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# Modelling and Pricing communication networks services in Markets for Bandwidth

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## Title

# Modelling and Pricing communication networks services in Markets for Bandwidth


## Abstract

The subject of this thesis is pricing and modelling of communication network services in bandwidth markets, more precisely connections between two or more geographical locations, subject to a certain Quality of Service requirements. First, it is shown how network topology leads to additional arbitrage opportunities, like the so called network or geographical arbitrage, and how it influences hedging strategies. A solution to such a pricing problem is proposed on the simplest bandwidth topology, the triangle network, both for pricing and hedging strategy based on a risk-minimization criteria. Such solution is then applied to various underlying price processes, from a Cox-Ross-Rubinstein type model to a more complex Geometric Brownian Motion (GBM). Future developments for price models including spikes as typical features of non-storable commodities are then discussed. Second, the foundations for a realistic extension of results found in triangle networks to a global telecommunication network are laid. In addition, estimates for the correlation function between traffic activities on distant routes are derived. More precisely, it is found an upper bound for the exponential decay rate of space and time two-point correlation functions. Lastly, the analysis moves to real data. The dynamics of different forward contract prices are linked with a price process dynamics for a forward contract with fixed maturity. Subsequently, a truncated increments variation technique is used to detect and remove spike prices from real data, in order to estimate parameter values for a GBM process. Then a simple model for the price process, consisting of a GBM with Poissonian spikes, is proposed and simulated in order to mimic the empirical data.

## Titre

# Modélisation et valorisation d'instruments financiers dans le marché des télécommunications

## Résumé

L'objet de cette thèse est la modélisation et la valorisation des services relatifs aux réseaux de télécommunications dans les marchés de largeurs de bande, en particulier entre deux ou plusieurs positions géographiques, sous certaines conditions de qualité du service. D'abord, nous montrons comment la topologie du réseau induit de nouvelles opportunités d'arbitrage, comme l'arbitrage géographique, et quelles sont les conséquences sur les stratégies de couverture. Nous proposons une solution basée sur un critère de minimisation du risque, aussi bien dans la valorisation que dans la couverture avec la topologie la plus simple, c'à dire le triangle. Ces résultats sont appliqués à plusieurs processus décrivant le prix, allant du simple modèle de Cox-Ross-Rubinstein au plus compliqué mouvement brownien géométrique (MBG). Nous discutons brièvement des développements futurs possibles vers des modèles de prix qui incluent des “spikes”, phénomènes typiques liés aux biens non stockables. Deuxièmement, nous posons les bases pour une généralisation de tels résultats de la topologie du triangle à une plus compliqué. Nous dérivons également des estimations de la fonction de corrélation entre des activités de trafic sur des parcours distants. En particulier nous déterminons une borne supérieure à la vitesse de décroissance exponentielle de la fonction de corrélation à deux points. En suite, l'analyse se concentre sur les données réelles. La dynamique des prix de plusieurs contrats à terme (“forward”) est liée à celle de l'allure dans le temps du prix d'un “forward” à échéance fixe. Nous estimons les paramètres de la série temporelle des prix nettoyée des “spikes”, par une technique de variation tronquée des incréments. En conclusion, nous proposons et simulons un modèle simple pour le processus des prix, composé d'un MBG et des “spikes” poissoniens.

## Titolo

# Modellizzazione e Prezzaggio di servizi su reti di comunicazioni nei Mercati per le Ampiezze di Banda

## Riassunto

L'oggetto di questa tesi di dottorato è la modellizzazione e il prezzaggio dei servizi su reti di comunicazioni nei mercati per le ampiezze di banda e nello specifico, servizi di connessione tra due o più posizioni geografiche sotto determinate condizioni della Qualità del Servizio. Dapprima, viene fatto vedere come la topologia della rete induce nuove forme di opportunità di arbitraggio, come l'arbitraggio geografico, e come questo si ripercuote sulle strategie di copertura. Viene proposta una soluzione basata su di un criterio di minimizzazione del rischio, sia al problema del prezzaggio che della copertura sulla topologia più semplice, il triangolo. Questi risultati vengono quindi applicati a vari processi di prezzo che vanno dal semplice modello Cox-Ross-Rubinstein al più complesso Moto Browniano Geometrico (GBM). Vengono brevemente discussi possibili sviluppi futuri verso modelli di prezzo che includono gli spikes, tipici dei beni non immagazzinabili. In secondo luogo, vengono gettate le fondamenta per un'estensione realistica di tali risultati dalla topologia del triangolo ad una più complessa. Inoltre, vengono derivate stime per la funzione di correlazione tra le attività di traffico su percorsi distanti. In particolare, viene trovato un limite superiore alla velocità di decadimento esponenziale della funzione di correlazione a due punti. Successivamente, l'analisi si sposta sui dati reali. La dinamica dei prezzi di diversi contratti forward viene legata a quella dell'andamento temporale del prezzo di un contratto forward con scadenza fissa. Vengono poi stimati i parametri della serie storica dei prezzi "pulita" dagli spikes tramite una tecnica di variazione troncata degli incrementi. Infine, viene proposto e simulato un semplice modello per il processo dei prezzi, composto da un GBM con spikes Poissoniani.

# Contents

## *Chapter 1*

# Introduction

Since the first bandwidth deregulation came from the Telecommunications Act of 1996, many energy companies realized that capacity could be treated as a commodity and they initiated to trade Bandwidth. In 1997 Band-X opened the way to trading bandwidth, followed in 1998 by RateXchange who offered to link buyers and sellers anonymously. In the same year, Enron announced that it was creating a new market for trading bandwidth. By the end of 1999 other energy brokers began to create their own bandwidth market desk, and formed The Association of International Telecommunications Dealers (AITD). In December of that year, Enron completed the world's first bandwidth trade. By the beginning of 2000, many carriers were welcoming a new era of commodity bandwidth trading. In May 2000, the first comprehensive index to measure telecommunications bandwidth prices was launched.

Year 2001 was the beginning of the fall: In January, TeleExchange suspended its operation and in March 30% decrease of Bandwidth prices since January 2001 was registered. In June 2001, Bandwidth.com suspended its trading activities and converted to a broker. Since October 2001, 17 companies, with a combined market capitalization of $96 billion went

bankrupt. December 2001 was the major turnover: Enron filed for Chapter 11 bankruptcy. Seeing Enron falling, other energy merchants suspended trading Bandwidth in 2002 and returned to their core businesses: electricity and gas.
However, this does not mean that bandwidth will not become a commodity that can be traded like others. After this experience, bandwidth showed to be different from other commodities such as coal or even electricity.
Other components such as Quality of Service (QoS) and risk management have to be taken into account, when dealing with bandwidth transactions. By March 2002, brokers continued to match buyers and sellers but suspended any trading activity. In October 2002, 47 carriers went bankrupt, trying to compete on bandwidth prices. By December 2002, prices had fallen 44% from January 2001 and were not likely to rise anytime soon.

As a summary it can be said that the following effects contributed to the collapse of the Bandwidth trading market: The decline in the Bandwidth prices induced a surplus (*i.e.* supply exceeding demand) in the bandwidth and made it a product of very low liquidity. All of this, made it hard for Bandwidth to be considered and treated like a tradable commodity. Also, technical incompatibilities between network operators were a major problem due to the non-existent unified set of quality of service standards. Finally, telecommunication companies did not use risk management tools to analyse their exposure to risks and determine how to best handle such exposure This made them oblivious to the risks of such a volatile market with huge price movements.

Advances in technology, amongst other catalysts, create and fuel markets for new products and commodities. A relatively recent addition to this background has been the market for information transfer, or data, between two geographical locations. Telecommunication networks facilitate such transfers, principally through the digital transmission of data via satellites or over fiber optic. Term bandwidth corresponds to the amount of data transferred on a given transmission path within a specified block of time. Therefore, bandwidth is a synonym for telecommunications capacity, and it is measured in units of bits per second. For example, the bandwidth required to transfer voice is approximately 64 Kbps. As the bandwidth required for transferring voice was the natural building block for earlier communication networks, it is denoted by DS-0, where DS stands for digital signal. Two other units are widely used. They are DS-1 which is known as a T1 line with 24 times the capacity of a DS-0; and DS-3 or T3 line which has 672 times the capacity of a DS-0. In the context of optical transmission, the bandwidth capacity is measured in units of OC-x, where OC stands for an optical carrier and x is an integer. Loosely speaking, an OC-1 is equivalent to a DS-3. In practice, the capacity of an OC-1 is 45Mbps, which is equivalent to 810 DS-0, instead of 672. The additional bandwidth is stored in the header for the purpose of signal synchronization. Most traded STM1/OC3 (STM is Synchronous Transport Module)155Mbps.

Managed bandwidth services are connections between two or multiple points on a certain capacity. Therefore the market for bandwidth is a mechanism facilitating the trading of transmission capacity between various geographical locations. While the focus of this work is not restricted to a specific medium of transportation, we often find it useful to think of bandwidth as the space available on a cable physically connecting two locations. A leasing contract on a point-to-point connection selects the cheapest path between two points. Therefore, a bandwidth option is a compound option. The underlying asset of these contract is a point-to-point telecommunications connection. Currently, in the bandwidth commodity market pure spot prices does not exist. All instruments in the market are derivative instruments. Thus, in practice the spot price refers to the forward price (leasing contract price) with shortest maturity and duration. At present, this means one-month forward price that starts after two months. Because of this, we assume in this work, that the spot prices can be approximated with the forwards of shortest maturity and duration. Nowadays, most bandwidth contracts fall under the realm of exotic options, where contract terms are negotiated among counter parties. For example, a contract may offer a network connecting several cities while guaranteeing a minimum amount of bandwidth. The contract may also give the customer the option to request additional capacity at a future time at a predetermined price. In this example, the customer guarantees the availability and the price of future bandwidth that may be needed. That is, the customer eliminates its exposure to risk deriving from bandwidth shortage or price fluctuations. In general, bandwidth contracts can be evaluated if it is possible to value their simpler counterparts, such as futures and options.

Therefore, the motivation for our study lays in the observation that bandwidth markets lack a strong mathematical models, as well as standardization and benchmarking of bandwidth contracts, all requirements for the process of bandwidth *commoditization* and the creation of a liquid market.

The structure of this thesis is the following:

In the second chapter we describe the bandwidth market in its general form. We summarize the historical background which leads to the current state of such markets. We then explain the basic rules of their functioning and the main criteria which "players" have to follow. Mostly, we dwell on peculiarities of bandwidth markets, such as network topology and its implications in pricing and hedging, Quality of Service (QoS) as a constant of all bandwidth contracts, markets dynamics and its conventions. Then, a comparison with other "standard" commodities is done, focusing the attention on new properties arising from the perishable feature. We do that in order to illustrate the system that is at the heart of our analysis. Moreover, we focus our attention on peculiar forms of arbitrage opportunities related to bandwidth markets, and in particular we discuss on a new form of arbitrage opportunity, named network arbitrage, which is the main subject of our thesis. Finally, we give an alternative interpretation of a specific contingent claim, a European call Option, as the penalty payment a buyer has to pay to rescind a bandwidth service contract. The right to rescind a bandwidth service contract is one which makes a European Call Options the perfect candidate for the most tradable derivative in such markets.

In the third chapter, we look at the basic features of bandwidth as underlying asset and related issues of derivatives pricing. Above anything else, we investigate the influence of bandwidth network on pricing European style contingent claims. We show how network topology leads to new forms of arbitrage opportunities, like the so called network or geographical arbitrage, and how it influences hedging strategies for pricing derivatives. A solution is proposed in order to solve such pricing problem on the simplest bandwidth topology, a complete graph of three nodes, the triangle network. Moreover we develop a risk-minimizing hedging strategy yielding "fair" prices for contingent claims in agreement with standard pricing formulas in a riskless world. Then, we apply our pricing and hedging strategy to various underlying price processes, starting from the traditional one-period binomial model, likewise the Cox-Ross-Rubinstein model [21], and ending up with a more complex Geometric Brownian Motion (GBM) stochastic process. Furthermore, we discuss future developments for price models including spikes, a typical features of non-storable commodities.

In the fourth chapter, we discuss decay of correlations in loss networks. In our models calls can be initiated between nodes at a uniformly finite distance. The study of the generator of call dynamics allows us to deduce estimates of exponential decay rates of correlations both in time and space.

In the fifth chapter we turn our attention to the analysis of the real price dataset. Since such a dataset contains prices of forward contracts with different maturities, we develop a model in order to link the dynamics of those prices to the dynamics of price process for a forward contract with fixed maturity on the same route. Moreover we propose a simple model for the price process consisting of a GBM with Poissonian spikes. We also estimate parameter values from real data for a GBM process, using a truncated increments variation technique to detect and remove spike prices. Finally, we simulate such a model in order to mimic the prices data.

