad \text{with } \delta > 1,$$

then the value functions for the unconstrained problem are given by

$$\bar{J}(t, z_0, r, \lambda, S, V) = \frac{V^{1-\delta}}{1-\delta} \bar{F}^\delta(t_0, z_0, z, V, S),$$

and

$$\bar{J}(t, r, \lambda, S, V) = \frac{V^{1-\delta}}{1-\delta} F^\delta(t_0, z, V, S).$$

It turns out that

$$\bar{F}(t_0, z_0, z, V, S) = z_0^{1/\delta} F^\delta(t_0, z, V, S)$$

and  $F$  is characterized as the solution of a linear partial differential equation, and is represented by means of Feynman-Kac formula, under suitable regularity conditions for the process  $(r(t), \lambda(t), S(t))$ . Therefore the value function  $J$  can be computed either with numerical schemes for linear partial differential equation or by means of Monte Carlo simulations.

The regularity conditions are satisfied, for example, when  $S$  is a Geometrical Brownian Motion, and  $(r(t), \lambda(t)) = (r(t), \lambda^{(c)}(t))$  is the bidimensional CIR model

$$dr(t) = a_r (b_r - r(t)) dt + \bar{\sigma}_r \sqrt{r(t)} d\bar{W}^r(t),$$

$$d\lambda^{(c)}(t) = a_\lambda \left( b_\lambda - \lambda^{(c)}(t) + c r(t) \right) dt + \bar{\sigma}_\lambda \sqrt{\lambda^{(c)}(t)} dW^\lambda(t),$$

with  $2a_r b_r > \bar{\sigma}_r^2$ ,  $2a_\lambda b_\lambda > \bar{\sigma}_\lambda^2$ , and  $c \geq 0$ . The latter condition may be interpreted as follows: the interest rate growth may affect the active population mortality intensity, for instance, a large interest rate may diminish health care and prevention (see Section 3.4). The above example may be generalize considering the translation  $(r(t) + \underline{r}, \lambda^{(c)}(t) + \underline{\lambda})$ , with  $\underline{r}, \underline{\lambda} \geq 0$ . In these models the death time  $\tau$  is finite, but is not bounded.

Furthermore in this cases  $\mathcal{Z}_{sup}^A(t)$  is a martingale, and the optimal consumption-investment strategies for the unconstrained problem are optimal also for the problem with the budget constraint.

We study the control problem also when the agent's portfolio, besides of the riskless and risk assets, consists of discrete time rolling both for ordinary bonds and longevity bonds. Rolling bonds are not anymore deterministic functions of time  $t$  and  $r(t)$  (and of  $\lambda(t)$  for longevity bonds), and depend also on the value of the interest rate (and of the mortality intensity for longevity bonds) at rolling times  $t_k \leq t$ . Nevertheless the consumption-investment optimization problem can still be put in a Markovian control problem setting.

The rest of this thesis is structured as follows.

In Chapter 1 we give a mathematical definition of basic financial concepts. We refer to Øksendal [20], Karatzas and Shreve [18], and Björk [2] for the basic notions in stochastic differential theory, the general results in stochastic calculus, and the arbitrage theory in continuous time, respectively.

In Chapter 2 the aim is focused on the problem of modelling (i) an arbitrage free family of zero coupon bond price processes (we follow the approach of Björk [2]), (ii) a discrete-time rolling bond price process, and we refer to Rutkowski [21].

In Chapter 3 we focus on the mortality risk and on modelling the survival function of the individual. We refer to Brémaud [6] and Duffie [8], for basic theory of point processes with a stochastic intensity, and modelling the dynamic mortality, respectively.

In Chapter 4 we face the problem of modelling an arbitrage free family of zero coupon longevity bond price processes. The previous zero coupon longevity bond is taken starting by Azzopardi-BNP Paribas [11] longevity bonds, while in the last section we introduce a new zero coupon longevity bond.

In Chapter 5 we focus on solving our optimal Markov control problem in different markets (included the market with rolling bonds and rolling longevity bonds) following the (stochastic) dynamic programming approach via the so-called Hamilton-Jacobi-Bellman equation and the verification technique (which is recalled briefly in Appendix D.1).

In Chapter 6 we take into account the case of a complete market with a CRRA investor, i.e., we consider the CRRA (Constant Relative Risk Aversion) utility function, and we solve

the optimal Markov control problem taking into account a particular factorization for the corresponding value function.

In Chapter 7 we present a numerical simulation for the interest rate and mortality intensity and we compute the price of derivatives with a Monte-Carlo methods for the CRRA case. We refer to Brigo and Alfonsi [4] and [5] for the the 1-dimensional implicit positivity-preserving Euler scheme, and to Glasserman [14] for Monte-Carlo methods.

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