Finally, in the last chapter we sum up and discuss the results of the thesis as well as we dwell on subjects for future research.

*Chapter 2*

# Bandwidth Markets

In this chapter we illustrate the main characteristics of Bandwidth Markets. We summarize the historical background which leads to the current state of such markets. We then explain the basic rules of their functioning and the main criteria which "players" have to follow. Mostly, we dwell on peculiarities of bandwidth markets, such as network topology and its implications in pricing and hedging, Quality of Service (QoS) as a constant of all bandwidth contracts, markets dynamics and its conventions. Then, a comparison with other "standard" commodities is done, focusing the attention on new properties arising from the perishable feature. We do that in order to illustrate the system that is at the heart of our analysis. Moreover, we focus our attention on peculiar forms of arbitrage opportunities related to bandwidth markets, and in particular we discuss on a new form of arbitrage opportunity, named network arbitrage, which is the main subject of our thesis. Finally, we give an alternative interpretation of a specific contingent claim, a European call Option, as the penalty payment a buyer has to pay to rescind a bandwidth service contract. The right to rescind a bandwidth service contract is one which makes a European Call Options the perfect candidate for the most tradable derivative in such markets.

## 2.1 Bandwidth Background

Markets today serve as mechanisms which facilitate the distribution of goods, services, securities and risks between what are assumed to be rational agents, attempting to maximise some measure of personal welfare. It is envisioned, perhaps ideologically, that these markets serve to ration items between competing agents to achieve a socially optimal allocation, however that may be defined. Frequently traded, or liquid products, have transparent prices which help to identify profitable, and perhaps socially optimal, areas for additional investment or research. Advances in technology, amongst other catalysts, create and fuel markets for new products and commodities. Telecommunication firms are now operating in a more competitive and open market following the deregulation of the telecommunication market in recent years. Consequently, interest has developed in a market for the transfer of information, or data, between geographical locations.

Telecommunication networks facilitate information transfer, mainly through the digital transmission of data via satellites, or over fiber-optic or coaxial cables. Regardless of the medium of transportation, we define bandwidth as the transmission capacity for data between two specific locations, a source and a sink, over some time interval. Therefore the market for bandwidth facilitates the trading of transmission capacity between various geographical locations. While the focus of this research is not restricted to a specific medium of transportation, or network resource, it is often useful to think of bandwidth as the space available on a cable physically connecting two locations. The specific time interval over which transmission capacity is offered is particularly important in this market. Markets often facilitate the trading of spot contracts for immediate purchases, and forward contracts for purchases at future points in time. The distinction between these two contracts is important in the market for bandwidth.

## 2.2 Features of Bandwidth Markets

For each network resource connecting two locations there is a finite transmission capacity, and bandwidth is an allocation of resource capacity over a defined time period. Several specific features of this product establish it as a unique commodity that is unique to any other.

Afterwards, the principle characteristics are discussed, which include: the non-storability of bandwidth, the influence of network structure on prices, and the quality of service. Individually such features may not be unique, with network structure an exception, and as each of the features are outlined in the following discussion, products with similar features are identified. To conclude this section, other differences are briefly pointed out.

### 2.2.1 Non-Storability

Bandwidth capacity is a non-storable resource, unlike other "standard" commodities such as oil, gas, gold, that can be stored and still remain tradable. The non-storable property, shared with commodities like electricity, airline tickets etc., means that, any free capacity not used today cannot be stored for future use, *i.e.* unused capacity has no value. For example, one cannot buy a one-month spot contract on a bandwidth exchange, store it for the next month and then sell it. In one month, the contract will be worthless. As such, in addition to being a consumable commodity, bandwidth is also described as perishable. Like a concert ticket or an airline seat, a perishable commodity is consumed by time, whether used or not.

In the storable case, inventories tend to smooth variations in supply and demand, hence in spot prices. Since Bandwidth has no storage mechanism, spot prices may exhibit sudden jumps or spikes both upwards and downwards, as already seen in electricity spot prices. As a consequence, spot prices in different time periods could quite likely be unrelated and there is no reason to expect they evolve by any well defined process, or to be Markovian.

Therefore, spot bandwidth prices are untradable, and certainly cannot be included in a self financing portfolio to hedge or construct derivative products.

The non-storability property also breaks the usual link between spot and future prices

$$F(t,T) = S(t)e^{(T-t)(r+y)}$$

where $F(t,T)$ is the future price at time $t$ for purchase of a good or a service at time $T$, $S(t)$ is the price at time $t$ of the commodity, $r$ is the market interest rate over $[t,T)$, and $y$ is the convenience yield, *i.e.* the storage cost. Exactly the lack of this last parameter, $y$, usually interpreted as the benefit that comes from actually holding the commodity, makes it impossible to establish a link between spot and future prices.

Nevertheless, the main business of telecom wholesale division at the moment are forward and future contracts, which represent the primary traded contracts for bandwidth capacity over-the-counter (OTC). Forward contracts, unlike spot Bandwidth, are storable in the sense that they have value from first sale up to delivery or financial settlement. is not a problem for derivative pricing.

### 2.2.2 Topology

The main feature of bandwidth capacity market is its network topology architecture. A few other commodities (especially, electricity) share the network feature with bandwidth, but for example, electricity transmission flow over a network is governed by Kirchhoff's laws, which make it a very different issue.

The basic design of the telecommunication network topology was based on the pre-existent telephony cable connections network. In order to avoid circuit switching congestions, tolerable by telephone calls, excess bandwidth capacities was reserved in the construction of telephony connections, enough to allow the use of the same cables for the primordial stage of data transfers.

After the advent of de-regulation, a more aggressive and cost efficient design principle has been adopted. There have been extraordinary capital investments and building of new network capacity between 1998 and 2001. As a consequence of overbuilding, it has been estimated that there are already connection cables for at least two or more suppliers on each significant route. Due to advances in technology, overcapacity and strong competition, bandwidth capacity prices do not depend strongly on distance. Nowadays, most long-distance telecommunications travelling along fixed links, travel over optical fibre. This kind of cables guide pulses of light with extremely high efficiency, they do not need any amplification mechanism, do not suffer from electrical interference and finally they offer higher capacity than traditional copper wires. Optical fibers have relevant cost factors in initial equipment cost and maintenance, but they do not have information transport cost. In this framework, there will be many different possible routes connecting the same two points (cities), allowing providers to choose the route to deliver the negotiated service based on their aggregate prices. The presence of more than one route connecting two points is the main concern over the network effects of the bandwidth markets. Such a network effect leads to a kind of arbitrage opportunity, the so called network arbitrage or geographical arbitrage, proper to bandwidth. In fact, in an efficient market, the purchase price of a route should not be less than the sale prices of all alternative routes, and the violation of this condition makes an arbitrage possible.

### *Network Topology*

There are several approaches to describing telecommunication networks. Due to their characteristics, *i.e.* a set of distribution points (distribution frames, distributors, etc.) and a set of different cables types to connect different points, graphs are a natural choice for modelling telecommunications networks, even though networks are not fully meshed (all nodes connected to each other).

Connection points translate into nodes and act as endpoints for point to point connections, *i.e.* a telecommunication link of fixed capacity between two endpoints without intermediate demarcation points. A direct point to point connection constitutes an edge. A sequence of edges that connect two nodes is called a path (route). Edges do not have a direction, but rather it is traversable in both directions, at the same cost and a fixed QoS (Quality Of Service).

General features of a part of the global telecommunication network have been analysed in

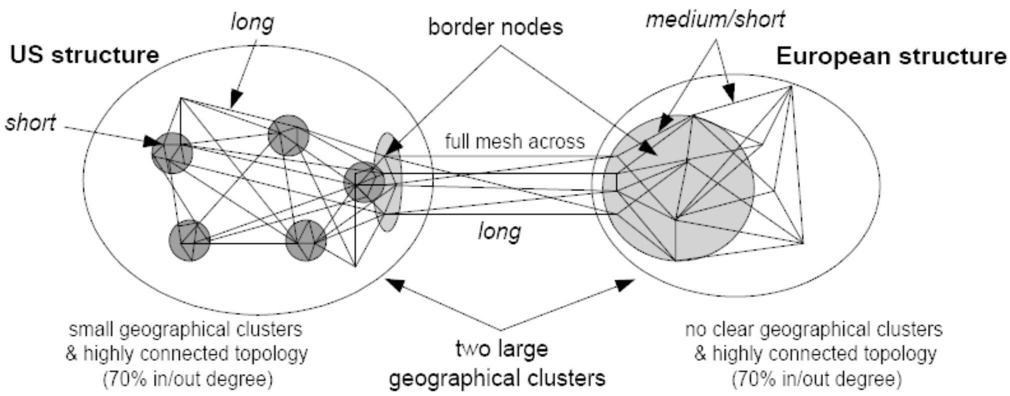



**Figure 2.1:** Graph of the different topologies exhibited by the US, European and Trans-Atlantic networks. [36].

[36]. Figure 2.1 illustrates a graph for the network topology of US, European and Trans-Atlantic. The graph was constructed by examining several incumbent and new entrant back-

bone networks and graphing the connection which could be offered by those providers as individual contracts between pooling points location.

After a first look, we can note that the US structure has a more heterogeneous structure with more or less fully meshed subnetworks in major metropolitan areas. Subnetworks have dominant nodes, which again form a close to fully meshed network. The European network is more homogeneous in its topology. Both the US and European topologies have border nodes, whereby the networks are mainly interconnected. The border nodes connect and form the Trans-Atlantic network, which has a topology of its own, too. Kenion and Cheliotis also analysed the presence of triangular subnetworks in such a topology. The network contained $187$ triangular subnetworks, of which $77\%$ are acute, and according to the authors, the network effects on price dynamics were found to be pronounced in acute triangles.

Those analysis, along with earlier literature on bandwidth price dynamics on triangle networks ([23, 10, 11]), are very relevant to this thesis. In fact, the model we study will be developed, at a first step, on a triangular network, which is the simplest topology where network effects arise in the pricing process.

### 2.2.3 Network Arbitrages

The issue of network arbitrage opportunities has been widely studied in literature []. Today's bandwidth markets exhibit such arbitrage opportunities. The network effects of the bandwidth markets have been widely studied in literature, *e.g.* see [10, 23, 37]. The main concern with network effects is the presence of network arbitrage opportunities. The simplest form of network arbitrage is realized by comparing forward contracts on the same capacity, maturity and duration. However, complex forms of arbitrage condition exist, too. These relate to combining different kinds of contracts. For example, in [34] a temporal arbitrage is analysed. This particular form of arbitrage opportunity could arise among bandwidth contratcs with the same capacity, the same QoS, but different duration, *i.e.* one could cover a one year forward contract with two consecutive six months contracts. However, a temporal arbitrage could be avoided by imposing a relationship among contracts of various durations as it has already be done in other markets, such as Electricity Markets.

The existence of network arbitrage opportunities, related to the telecommunication topol-

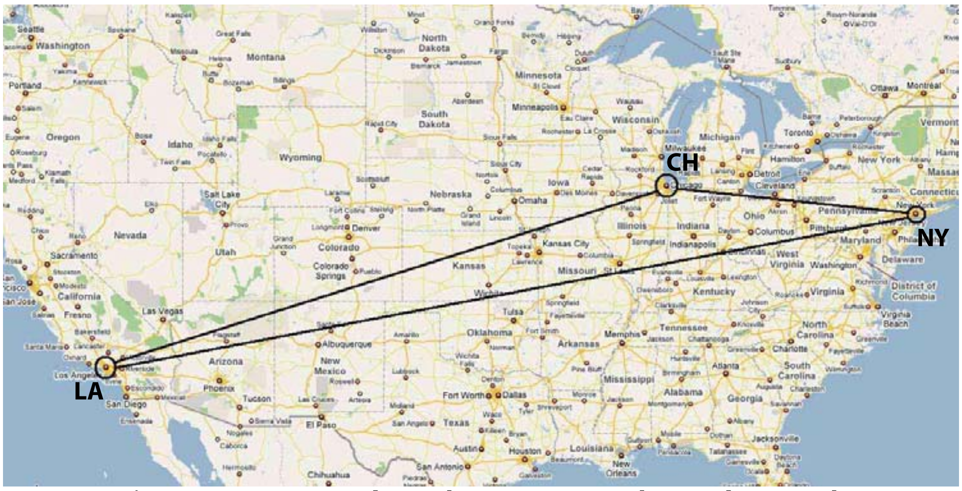


**Figure 2.2:** Los Angeles - Chicago - NewYork triangle network

ogy, has been shown as true effects of the bandwidth market and may arise for a very short time period even in mature markets.

Table (2.1) shows an example of a possible arbitrage opportunity. The data presents sale

**Table 2.1:** Example of possible network arbitrage opportunity in the LosAngeles - Chicago - NewYork triangle connections. Data Source: Telegeography, Connection type: DS-3 (45Mbts), Date: October 2005.

| | | | |
|---|---|---|---|
| Direct Routing | LA - NY | \$3.282 | |
| Alternative Routing | LA - CH | \$1.129 | |
| Alternative Routing | CH - NY | \$956 | |
| Savings | | \$1.197 | 36% |

offers from Telegeography pricing sheet from October 2005. Prices are not binding offers price but rather price indications. The prices indicate that it is possible a cost saving of $36\%$ by making a Los Angeles - New York connection passing by Chicago.

There might be logical explanations for the price discrepancy. For example, the last mile is usually the most expensive part of the connection, and the offer for the New York - Los Angeles connection might be between different connection points than the alternative route. Furthermore, there are several reasons which should be taken into account. Firstly, there is no continuously traded market currently and hence it is not possible to realize any deals instantaneously. Secondly, the price quotes normally do not define the connection points. *i.e.* the product is poorly defined, and the last mile is a considerable cost in establishing the connection. Lastly, quality of service requirements constrain the construct of the alternative routing, because re-routing may deteriorate quality of service.

However, a possible $36\%$ cost saving should be reason enough for any manager to look into the pricing a bit more closely.

### 2.2.4 Quality of Service

The term Quality of Services (QoS) is used, in this thesis, as a measure of the guaranteed service quality, rather than referring to the ability to reserve resources. Bandwidth is a product and as such certainly requires some form of guarantee. Furthermore, when we consider possible substitutions of routes connecting two fixed points, we do assume that all those different routes must have the same QoS, as a parameter identifying different type of products. QoS comprises a set of quality requirements on all the aspects of a connection, such as service response time, loudness levels, signal noise ratio, packet dropping probability, bit error rate, and so on. See [6, 7, 8] for a survey and an overview of some QoS mechanisms and related topics such as resource allocation and congestion control. Since each requirement has a different QoS metric, a path consisting of multiple links accumulates different QoS metrics in different ways. For example, loss ratio and reliability (the probability of no failure in the network) are multiplicative metrics while delay and jitter (delay variation) are additive. All of those different QoS metrics are obviously correlated, since, for example, if a network experiences a failure, all of QoS will increase at the same time.

On a public network it is quite hard to refer to QoS, and it is necessary to deal with the weakest link or network hop, but on a private network QoS is one of the key parameters on the same level of the Bandwidth capacity. The idea behind QoS consists in handling data traffic differently, based on the kind of application and the data being sent. An application doing remote system backup has very different quality demands on delay and packet loss than an application which sends live video to a costumer. When link network becomes congested, data packets have to be delayed, or dropped, and it is the QoS policy that determines which packets to prioritize, and which to delay or drop. The question of pricing such services is central, since some services benefit from QoS at the expense of others. Without any associated cost, there is no reason to simply demand the highest possible service level at all the time. Nowadays, most bandwidth leasing contracts are based on long-term agreements, where the QoS is specified, but there is no associated cost to the QoS level.

However, even if a network consists of links with large QoS, routers have limited resources and congestions at nodes can have a negative impact on end-to-end QoS.

In this thesis we do not address QoS issues since it is an external parameter that we consider fixed for any leasing bandwidth contract. Regarding congestion issues, we assume that in any case only a small percentage of existing network capacity will be traded in the market, so as to avoid the possibility of congestions at some nodes.

Moreover, recent market developments have started the process of a more transparent and standardised market for bandwidth capacity. In fact, traders and brokers entering this market have opened a lot of discussion in order to standardise the contractual terms of such leases, and associated instruments. Among of all master agreements proposed, the most widely accepted is commonly referred to as the Bandwidth Trading Organization (BTO) master agreement [9]. This agreement regulates most of the issues of bandwidth leasing contracts so that many technical and QoS concerns are no longer at issue.

## 2.3 Bandwidth Contracts

The entry of Internet service providers and energy companies into the market for wholesale bandwidth has fuelled interest in an open market for trading transmission capacity. Previously, spare transmission capacity was privately traded between large interconnected telecommunication companies, or sold by marketing departments to large wholesale customers. Recent market developments have advanced the possibility of a more transparent and standardised over-the-counter market for capacity. This has been aided by investment banks, particularly those who have financed expensive network expansions, which have an interest in observing market values for networks, and in facilitating mechanisms to manage the risks of their loans.

The main “commodity” of interest amongst such players has been the lease of space on coaxial and fibre optic cables connecting major industrialised cities worldwide. Particular to each contract are the: origination and termination cities, type of connection, installation date, length of contract and the price.

However, for bandwidth to be traded as a commodity there must be a number of derivative financial instruments (such as forward contracts and options) to help market participants manage risk and guard against price fluctuations. In addition to sellers and buyers of capacity, financial institutions also have a stake, in creating and trading financial instruments related to bandwidth:

- Capacity providers and tier 1 carriers want to protect themselves against changes in the market price of capacity, which ultimately affects their bottom line.

- Carriers and telecommunication products resellers in possession of long-term contracts on specific destinations want to hedge some of the inherent risk of their contracts by purchasing derivatives on those routes before the contract expires. These derivatives may either lock-in lower prices or provide insurance protection against adverse price changes. Operators without such long-term contracts want to provision enough bandwidth for their clients as well as limiting their price exposures by locking-in future capacity prices and availability.

- Portfolio managers want to protect their telecommunication funds against major changes in the pricing structures in the industry. Professional speculators with industrial expertise want to profit from market inconsistencies using better knowledge of market trends.

Financial risk management instruments enabled carriers and market participants to control their costs and to protect themselves against the risks of a long-term contract as conditions change.

However, forward contracts on a perishable commodity, e.g. bandwidth or electricity, are not perishable. Arbitrage reasoning can be applied to price forward contracts at future times, if we know the contract price now. In effect, the price of a forward contract should not exhibit any systematic price movements. These should already have been taken into account in the forward price.

### 2.3.1 Forward Arbitrage

Since Bandwidth underlyings consists mostly of forward contracts with different durations, the first natural arbitrage opportunity we have to deal with is the Temporal Arbitrage, also called Forward Arbitrage.

The Forward arbitrage is an arbitrage opportunity, which arises when two or more forwards of different maturity and/or duration, with the same starting dates, are mispriced in time in such a manner that an arbitrageur may profit without risk. In other words, this arbitrage condition states that the price of a forward contract with multiple payouts over time, *i.e.* the constant monthly payments common in bandwidth contracts, has a price equal to the sum of the prices of forward contracts on each single payout. Conversely, if such an arbitrage condition is not fulfilled, a "frictionless market" would present an opportunity for arbitrage. From a formal point of view, if $F(t_0, T_d, T_f)$ denotes the price of a forward contract drawn up at $t_0$ with delivery period starting at time $T_d$ and ending at time $T_f$, the temporal no arbitrage relationship reads

$$F(t_0, T_d, T_f) = \sum_{i=0}^{n-1} F(t_0, T_i, T_{i+1})$$

where

$$T_d = t_0 < t_1 < \ldots < t_n = T_f$$

Such an arbitrage has been studied, for the first time in a bandwidth context, by Chiu and Crametz [34]. They examine the forward arbitrage issue by comparing several spot price contracts, *i.e.* the shortest one-month forward contracts, with constant monthly payment forward contracts. In the light of the calculations used in the computations of the first bandwidth price indexes[1], they provide a comparison principle that links prices of forward contracts with different duration avoiding risk-free profit opportunities. Nevertheless, Chiu and Crametz [34] show that forward prices derived by the no-forward arbitrage condition from spot prices, which fulfil the no-network arbitrage condition, do not necessary preserve such a condition. In other words, composite arbitrage could be hard to avoid in a realistic world. Later, Reinman and Sweldens [35] tackle the composite arbitrage problem in bandwidth

[1] RateXchange's revealed Price Index (RPI) and Revealed Forward Price Index (RFP)

markets explaining how the apparent pricing anomaly can be easily avoided taking forward prices as basic data for deriving spot prices rather than vice versa. Such an approach is motivated by the numerical instability of the relationship derived in [34], and by the demonstration that if forward prices fulfil the no network arbitrage condition, then the spot prices derived from them are also guaranteed to fulfil such a no arbitrage condition.

In this thesis, we only deal with the network arbitrage, whose implications have more impact on derivative pricing models, leaving the forward arbitrage issue for future researches, when the wholesale bandwidth market will be liquid enough to point out such problems.

### 2.3.2 Capacity Conventions

Bandwidth capacities traded in the market have a specific format and size, depending on the type and quality of the cable, so as on the equipment installed at either end of the cable making possible to transmit and receive data.

For example, the bandwidth required to transfer voice is approximately 64 Kbps, it is also named as the capacity of one voice frequency equivalent channel. As the bandwidth required for transferring voice was the natural building block for earlier communication networks, it is denoted by DS-0, where DS stands for digital signal, and connections are often expressed in terms of how many DS-0's they can carry. In table 2.2, there are listed the most common types of connections, linked with their sizes (Bit Rate). Two other units are widely used. They are DS-1 which is known as a T1 line with 24 times the capacity of a DS-0; and DS-3 or T3 line which has 672 times the capacity of a DS-0. In the optical space, the bandwidth capacity is measured in units of OC-x, where OC stands for an optical carrier and x is an integer. Loosely speaking, an OC-1 is equivalent to a DS-3. In practice, the capacity of an OC-1 is 45Mbps, which is equivalent to 810 DS-0, instead of 672. The additional bandwidth is stored in the header for the purpose of signal synchronization. The most traded capacity types are the STM1/OC3 (STM is Synchronous Transport Module) 155Mbps.

There is an important issue to point out, namely, that the capacity types that are traded in the market are not divisible. In other words, a STM-4 connection cannot be substituted with four STM-1 connections even if they can carry the same amount of data. This is due to the cost of the extra equipment needed first to multiplex the four STM-1 signals into one STM-4

**Table 2.2:** Common Exchanged Bandwidth Capacity Types, from www.telegeography.com. (Type1/Type2 is for North American vs. International convention)

| Capacity Type | Bit Rate | No. of DS0s |
|---|---:|---:|
| DS-0 | 64 Kbps | 1 |
| DS-1(T-1) | 1.5 Mbps | 24 |
| E-1 | 2.0 Mbps | 32 |
| DS-3(T-3) | 45 Mbps | 672 |
| OC-3/STM-1 | 155.5 Mbps | 2016 |
| OC-12/STM-4 | 622.1 Mbps | 8064 |
| OC-48/STM-16 | 2488.3 Mbps | 32256 |

before transmission and secondly to separate the signals afterwards.

## 2.4 Comparison with Other Commodities

Much can be discovered about bandwidth trading market dynamics, required infrastructure, problems and limitations by comparing it to trading in two other commodities: electricity and natural gas. In addition to providing a useful contrast, these two markets offer lessons in management and regulation that can shape the development of bandwidth market and offer a glimpse into its future.

### 2.4.1 Storability and Transportability

Bandwidth, electricity and natural gas differ primarily along two dimensions: storability and transportability. A commodity is considered storable if it persists over time and yet remains tradable. Storable commodities allow the creation of inventories to smooth supply and demand fluctuations. A commodity is transportable to the extent that it can be moved from one location to another and remains tradable. Gold is, for example, both highly transportable and highly storable. A right of way along a segment of railway is neither transportable nor storable: it cannot be moved and the “right” at any moment in time vanishes if it is not used.

Natural gas has the highest level of storability and transportability among the three commodities considered, but it is still below that of a commodity such as gold. Natural gas can be moved from one part of the country to another, but is largely constrained by the topology and carrying capacity of the pipeline network. The ability to store natural gas and to transport it through alternative (albeit less efficient) means mitigates some of the network constraints.

On the other hand, electricity is not storable (at least not on the scale being considered here), but, the raw materials used in generation are. Generation capabilities can also be brought online as necessary beginning with the most cost-efficient, which provides a degree of flexibility in managing supply. When considered together, this provides a level of storability below natural gas. Transportability is also more restricted than natural gas. Electricity can be moved from one location to another, but it is bound to the topology and capacity limits of the transmission network. Bandwidth is defined entirely by the topology and capacity of its network. The network is not the medium for distribution of a commodity as it is with electricity and natural gas, but is the commodity itself. As such, it is not transportable: bandwidth cannot be moved outside the confines of its physical media. Nor is it storable: a unit of bandwidth that is not used in the time it exists is lost forever.

Natural gas, electricity and bandwidth share another common characteristic: local monopolies of consumer distribution networks. All three are undergoing similar separation of the competitive components that are the subject of trade and the monopoly access components.

### 2.4.2 Market Dynamics

Differences in storability and transportability contribute to different market dynamics. Commodities with low degrees of storability and transportability typically develop into markets with the following characteristics [2]:

- High price volatility
- Brokered instead of exchange-style transactions
- Expensive and illiquid risk management instruments.

These characteristics are not, however, necessary or inevitable. It is certainly possible to alter market dynamics through the introduction of infrastructure that relaxes transportability and storability constraints. But there are other factors that contribute to the style of market that develops. A commodity with many suppliers and buyers work to counter the above characteristics. The evolution of electricity pricing and an electricity trading market offers an example of altering market dynamics with its origin in industry deregulation and government rulings requiring open access to transmission systems in 1996 [1].

Prior to restructuring, electricity pricing was based on a contract path between small local zones. The rate reflected a fictional transmission path agreed upon by transaction participants. However, contract path pricing does not reflect actual power flows through the transmission grid, including loop and parallel path flows. [3].

This pricing strategy was required in part because the transmission system was not equipped with devices that could provide exact power flow control between facilities. This pricing did not consider (or compensate) the actual transmission facilities used and ignored congestion and capacity limits that might make delivery of the contracted electricity impossible. In terms of the bandwidth market, this pricing strategy was equivalent to the flat rate most consumers pay for the best-effort routed delivery of data between a source and destination across the Internet. The fact that much of the Internet infrastructure lacks the ability for dynamic re-provisioning and QoS guarantees is analogous to the flow control problems of the electric transmission grid.

Contract path pricing has since evolved into flow-based pricing due to improvements in both the infrastructure and the creation of trading platforms that can manage complex transactions. Flow-based pricing incorporates many factors in determining the cost of a transaction including the megawatts of electricity, the distance travelled, the parallel paths used to deliver the electricity, the actual transmission facilities used on a hop-by-hop basis, and loops. Such pricing allows all suppliers to the production and delivery of electricity to be compensated. The commodity is no longer electricity, but the pattern that describes its source, destination, timing and delivery. The agreement is the fundamental unit of trade and upon this is built derivatives for managing risk.

By analogy, flow-based pricing is similar to a virtual circuit (such as that created with MPLS) along a specific path with a contracted quality of service. The mapping between the price of a contract and the components that provide the service are evolving just as it did in the electric industry due to improvements in the infrastructure and the creation of efficient platforms for handling complex transactions.

### 2.4.3 Interconnections and Pooling Points

Pooling points are hubs where buyers and sellers connect their networks so that they can participate in bandwidth exchanges. They are frequently located in metropolitan areas (such as New York and London) where trading participants are already likely to have high-speed, high-capacity loops

Each pooling point contains secure, managed space with digital cross connects and monitoring systems. Pooling points are interconnected either with a private network or an overlay network through other service providers and also include connections with the trading infrastructure described in the next section. How a buyer or seller connects to a pooling point depends on the class of bandwidth involved. For example, Band-X requires a Gigabit Ethernet connection to one of its pooling points to participate in Internet access exchanges. The cross connects in the pooling point can be dynamically programmed to connect bandwidth providers and consumers by creating virtual private networks that isolates the traffic of the interested parties and provides security. They also provide the means to monitor the quality of service on established connections and to enforce service level agreements as part of a traded contract. When true QoS guarantees are not available such as the case of Internet access, the pooling point must still provide hardware to monitor the overall quality of the service being supplied. For example, Band-X dedicates one server to each Internet service provider to monitor traffic across the service provider's network. The servers collect information on throughput, latency, hop count and packet loss to a wide range of Internet hosts to ensure the service provider delivers an agreed upon service. Redundancy and backup systems are necessary components to deliver the reliability needed by trading participants. However, it should also be recognized that, despite any local reliability, connecting to and depending on a single pooling point represents a single point of failure for both a consumer and provider of bandwidth. For this reason, it is not uncommon for participants to connect to multiple pooling points.

Bandwidth capacity price declines are certainly due to an oversupply, but by some measures a bandwidth shortage actually exists: “Although there is a surplus in bandwidth capacity due to new wholesale build outs where carriers are using trading to offload their excess bandwidth, the available bandwidth connection points, in many instances, does not match the buyers’ needs. In other words, sellers want to trade their excess bandwidth capacity, but all potential buyers are not interconnected” [5]. This suggests that the development of pooling points is still a vital part of delivering bandwidth trading and that doing so may help reduce the oversupply by increasing demand.

The network structure and the market’s optimal point-to-point routing selection introduce new arbitrage conditions like the so called geographical or network arbitrage. For example, suppose that a monthly forward for January, $2009$, connecting Los Angeles to New York with a STM1-OC3 circuit is offered at $7000. Furthermore, suppose that two other circuits are offered with the same capacity, QoS, expiration date, and duration, but with different endpoints: Los Angeles-Chicago at $4300 and Chicago-New York at $1800. Then it would be possible to extract a profit by selling a Los Angeles-New York contract and replicating it by buying two contracts, Los Angeles-Chicago and Chicago-New York. We refer to this riskless opportunity by network arbitrage.

The network arbitrage condition states that the market price for any point-to-point capacity must be the minimum price over all possible routings. Therefore, the bandwidth contracts cannot be hedged by using the underlying asset like usual derivative instruments and arbitrage argument is possible only between different bandwidth contracts. While traditional methods of mathematical finance may approximate price fluctuations for long term capacity contracts, they fail to describe short term congestion price fluctuations adequately.

## 2.5 Telegeography Dataset Sample

Here we present an example of a typical dataset for Bandwidth contracts specification. We get such a dataset from the marketplace www.telegeograpy.com where the data are organized as follows

- City One First end-point

- Country One (optional display) Associated first end-point country
- City Two Second end-point
- Country Two (optional display) Associated second end-point country
- Region (optional display) A route's regional grouping
- Posted Date Date the price point was made available
- Contract Type Leases or IRU's
- Bandwidth Type One of the capacity types available (e.g., E-1)
- Bandwidth Capacity Capacity in Mbps
- Lease Install Price Installation price (USD)
- Lease Monthly Price Recurring monthly costs (USD)
- IRU Price IRU price (USD)
- IRU Install Price Installation price (USD)
- IRU Annual O&M Annual operations and maintenance charges (USD)
- Notes Additional notes about the price point

It is important to note that installation charges vary significantly between carriers and often are negotiable. Historical data points do not in all cases include install charges; however, assuming an additional upfront payment of the single month's recurring charge is a useful estimate.

Some price points may reflect an extremely low recurring charge but may have a higher than average upfront fee.

# Chapter 3

# Bandwidth Options Pricing

In this chapter we look at the basic features of bandwidth as underlying and related issues on derivatives pricing. Above anything else, we investigate the influence of bandwidth network on pricing European style contingent claims. We show how network topology leads to new arbitrage opportunities, like the so called network or geographical arbitrage, and how it influences hedging strategies for pricing derivatives. First of all, we focus our attention on pricing problems arising on the simplest bandwidth topology, a complete graph of three resources, the triangle network. We propose a solution to such pricing problems derived from financial theory of multi-asset options. We also develop a hedging strategy based on a risk minimization criteria yielding "fair" prices for contingent claims in agreement with standard pricing formulas in a riskless world. We then apply our pricing and hedging strategy to various underlying price processes, starting from the traditional one-period binomial model, *i.e.* the Cox-Ross-Rubinstein model [21]. Subsequently, we extend our model to a multiperiod framework introducing a Binomial Pyramid Lattice as a two dimensional binomial model. Finally, we consider underlying assets following a Geometric Brownian Motion and we propose an analytical solution based on the exchange option theory, see [31].

## 3.1 Bandwidth Underlying

Unlike others markets for non-storable commodities, such as electricity markets, where a spot price does exist and therefore different derivative contracts can be created on it, in bandwidth markets the simplest asset is already a contract. More precisely, the basic contracts exchanged on such a markets are leasing contracts for point-to-point connections with a fixed capacity (transfer mode and bit rate), a set of guarantees on QoS and a start date associated with a duration period, *i.e.* forward contracts.

The inability to store bandwidth and hence the inability to hedge derivative contracts with the underlying is the main reason why the market is incomplete. Nevertheless, there has to be a consistency between prices of different options in order to leave the market free of arbitrage. Thus, trading of bandwidth as a commodity requires standardization of traded products. Yet, no master level agreement on the bandwidth contracts has been done.

Therefore, in order to proceed to stochastic modelling making use of all its financial mathematics instruments, we refer to the forward contract price as the price of the underlying asset, which is storable.

In this way, for a fixed delivery time, the price evolution of a forward contract is the price evolution of the underlying, our spot price.

$$S(t, T_e, T_d, D) = S(t)$$

where $T_d$ is the delivery time, when the bandwidth is available to the owner of the contract, time $t$ varies between $t_0$, the time the contract is signed, and $T_e$, the maturity time, and $D$ refers to the duration period. Usually, in financial markets $T_e$ and $T_d$ are the same time, because one can buy at any time the underlying, but in bandwidth markets there is a minimum delay time between signing a connection contract and practically making use of the connection, because of technical obstacles.

### 3.1.1 Options Pricing via Physical Hedge

Due to the peculiarities of bandwidth markets, moreover its incompleteness, we choose a physical hedge approach to price derivative products. This kind of approach is often used in other capacity markets such as energy, gas, etc., where, at delivery time, the ownership of a contract that gives the rights to physically use the connection to supply the arranged service, is required. The basic idea consists in buying different forward contracts, with the same QoS, to cover possible fluctuation on bandwidth prices in order to hedge financial derivatives.

Assume a provider intends to sell at time $t = t_0$ an option (any types) on a connection contract between two pooling points, starting at time $t = T_d$ and ending at time $t = T_f$. In order to hedge such a derivative and at the same time to price it, he could buy, at $t_0$, a forward contract for each route connecting the two pooling points, direct and alternative, with the same QoS and delivery period, for a portion of the bandwidth capacity fixed by the contract service. The cheapest route through which the connection will be delivered, will depend on the latest market prices available, *i.e.* the shortest forward contract available that we fix at time $T_e$. At that time, the provider will choose the cheapest route, he will sell the portions of bandwidth for the link that he will not use at the current price market, and finally he will buy the remaining portion of bandwidth on the cheapest route at its current market price.

We can summarize this temporal scheme in figure 3.1.

Where we named with *Shortest Forward Time*, the shortest time interval between the time

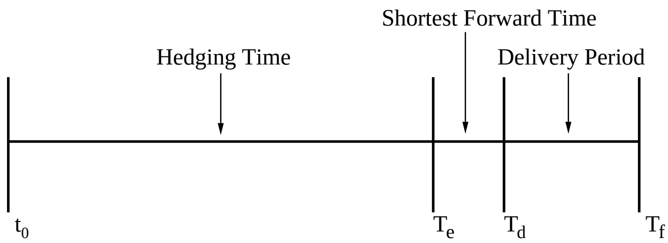



**Figure 3.1:** $t_0 =$ initial time; $T_e =$ exercise time; $T_d =$ delivery time; $T_f =$final time

of delivery of the connection service, $T_d$, and the maturity time, $T_e$, among all the forward contracts available on the market.

## 3.2 Network Arbitrages

In the light of the above considerations, we start considering the simplest network topology where network arbitrage can occur, the case of a complete graph of three resources (the triangle network), figure 3.2, in order to enable inspection of the basic properties without tedious calculations. All the point-to-point connections have the same capacity level and

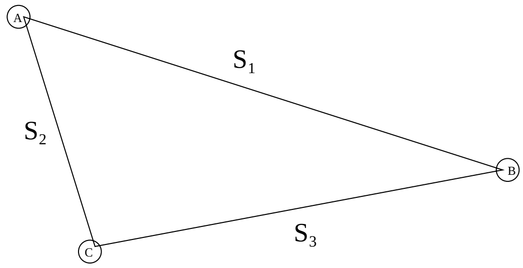



**Figure 3.2:** The triangle network

QoS. At this initial step, we assume a capacity level much larger than that sold in the market, just to avoid congestion problems which will be discussed in a successive step. In such a model, we assume that the price, $S_i$ of each edge is a random variable following the same probability distribution. In this topology, the market spot price, $Y$ for the connection of two fixed nodes is the minimum between the price of the direct edge and the price of the alternative route, *i.e.* the sum of the price of the remaining two edges. For example, the market spot price between the vertices $A$ and $B$ is given by

$$Y_1 = \min\{S_1,\, S_2 + S_3\} \quad .$$

The cheapest path selection is called network arbitrage condition, it can be seen as the optimal behaviour of telecommunications companies. Let us denote the price of the alternative route with $X_1 = S_2 + S_3$, then we have

$$Y_2 = \min\{S_2,\, X_2\} \quad \text{where} \quad X_2 = S_1 + S_3$$
$$Y_3 = \min\{S_3,\, X_3\} \quad \text{where} \quad X_3 = S_1 + S_2$$

According to this definition, the market spot prices are free of network arbitrage ($Y_1 + Y_2 \geq Y_3, Y_2 + Y_3 \geq Y_1, Y_1 + Y_3 \geq Y_2$). The first feature we note is that, independently of the probability distribution of $S_i$, even assumed independent, the respective $Y_i$ would not be, because the market prices depend on the point-to-point routings.

## 3.3 Pricing and Hedging in a Bernoullian World

To better point out the differences between Bandwidth markets and classical financial markets, caused by the network topology, we consider the simplest market model, *i.e.* the binomial asset pricing model. As in a didactic exercise, we start looking at the one period market. In such a model we model forward contract prices referred to a single link connection, our spot prices, in discrete time, assuming that from initial time, $t_0$, to the first step time, $t_1$, the price will change to one of two possible values according to

$$\begin{array}{l} S_0 = S_{t=t_0} \\ 0 < a < S_0 < b \end{array} \quad \text{and} \quad S_1 = S_{t=t_1} = \begin{cases} b & \text{with probability} \quad p \in [0,1] \quad (S_1(\omega_1)) \\ a & \text{with probability} \quad 1-p \qquad (S_1(\omega_2)) \end{cases} \tag{3.1}$$

where $\omega_1$ and $\omega_2$ represent all possible scenarios of the random variable $S_1$, and as usual we name the sample space $\Omega = \{\omega_1, \omega_2\}$.
To complete our first market model, we introduce a *money market* with interest rate $r$, for both borrowing and lending money: a fixed quantity invested in the money market at time $t_0$, becomes $(1+r)$ times that quantity at $t_1$. We also make the natural assumption on $r$ that has to satisfies the following condition

$$a < (1+r)S_0 < b$$

This condition ensures that our model makes sense. In fact, if the condition does not hold, it leads to arbitrage opportunities. For example, if $(1+r)S_0 \leq a$, one could borrow money at interest rate $r$ and invest in the stock, since even in the worst case, the stock price rises at least as fast as the debt used to buy it.
Furthermore, keeping in mind the nature of this underlying, we define the initial price $S_0$ as the discounted expectation of the forward contract price at time $t_0$

$$S_0 = \frac{1}{r+1}\mathbb{E}[S_1] = \frac{1}{r+1}\left[bp + a(1-p)\right] \tag{3.2}$$

where $S_0$ is the price at time zero of a forward contract that expires at time $t_1$. Besides this, the definition (3.2) also known as the *First fundamental Theorem* of Asset Pricing, uniquely identifies $p$ as the risk neutral probability leading ourselves in a risk-neutral world, so as to postpone any issue related to the search of a risk neutral probability measure.

### *3.3.1 Option over a single link*

As a first example of pricing and hedging a derivative on this underlying, we consider a European Call Option, with strike price $K$ with $a < K < b$, and exercise time $T_e = t_1$. This

option confers the right to buy the forward contract at time $t_1$ for $K$ price, and so its value is $S_1 - K$ at time $t_1$ if $S_1 - K$ is positive and zero otherwise.
We denote with

$$C_1(\omega) = (S_1(\omega) - K)^+ \triangleq \max\{S_1(\omega) - K, 0\}$$

the value of the payoff at the exercise time.
Our first goal is to find the "fair" price of this derivative at time $t_0$. Following the traditional way, we have to build up a portfolio, formed by a mix of underlying and money market, whose value at $t_1$ is equal to the option payoff, so that we can define its initial value at time $t_0$ as the "fair" price of the option. In order to do that, if we denote with $C_0$ the value at which we sell the option at time $t_0$, we can use the money gained from the sell of the option to buy a forward contract with the same delivery period, the same QoS, and on the same link, for a portion, $\alpha$, of the total bandwidth capacity settled in the option contract. We could invest the difference $(C_0 - \alpha S_0)$ in the money market. Of course, if $C_0 > \alpha S_0$ then we put the difference in a bank account that will give us $(1 + r)(C_0 - \alpha S_0)$ at time $t_1$, otherwise we borrow the difference from the bank at the same interest rate and the debt will be $(1 + r)(C_0 - \alpha S_0)$ at $t_1$. Finally, we impose that the value of such a portfolio has to be equal to the option payoff. In other words,

$$\begin{aligned} C_1(\omega_1) &= \alpha S_1(\omega_1) + (1 + r)(C_0 - \alpha S_0) \\ C_1(\omega_2) &= \alpha S_1(\omega_2) + (1 + r)(C_0 - \alpha S_0) \end{aligned} \tag{3.3}$$

These are two equations in two unknowns, $\{\alpha, C_0\}$,

$$\begin{cases} b - K &= \alpha b + (1 + r)(C_0 - \alpha S_0) \\ 0 &= \alpha a + (1 + r)(C_0 - \alpha S_0) \end{cases}$$

which can be easily solved, leading for $\alpha$

$$\alpha = \frac{C_1(\omega_1) - C_1(\omega_2)}{S_1(\omega_1) - S_1(\omega_2)} = \frac{b - K}{b - a}$$

Going on with the well known basic financial concepts, this result is the discrete version of the *delta-hedging formula* for options, according to which the number of shares of an underlying asset a hedge should hold is the derivative of the value of the Option payoff with respect to the price of the underlying asset. In our specific case, we should buy at time $t_0$ a forward contract with the same QoS, exercise time, link, and delivery period of the connection settled for the option, but for a portion $\alpha$ of the capacity we have to deliver.

To complete the solution of (3.3), we substitute the $\alpha$ computed before into one of the equations of (3.3). After few elementary steps we get

$$C_0 = \frac{1}{r+1}\left[\left(\frac{S_0(1+r) - S_1(\omega_2)}{S_1(\omega_1) - S_1(\omega_2)}\right) C_1(\omega_1) + \left(\frac{S_1(\omega_1) - S_0(1+r)}{S_1(\omega_1) - S_1(\omega_2)}\right) C_1(\omega_2)\right] \quad (3.4)$$

which is the arbitrage price of a European Call Option in the classical Binomial Asset Pricing Model.

Substituting the relative values for $C_0(\omega)$ and $S_1(\omega)$ we get

$$C_0 = \frac{1}{(1+r)}\frac{b-K}{b-a}[(1+r)S_0 - a]$$

Recalling now the expression (3.2) for $S_0$, we get for the value of the option

$$C_0 = \frac{1}{r+1}(b-K)p$$

We go back again to the equation (3.4) and, substituting the values of $S_1(\omega)$, we can define the factors in brackets as follows

$$\widetilde{p} \triangleq \frac{S_0(1+r) - a}{b-a} \qquad \text{and} \qquad \widetilde{q} \triangleq \frac{b - S_0(1+r)}{b-a} = 1 - \widetilde{p}\,. \quad (3.5)$$

The "weights" just defined, due to the conditions required for the parameters, satisfy

$$\widetilde{p}, \widetilde{q} \geq 0\,, \quad \widetilde{p} + \widetilde{q} = 1 \qquad \text{and} \qquad \widetilde{p}, \widetilde{q} \in [0,1]\,,$$

therefore we can regard them as probabilities of the states $\omega_1$ and $\omega_2$, *i.e.* the risk-neutral probabilities.

Since we started already the analysis with the probability $p$ as risk neutral probability for the forward contract price $S_t$, we would like to get the same probability even for other derivative securities on the same market. In order to verify that, we substitute the expression for $S_0$ in 3.5, getting

$$\widetilde{p} = \frac{(r+1)\frac{1}{r+1}[bp + a(1-p)] - a}{b-a} = p\,, \quad \widetilde{q} = \frac{b - (r+1)\frac{1}{r+1}[bp + a(1-p)]}{b-a} = 1 - p$$

which is obviously correct.

As a double check, we compute again the "fair" price using the standard pricing formula in the riskless world, *i.e* the expectation of the option payoff under the risk-neutral probability $p$,

$$C_0 = \frac{1}{r+1}\mathbb{E}[C_1(\omega)] = \frac{1}{r+1}\left[(b-K)\cdot p + 0\cdot(1-p)\right] = \frac{1}{r+1}(b-K)p$$

that is the risk neutral valuation formula.

### 3.3.2 Option over two links

According to paragraph (2.2.3), topological effects arise whenever there are at least two alternative routes connecting two nodes (cities). In order to investigate the impact that the cheapest path selection has on pricing derivatives, we consider the simplest example of a network topology allowing an alternative route to the one we examined in the previous section.

Figure (3.3) displays such a network, where we assume that each link has the same properties, *i.e.* same QoS and unlimited capacity. In addition, we also assume that processes, driving prices of forward contracts over each single link, have the same probability distribution and they are independent of each other.

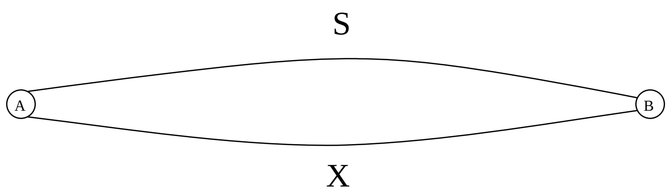



**Figure 3.3:** The two-links network

In such a topology the link, through services are delivered, is no longer uniquely determined. Consequently, we need to impose the cheapest path selection condition (see section 2.2.3) to ensure the absence of network arbitrages. Then, the forward arbitrage-free price of the underlying for every derivative contracts, reads

$$Y = \min\{S,\, X\}$$

For simplicity, we assume that the price $X$, of a forward contract on the alternative link, follows the same binomial model of $S$, *i.e.* $S_0 = X_0$, $S \sim X$ and they are independent of each other.

In such a model, the state space is the set of all pairs s.t.

$$\Omega = \{(\omega_S,\, \omega_X)|\, \omega_S,\, \omega_X \in \{\omega_1,\, \omega_2\}\}$$

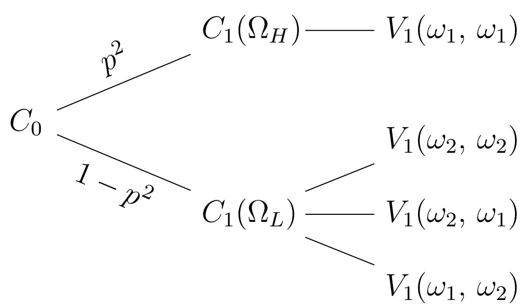
As before, we consider a European Call Option on the fair underlying $Y$, and its payoff rewrites

$$C_1(\omega_S,\, \omega_X) \triangleq \max\{\min\{S(\omega_S),\, X(\omega_X)\} - K,\, 0\}$$

Such a kind of option belongs to the family of Exotic Options, more specifically to min-max options or rainbow options (where each asset is referred to as a colour of the rainbow). Two assets rainbow options were first independently priced by Johnson[13] and Stulz[12] who obtained the formula for a call(put) option on the minimum(maximum) of two risky assets. Later on, Boyle[14] and Rubinstein[15] derived closed form solutions for American style options and suggested many algorithms for numerical solutions.

This options can be perfectly hedged with dynamic hedging strategy, and require complex portfolios (even including vanilla options) for a static hedge as in one step time. The problem we deal with is the following:

If we build a hedging portfolio, as in the Binomial Pricing Model, buying a forward contract for a portion $\alpha$ of the first link (the one with price $S$, that from now on we name Direct Path) and another one for a portion $\beta$ of the second link, (the one with price $X$, which we name Alternative Path), we get the same $2$ values of the option payoff as in a single link case, but with $4$ different combinations of forward prices leading to as many portfolio values, *i.e.*




where $\Omega_H$ and $\Omega_L$ refers to the subsets of the global state space $\Omega$, respectively to the one the option is exercised and the one the option is not, *i.e.* $\Omega = \Omega_H \oplus \Omega_L$ . Substituting the expressions of each portfolio, we get

$$C_1(\omega_S,\,\omega_X) = \begin{cases} C_1(\omega_1,\omega_1) & = \quad b-k = \alpha b + \beta b + (C_0 - \alpha S_0 - \beta X_0) \\ \left.\begin{array}{c} C_1(\omega_1,\omega_2) \\ C_1(\omega_2,\omega_1) \\ C_1(\omega_2,\omega_2) \end{array}\right\} & = \quad 0 \quad = \begin{cases} \alpha b + \beta a + (C_0 - \alpha S_0 - \beta X_0) \\ \alpha a + \beta b + (C_0 - \alpha S_0 - \beta X_0) \\ \alpha a + \beta a + (C_0 - \alpha S_0 - \beta X_0) \end{cases} \end{cases}$$

Therefore, the network arbitrage condition leads to the lack of a one-to-one correspondence between the option payoff and the hedging portfolio value. The violation of such a correspondence univocally compromises the identification of a perfect hedging strategy, at least using only a portfolio of the two-risky assets. In such a “world”, a contingent claim cannot be perfectly replicated, hence the issue turns into choosing a criterion which defines an optimal hedging strategy yielding a fair price for the derivative.

A possible solution to this problem could be to define a risk measure, and then looking for a hedging strategy that minimizes the risk. In our case, it is also possible to build up an alternative hedging strategy leading on the one hand to a portfolio whose discounted value at time $t = 0$ is equal to the expected value of the option payoff with respect to the risk-neutral probability measure. On the other hand, it is possible to find a particular risk measure that is minimum for such a strategy as we show in section 3.3.3.

On the basis of the Binomial Pricing model, we associate each different payoff of an option with an “expected portfolio” defined as the expected value of all portfolios corresponding to the same option payoff. In this way, we get as many portfolios as option payoffs and therefore we can make use again of the Binomial Pricing Theory.

Since the portfolio value at time $t = 1$, can be represented as

$$V_1(\omega_S,\omega_X) \triangleq \alpha S_1(\omega_S) + \beta X_1(\omega_X) + (C_0 - \alpha S_0 - \beta X_0)$$

with the condition $V_0 = C_0$, we compute the two expected portfolios corresponding to the two option payoffs

$$\begin{aligned} \widetilde{V}_1(\Omega_H) &\triangleq \mathbb{E}[V_1(\omega_S,\omega_X)|(\omega_S,\omega_X) \in \Omega_H] \\ \widetilde{V}_1(\Omega_L) &\triangleq \mathbb{E}[V_1(\omega_S,\omega_X)|(\omega_S,\omega_X) \in \Omega_L] \end{aligned}$$

where $\Omega_H = (\omega_1,\omega_1)$ and $\Omega_L = \{(\omega_1,\omega_2),(\omega_2,\omega_1),(\omega_2,\omega_2)\}$ refer to a partition of the global state space, $\Omega$. Respectively, $\Omega_H$ denotes the set of the states corresponding to the payoff $b - K$, while $\Omega_L$ the one whose payoff is $0$.

At this point, we are able to build up the system of equations allowing us to determine the optimal parameters of the hedging portfolio and hence the option fair price. In order to simplify the calculation, it is helpful to look at the symmetry of the problem. Due to both symmetries of the specific network topology and price process distributions, it is possible to reduce the set of parameters from $2$ to $1$, imposing the condition $\alpha = \beta$. In fact, the solution could not depend of which path one identifies as the direct or alternative path since they are identical. The system of equations we have to solve become then

$$\begin{cases} b - k &= \alpha b + \alpha b + (C_0 - \alpha S_0 - \alpha X_0) \\ 0 &= \alpha(a+b)\frac{2p(1-p)}{1-p^2} + 2\alpha a \frac{(1-p)^2}{1-p^2} + C_0 - \alpha S_0 - \alpha X_0 \end{cases}$$

whose solutions read

$$\begin{cases} \alpha &= \frac{b-K}{2(b-a)}(1+p) \\ C_0 &= \frac{b-K}{b-a}\left[S_0(1+p) - (bp+a)\right] \end{cases}$$

and keeping in mind that $S_0 = \mathbb{E}[S_1(\omega)] = bp + a(1-p)$ we get for the European Call price

$$C_0 = (b-K)p^2 \tag{3.6}$$

In order to bring up the value of the solution, we compute the price of the option directly by the risk neutral valuation formula, *i.e.* the expectation of the option payoff under the risk neutral probability,

$$C_0 = \mathbb{E}[C_1(\omega_S, \omega_X)] = (b-K)\cdot p^2 + 0 \cdot (1-p^2) = (b-K)p^2$$

This result confirms the validity of our choice for the optimal hedging strategy which, even if it does not perfectly hedge the derivative, leads for construction to the fair price for the option minimizing the risk of losing money.

The figure (3.4) shows how the price of a bandwidth derivate depends on network topology. The introduction of an alternative path for delivering the service, lowers the fair price of an option as a function of the parameter $p$. This variation comes from the change of underlying which, because of the condition of absence of geographical arbitrage, has a lower expected value, in fact

$$\mathbb{E}[Y] = \mathbb{E}[\min\{S_1, X_1\}] = bp^2 + a(1-p^2) < bp + a(1-p) = \mathbb{E}[S_1]$$

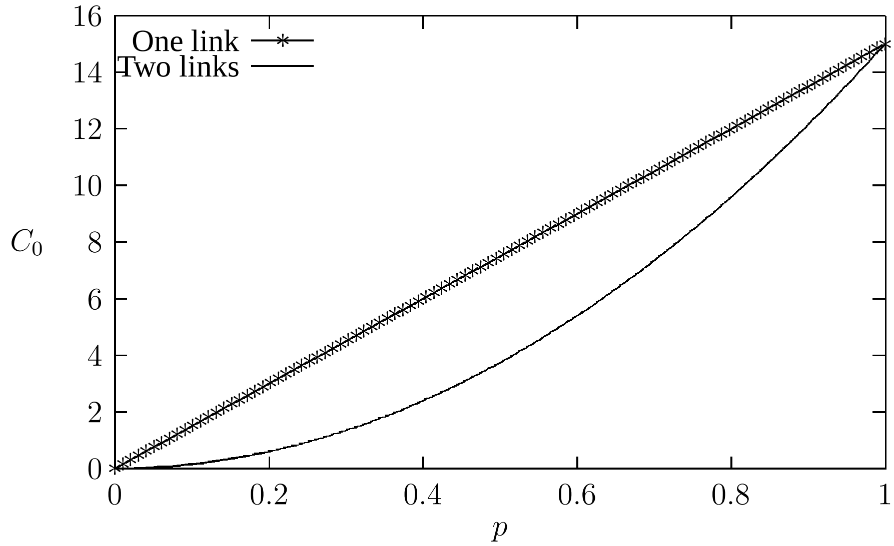



**Figure 3.4:** European Call Option price in a one-link topology vs two-link topology

### ***Put-Call Parity Relationship***

In order to avoid further arbitrage opportunities, and, at the same time, to find a fair price for an European Put Option, we poof the Put-Call parity for such a pricing model. Put-Call parity is a relationship that must exist between the prices of European Call and Put options on the same underlying with the same strike price and expiration time. We denote respectively the prices of European Put and Call Options on the minimum of the two price processes $S$ and $X$, with both strike price $K$ and time to maturity $T_e - t \triangleq \tau$, with $P(S, X, K, \tau)$ and $C(S, X, K, \tau)$.

The Put-call parity relationship writes

$$P(S, X, K, \tau) = C(S, X, K, \tau) - C(S, X, 0, \tau) + Ke^{-r\tau} \tag{3.7}$$

where $C(S, X, 0, \tau)$ refers to call options with $0$ strike price and $r$ denotes the constant market interest rate.

To prove this relationship, we consider two portfolios consisting of:

- Portfolio A. A put option on the minimum of the two underlyings with strike price $K$
- Portfolio B. A call option on the minimum of the two underlyings with strike price $K$, a short position on a call option on the minimum of the two underlyings but with strike price $0$ and a discount bond which pays $K$ at maturity.

We have to show that the two portfolios have the same value at maturity in any state of the world, if it holds they must have the same value at initial time.

- $\min(S, X) = S < K$. Portfolio A is equal to $K - S$ whereas Portfolio B pays $0 - S + K = K - S$
- $\min(S, X) = X < K$. Portfolio A pays $K - X$ and again Portfolio B has the same value of A, *i.e.* $0 - X + K = K - X$
- $\min(S, X) > K$. Portfolio A is worth $0$ whereas Portfolio B is equal to $\min(S, X) - K + \min(S, X) + K = 0$ for any $S, X, K \in \mathbb{R}^+$

Since the two portfolios have the same value at maturity, they must have the same initial value, then the put-call parity relationship holds, and equation (3.7) defines the price of a put on the minimum of the two risky assets.

### 3.3.3 Risk-minimization Hedging Strategy

The use of minimization-risk criteria to determine a hedging strategy is a widespread habit in practice. They were first introduced by Föllmer and Sondermann [16] and developed further by Schweizer [17, 18].

Real markets are often incomplete, they have transaction costs and only discrete hedging times are allowed. For example, even in a Black-Scholes framework, where positions are only instantaneously risk-free, minimization-risk criteria are needed when one faces the reality.

Due to such issues, different criteria of risk minimization coupled with various risk measures have been developed in financial theory. Among many, we focus our attention on quadratic risk minimization criteria, because of their well studied properties and explicit functions, see [19] and [20] for a survey, and in particular, on the quadratic local risk-minimization which consists in minimizing the hedging cost of a strategy at each time step.

In practice, the local risk-minimization criterion relies on controlling the one step incremental risk, by minimizing the $L^2$-norm $\mathbb{E}[(C_{k+1} - C_k)^2|\mathcal{F}_k]$ for all $k$, where $C_k$ denotes the cumulative cost of the hedging strategy up to time step $k$. In order to make such a criterion suitable for our model, we have to slightly modify it without reducing its efficacy, so as to

include a second risky asset and mainly the absence of the one-to-one correspondence between the option-payoff and the portfolio value.
In our model, the cost of a hedging strategy at each time step is linked to the difference between the option payoff and one of the corresponding portfolio values, since there could be many of them for any payoff. Looking at the case of an option over two links, see paragraph (3.3.2), we have at time $t = 1$, three different portfolio values linked up to the option payoff $C_1(\Omega_L)$. Therefore, we define the cost associated with the hedging strategy at that time step with the difference

$$(C_1(\Omega_L) - V_1(\omega_S, \omega_X)) \qquad \text{with} \qquad (\omega_S, \omega_X) \in \Omega_L$$

Keeping in mind the quadratic risk minimization criteria, we set out a risk measure to minimize by looking at the average quadratic cost in such a way

$$\mathbb{E}[(C_1(\Omega_L) - V_1(\omega_S, \omega_X))^2 | \mathcal{F}_0] \tag{3.8}$$

In order to minimize such an expectation and at the same time to build up a hedging strategy suitable with the Binomial Pricing Theory, we first introduce an "expected portfolio" associated with the option payoff, that in this case writes

$$\widetilde{V}_1(\Omega_L) \triangleq \mathbb{E}[V_1(\omega_S, \omega_X) | (\omega_S, \omega_X) \in \Omega_L] \tag{3.9}$$

Then we impose the condition

$$\widetilde{V}_1(\Omega_L) = C_1(\Omega_L)$$

as a restriction over the portfolio parameters. Such a restriction ensures, from one hand the minimization of (3.8) since the average quadratic cost plays the role of a "variance" that is minimized by the mean, and from the other hand the applicability of the Binomial Pricing Theory since we get the one-to-one correspondence (option payoff - portfolio value) back.
Such a methodology strengthens what we already did in pricing an option over two links, see paragraph (3.3.2), and we will further extend it first to the multiperiod case and then to the triangle network case.

# 3.4 Multiperiod Model

In this section, we extend the one-period securities models to the respectively multiperiod versions. In order to get the mathematical background ready, we introduce $N+1$ trading dates, $t_0 = 0, \ldots, N+1 = T_N$ and a time step $\delta t = t_{n+1} - t_n = (T_N - t_0)/(N+1)$. Furthermore, similar to the one-period version, we have a finite sample space $\Omega$ of $M$ elements, $\Omega = \{\omega_1, \ldots, \omega_M\}$, which represents the possible states of the world.

We extend, also, the concept of interest rate, $r$, as the constant interest rate over the time interval (n,n+1).

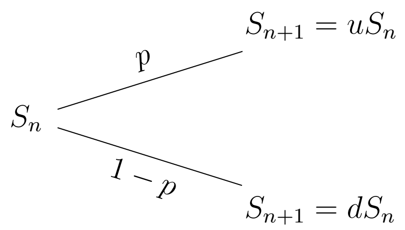
The key element is to deal with the concepts of arbitrage opportunities and risk neutral valuations in a multiperiod framework. This can be done by specifying how the investors learn about the true state of the world on intermediate trading times up to the maturity. In order to show how, we build up an information structure which models how information is revealed to investors in terms of subsets of the sample space $\Omega$.

## 3.4.1 Option over Single Link - Multiperiod

First of all, we look at the one period model for a single link as described in section (3.3). In order to do that, we follow a standard multiperiod model, in particular the model of Cox-Ross-Rubinstein [21].

We assume that the market contain a riskless asset whose rate of return, $r$, is constant thought time periods and state of the world and a risky asset, stock, whose rate of return between the time $t_n$ and the time $t_{n+1}$ can be either $u$, or $d$ with yet $d < r+1 < u$.

For the sake of simplicity, we rewrite what already done previously, see Eq. 3.1, where the model for the price had fixed values to a multiplicative model,




where $u, d$ are defined coherently with 3.3 and they still are functions of the parameter $p$, so as to compare the results with the one period case. Keeping in mind the one period model, we get

$$u : u(p) = \frac{S_1(H)}{S_0} = \frac{b(r+1)}{[bp + a(1-p)]} \tag{3.10a}$$

the same for $d$

$$d : d(p) = \frac{S_1(L)}{S_0} = \frac{a(r+1)}{[bp + a(1-p)]} \tag{3.10b}$$

In such a way, we have the same degree of freedom to choose the parameter $p$, which can be determined with a fit of the dataset.

As in the one period case, the no arbitrage condition leads to a different probability measure independent of time step.

$$\widetilde{p} = \frac{(r+1) - d}{u - d} \qquad \text{and} \qquad 1 - \widetilde{p} = \frac{u - (r+1)}{u - d}$$

Under such a measure, since the price $S_n$ depends only on the number of up-moves, is a random variable which follows a binomial distribution with parameters $n$ and $\widetilde{p}$:

$$P(S_n = u^j d^{n-j} S_0) = \binom{n}{j} \widetilde{p}^j (1 - \widetilde{p})^{n-j}$$

where $j$ denotes the n, and

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} \quad .$$

Then, regarding the value of a European Call Option, we use directly the standard pricing formula as in section 3.3 which after $N$ steps reads:

$$C_0 = \frac{1}{(1+r)^N} \mathbb{E}^Q[(S_N - K)^+]$$

and hence

$$C_0 = \frac{1}{(1+r)^N} \sum_{j=0}^{N} \binom{N}{j} \widetilde{p}^j (1 - \widetilde{p})^{N-j} (u^j d^{n-j} S_0 - K)^+$$

where $Q$ is the unique martingale measure obtained from the probabilities $\widetilde{p}$.

Commonly, it can be written in a more suitable way. We introduce a useful number, $\zeta$, which is the minimum number of up-moves that are required over $N$ periods in order to have the price $S_N$ greater than the strike price $K$, *i.e.*

$$\zeta = \inf\{j \in \mathbb{N} | u^j d^{n-j} S_0 > K\}$$

and in others words

$$\zeta = \left\lceil \frac{\ln K/S_0 d^N}{\ln u/d} \right\rceil$$

where $\lceil \cdot \rceil$ denotes the next integer approximation. We than obtain after few algebra

$$C = S_0 D\left(N, \zeta; \frac{\widetilde{p}u}{1+r}\right) - \frac{K}{(1+r)^N} D(N, \zeta; \widetilde{p}) \tag{3.11}$$

where

$$D(N, \zeta, \widetilde{p}) = \sum_{j=\zeta}^{N} \binom{N}{j} \widetilde{p}^j (1-\widetilde{p})^{N-j}$$

is the probability that a random variable with the binomial distribution of parameters $N$ and $\widetilde{p}$ takes values greater than $\zeta$, *i.e.* the complementary binomial distribution.
Looking at the Eq. (3.11), the first term gives the discounted expectation of the asset (bandwidth) price at expiration given that the call expires "in the money" and the second term gives the present value of the expected cost incurred by exercising the call, where the expectation is taken under the martingale risk neutral measure.
We consider now the hedging strategy issue. We build up a self-financing portfolio as in paragraph 3.3, whose value at each time step is defined as

$$V_n = x_n(1+r) + \alpha_n S_n$$

where $x_n$ is the amount of money invested in a bank account at time step $n-1$ and kept until time step $n$ and, as before, $\alpha_n$ is a forward contract on the portion of the total bandwidth capacity settled in the option contract. We also define the portfolio strategy as the follow stochastic process

$$\{h_n = (x_n, \alpha_n);\ n = 1, ..., N\}$$

As already done in one period binomial model, we compute the parameter values at each step, starting from the last ones, and hence equalling the portfolio value at time $t = T$ to the option payoff.

$$V_T = (S_N - K)^+$$

and $V_n$ can computed recursively. In order to write down an algorithm, we index each node of the tree by a pair $(n, j)$ where $n$ represent the time step and $k = 0, ..., n$ is the number of up-moves occurred until $n$, than

$$V_n(j) = \frac{1}{1+r}\left[\widetilde{p}V_{n+1}(j+1) + (1-\widetilde{p})V_{n+1}(j)\right]$$

where $V_n(j)$ is the value of the portfolio corresponding to the underline $S_n = S_0 u^j d^{N-j}$. More precisely, substituting the value of $\widetilde{p}$ and the explicit expression of the portfolio, we get

$$\begin{cases} x_n(j) = \frac{1}{1+r}\frac{uV_n(j)-dV(j+1)}{u-d} \\ \alpha_n(j) = \frac{V_n(j+1)-V_n(j)}{(u-d)S_{n-1}} \end{cases}$$

as in the one period case.

### 3.4.2 Option over Two Links - Multiperiod

We now consider the two links problem from the pricing point of view. We extend, as already done for the single link case (3.3.2), the one period two links pricing model to a multiperiod. Therefore, we have to define a lattice for approximating its behaviour in discrete time.
We consider $S$ and $X$ (*i.i.d*) as in 3.3.2. Remembering the definitions of $S_0$ e $X_0$ e $u$ e $d$ in single link multiperiod, the parameter $p$ refers to the equivalent martingale probability.
Hence, we can directly extend to a multiperiod framework, and we compute the value of a

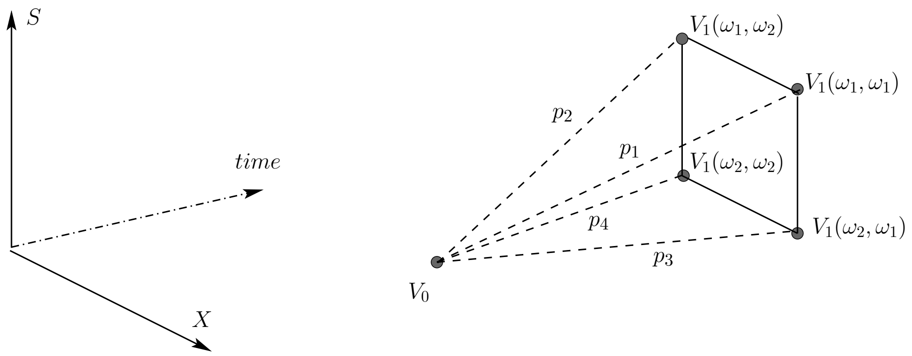



**Figure 3.5:** One Step Binomial Pyramid

European option at a fixed node as the discounted expectation of the four values at the next step. In other words, starting from the last we have

$$V_N(S_N, X_N) = C_N(S_N, X_N, K)$$

Payoff values of the option can be calculated for each final node since each of these contains a value for $S_N$ and $X_N$. It is sufficient to discount these back through the "tree" in the classical way, remembering that the values at four nodes are needed for each step back (rather than two in the single asset tree).

Therefore, the general algorithm to go backwards writes

$$V_n(S_n, X_n) = \frac{1}{r+1}\left[p_1 V_{n+1}(u_S S_n, u_X X_n) + p_2 V_{n+1}(u_S S_n, d_X X_n) + \right. \tag{3.12}$$

$$\left. + p_3 V_{n+1}(d_S S_n, u_X X_n) + p_4 V_{n+1}(d_S S_n, d_X X_n)\right] \tag{3.13}$$

For example, if we apply it to the one step case, illustrated in Fig. (3.5), we obviously get the same result already found, see Eq.(3.6)

$$V_0(S_0, X_0) = \widetilde{p}^2(b-K) + \widetilde{p}(1-\widetilde{p})\cdot 0 + \widetilde{p}(1-\widetilde{p})\cdot 0 + (1-\widetilde{p})^2 \cdot 0 = \widetilde{p}^2(b-K)$$

where we also remembered that $S$ and $X$ are ***i.i.d.*** and than $u_S = u_X = u(p)$ and $d_S = d_X = d(p)$ calculated respectively in Eq.(3.10a, 3.10b).

**Two Step Binomial Pyramid**

In order to get a general result which can be easily extended to the continuum, so as to compare to Black-Scholes results, we change our approach to such lattices, assuming that they are discretization of a continuous two dimensional stochastic process, where $S_n$ and $X_n$ are the binomial discretization of two Geometric Brownian Motion. Therefore, we consider

This can be done in two ways: first, we can estimates mean and variance from market data in a risk neutral world and subsequently we impose such conditions to compute the probabilities of the binomial pyramid.

**General case**

Independently from the stochastic process we choose to describe market data, we estimates the first two moments of the empirical probability distribution and we adapt the parameters of the Binomial Pyramid.

For the sake of simplicity, in order to link the empirical local mean and variance to the discrete model, we consider log-normal returns instead of the single asset bandwidth price, *i.e.* $s_n = \ln\frac{S_{n+1}}{S_n}$ and $x_n = \ln\frac{X_{n+1}}{X_n}$ where $n = 0, \ldots, N$. In such a way, the step sizes in both

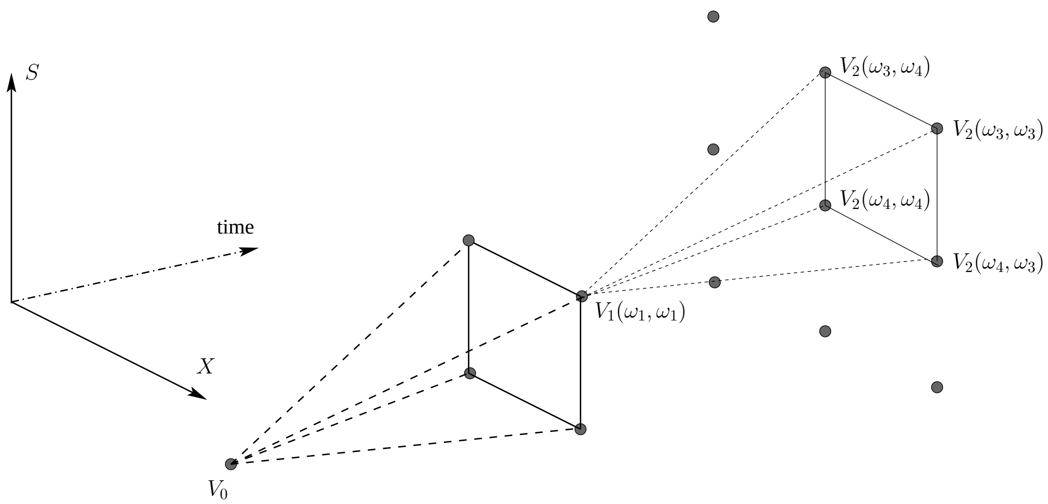



**Figure 3.6:** Two Step Binomial Pyramid

dimensions are of constant sizes, rather than proportional to prices, and this notation simplifies the computation of the probabilities.

$$\mathbb{E}[s_n] = m_s \quad \text{Var}(s_n) = \sigma_s^2$$

$$\mathbb{E}[x_n] = m_x \quad \text{Var}(x_n) = \sigma_x^2$$

where obviously $\delta t = t_{n+1} - t_n$. Since in a general model the two price variables can be correlated, we introduce a local correlation coefficient $\rho$, such that,

$$\rho = \mathbb{E}[s_n, x_n]$$

which we assume constant thought time.

We also adopt the convenient tree-symmetry condition choice,

$$u_S = \frac{1}{d_S} \qquad u_X = \frac{1}{d_X}$$

for convenience we write $\gamma_s = \ln u_S$ and $\gamma_x = \ln u_X$ that spares us a bit of algebra.

At this point, we are able to define the bivariate joint distribution function of the pair $(s_n, x_n)$ as

$$(s_n, x_n) = \begin{cases} \gamma_s & \gamma_x & \text{with prob. } p_1 \\ \gamma_s & -\gamma_x & \text{with prob. } p_2 \\ -\gamma_s & -\gamma_x & \text{with prob. } p_3 \\ -\gamma_s & \gamma_x & \text{with prob. } p_4 \end{cases}$$

where the unknown probability states have to be determined by equating the discrete first two moments to the corresponding moments of the empirical distribution, with an accuracy, including the correlation coefficient, we get

$$\gamma_s(p_1 + p_2 - p_3 - p_4) = m_S \tag{3.14a}$$

$$\gamma_s^2(p_1 + p_2 + p_3 + p_4) + O(\delta t) = \sigma_S^2 \tag{3.14b}$$

$$\gamma_x(p_1 - p_2 - p_3 + p_4) = m_X \tag{3.14c}$$

$$\gamma_x^2(p_1 + p_2 + p_3 + p_4) + O(\delta t) = \sigma_X^2 \tag{3.14d}$$

$$\gamma_s\gamma_x(p_1 - p_2 + p_3 - p_4) = \rho \tag{3.14e}$$

Writing the $O(\delta t)$ terms in variance equations, we supposed that the mean is a linear function of $\delta t$ as in the most financial stochastic processes having a drift term. Actually, if we look at a single asset, *e.g.* $s_n$, the one step mean is $p\gamma_s - (1-p)\gamma_s$, and after $N$ steps we have

$$\mathbb{E}[S_N] = N(p\gamma_s - (1-p)\gamma_s) = \frac{T}{\delta t}(p\gamma_s - (1-p)\gamma_s) = \mu_s T \Longrightarrow$$
$$(p\gamma_s - (1-p)\gamma_s) = \mu_s\delta t = \mathbb{E}[s_n]$$

For this reason, the mean square term of the variance has to be omitted if we seek an approximation up to $O(\delta t)$. Furthermore, the entire Ito analysis is based on rejection of terms of order higher than $\delta t$.
Finally, so that equations (3.14b) and (3.14d) are consistent, we must set

$$\gamma_s^2 = \sigma_S^2 \qquad \text{and} \qquad \gamma_x^2 = \sigma_X^2$$

which is not a strained choice, as we will show below in a Black-Scholes environment. After these consideration we achieve the following four independent equations for the “pyramid” probabilities

$$\begin{aligned}
p_1 + p_2 - p_3 - p_4 &= \frac{m_s}{\gamma_s}\\
p_1 - p_2 - p_3 + p_4 &= \frac{m_x}{\gamma_x}\\
p_1 + p_2 + p_3 + p_4 &= 1\\
p_1 - p_2 + p_3 - p_4 &= \frac{\rho}{\gamma_s\gamma_x} \quad .
\end{aligned}$$

The above system of equations is a linear algebraic system which solution gives

$$p_1 = \frac{1}{4}\left(1 + \frac{m_s}{\gamma_s} + \frac{m_x}{\gamma_x} + \rho\right) \tag{3.15a}$$

$$p_2 = \frac{1}{4}\left(1 + \frac{m_s}{\gamma_s} - \frac{m_x}{\gamma_x} - \rho\right) \tag{3.15b}$$

$$p_3 = \frac{1}{4}\left(1 - \frac{m_s}{\gamma_s} - \frac{m_x}{\gamma_x} + \rho\right) \tag{3.15c}$$

$$p_4 = \frac{1}{4}\left(1 - \frac{m_s}{\gamma_s} + \frac{m_x}{\gamma_x} - \rho\right) \tag{3.15d}$$

These probabilities are the discrete martingale probabilities that joint with the Binomial Pyramid Algorithm define by Eq. (3.12), allow us to price any derivative depending on two assets.

**Example CRR-GBM**

A practical example of the above lattice pricing method is the bivariate CRR model. As for the one asset model, where it is an approximation of a Geometric Brownian Motion. also in the two assets case, the binomial Pyramid is an approximation of a Bivariate Geometric Brownian motion.

Following the CRR continuous limit description, we assumed that the continuous counterpart of the underlying bandwidth prices is normally distributed in such a way

$$\ln \frac{S_{t+\delta t}}{S_t} \sim N\left[\left(r - \frac{\sigma_s^2}{2}\right)\delta t,\ \sigma_s^2 \delta t\right]$$

and similarly for $X_t$ with the corresponding $\sigma_x$ parameter.

At the same way, we can compute the discrete parameters for both $S_n$ and $X_n$ getting

$$u_i = e^{\sigma_i\sqrt{\delta t}},\ d_i = e^{-\sigma_i\sqrt{\delta t}} \quad \text{and} \quad \gamma_i = \sigma_i\sqrt{\delta t}$$

where $i = s, x$

Tracing the steps to lead the lattice probabilities (3.15), we get

$$p_1 = \frac{1}{4}\left[1 + \sqrt{\delta t}\left(\frac{r - \frac{\sigma_s^2}{2}}{\sigma_s} + \frac{r - \frac{\sigma_x^2}{2}}{\sigma_x}\right) + \rho\right] \tag{3.16a}$$

$$p_2 = \frac{1}{4}\left[1 + \sqrt{\delta t}\left(\frac{r - \frac{\sigma_s^2}{2}}{\sigma_s} - \frac{r - \frac{\sigma_x^2}{2}}{\sigma_x}\right) - \rho\right] \tag{3.16b}$$

$$p_3 = \frac{1}{4}\left[1 - \sqrt{\delta t}\left(\frac{r - \frac{\sigma_s^2}{2}}{\sigma_s} + \frac{r - \frac{\sigma_x^2}{2}}{\sigma_x}\right) + \rho\right] \tag{3.16c}$$

$$p_4 = \frac{1}{4}\left[1 + \sqrt{\delta t}\left(\frac{r - \frac{\sigma_x^2}{2}}{\sigma_x} - \frac{r - \frac{\sigma_s^2}{2}}{\sigma_s}\right) - \rho\right] \tag{3.16d}$$

**Another Way**

An alternative, popular arrangement in setting the parameters in a bidimensional tree, is to start the other way round, *i.e.* choosing the transition probabilities to be equal, $p = 1/2$ and deriving a compatible pair of up and down jumps, see [21]. The same can be done for a three-dimensional tree as in the Binomial Pyramid model, choosing the four probabilities equal to $1/4$ and finding each of the nodal values corresponding to these probabilities, [22].

**Uncorrelated Assets**

We firstly consider the case of two uncorrelated bandwidth price processes. Using the same notation of the earlier sections, we name the one step up-moves with suffix $()_u$ and respectively the down-moves with pedix $()_d$. At this point we just have to match local means (drift) and variance (volatilities) to the tree, so that we can write,

$$\mathbb{E}[s_n] = m_s\delta t = \tfrac{1}{2}s_u + \tfrac{1}{2}s_d$$
$$\text{Var}(s_n) = \tfrac{1}{4}s_u^2 + \tfrac{1}{4}s_d^2$$

which solves to

$$s_u = m_s\delta t + \sigma_s\sqrt{\delta t} \quad \text{and} \quad s_d = m_s\delta t - \sigma_s\sqrt{\delta t}$$

and similarly for $x_n$

$$x_u = m_x\delta t + \sigma_x\sqrt{\delta t} \quad \text{and} \quad x_d = m_x\delta t - \sigma_x\sqrt{\delta t}$$

**Correlated Assets**

If the two bandwidth prices are correlated, it is a bit more difficult to find the nodal values since the value of $x_n$, for example, will depend on the value of $s_n$. We suppose that $s_n$ can move up ($s_u$) or down ($s_d$) with equal probability as before, while $x_n$ can move to $x_A$ or $x_B$ if $s_n$ moves up or respectively to $x_C$ or $x_D$ if $s_n$ moves down. We have to impose the condition $x_A x_D = x_B x_C$ in such a way to get a recombining tree.

On the basis of the results found in the uncorrelated case, we write the Wiener processes approximating the binomial assets moves for the correlated case in such a way

$$\begin{aligned} s_n &= m_s \delta t + \sigma_s \sqrt{\delta t} z_s \\ x_n &= m_x \delta t + \sigma_x \sqrt{\delta t} z_x = m_x \delta t + \sigma_x \sqrt{\delta t} (\rho z_s + \sqrt{1-\rho^2} z_y) \end{aligned}$$

where $z_s$ is the binomial random variable associated to $s_n$, *i.e.* $z_s = 1$ with probability $p$ and $z_s = -1$ with probability $(1-p)$, $z_x$ is the correlated random variable associated to $x_n$ and $z_y$ is a third binomial random variable independent of $z_s$ and $z_x$.

A heuristic procedure can be used to find the nodal values, putting $z_3$ equal to $+1$ and $-1$ in correspondence to each of the independent values for $z_s$, leading to

$$s_u = m_s \delta t + \sigma_s \sqrt{\delta t} \quad \text{and} \quad s_d = m_s \delta t - \sigma_s \sqrt{\delta t}$$

as for the uncorrelated case, and

$$\begin{aligned} x_A &= m_x \delta t + \sigma_x \sqrt{\delta t} (\rho + \sqrt{1-\rho^2}) \\ x_B &= m_x \delta t - \sigma_x \sqrt{\delta t} (\rho - \sqrt{1-\rho^2}) \\ x_C &= m_x \delta t - \sigma_x \sqrt{\delta t} (\rho + \sqrt{1-\rho^2}) \\ x_D &= m_x \delta t + \sigma_x \sqrt{\delta t} (\rho - \sqrt{1-\rho^2}) \end{aligned}$$

for the four different correlated values.

### 3.4.3 Option over a triangle network

To better examine the geographical arbitrage constriction, we consider a more realistic telecommunication subnetwork, *i.e.* the triangle network, as in Figure 3.2.

Starting again with the one step case, we model the forward contract price of a single link with the binomial variable

$$S_i = \begin{cases} b & \text{with probability} \quad p \in [0,1] \\ a & \text{with probability} \quad 1-p \end{cases} \qquad i = 1, 2, 3\,.$$

We impose the condition $b \geq 2a$ so that, the price for the alternative route can be lower than the direct one. In this specific case, the variables $X_i$ and $Y_i$ are trinomial random variables distributed according to the following laws

$$X_i = \begin{cases} 2b & \text{with prob.} & p^2 \\ a+b & " & 2p(1-p) \\ 2a & " & (1-p)^2 \end{cases}, \quad Y_i = \begin{cases} b & \text{with prob.} & p^2(2-p) \\ 2a & " & p(1-p)^2 \\ a & " & (1-p) \end{cases}.$$

To show that the $Y_i$'s are not independent variables, we compute the respective covariance

$$\mathrm{cov}(Y_1, Y_2) = \mathbb{E}[(Y_1 - \mathbb{E}[Y_1])(Y_2 - \mathbb{E}[Y_2])] = \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_1]^2 \text{ (with } \mathbb{E}[Y_1] = \mathbb{E}[Y_2])$$

computing the first term

$$\mathbb{E}[Y_1 Y_2] = a^2\mathbb{P}(Y_1 = a, Y_2 = a) + 2ab\mathbb{P}(Y_1 = a, Y_2 = b) + 4a^2\mathbb{P}(Y_1 = 2a, Y_2 = 2a) +$$
$$+4a^2\mathbb{P}(Y_1 = a, Y_2 = 2a) + 4ab\mathbb{P}(Y_1 = 2a, Y_2 = b) + b^2\mathbb{P}(Y_1 = b, Y_2 = b) =$$

$$= a^2\mathbb{P}(S_1 = a, S_2 = a) + 2ab\mathbb{P}(S_1 = a, S_2 = b \cup \{S_1 = a, S_3 = a\}^c) + 4a^2\mathbb{P}(\emptyset) +$$
$$4ab\mathbb{P}(\emptyset) + 4a^2\mathbb{P}(S_1 = a, S_2 = b \cup \{S_1 = a, S_3 = a\}) + b^2\mathbb{P}(S_1 = b, S_2 = b) =$$

$$= a^2(1-p)^2 + 2ab(1-p)p^2 + 4a^2p(1-p)^2 + b^2p^2$$

then

$$\mathrm{cov}(Y_1, Y_2) = a^2(1-p)^2 + 2ab(1-p)p^2 + 4a^2p(1-p)^2 + b^2p^2 - [bp^2(2-p) +$$
$$+2ap(1-p)^2 + a(1-p)]^2$$

which is not $0$ under the parameter conditions $0 < p < 1$ e $b > 2a$.

As first application, we look for a fair price for a Bandwidth option on this network. In addition to the routing options described in the introduction section some bandwidth service contracts include also other option type characteristics. For instance, the seller can have a right to disconnect the service for a predefined penalty payment. These rights can be modelled as bandwidth options and, therefore, the understanding of option pricing is important in the bandwidth markets even though traded option contracts do not exist. In the same way a buyer can choose whether to buy the service at a fixed price (strike price). In both cases, a bandwidth option is equivalent to a European call option.

For simplicity, we consider a single-period market model with null interest rate, $r = 0$, with start time, $t = t_0$ and end time $t = T$. Then according to our time steps conventions, the value at maturity for an option on a contract for the connection A-B is

$$C(Y_1(T_e)) = (Y_1(T_e) - K)^+ \quad ,$$

where $K$ is the *strike price*, that satisfies, in this specific example, the condition $a < K < 2a$.

In the following, we summarize these different values as a function of the r.v. $Y_1(T_e)$,

$$C(Y_1(T_e)) = \begin{cases} b - K & \equiv \ C(H) \qquad \text{with prob. } p^2(2-p) \\ 2a - K & \equiv \ C(M) \qquad \text{"} \qquad p(1-p)^2 \\ 0 & \equiv \ C(L) \qquad \text{"} \qquad (1-p) \quad . \end{cases} \tag{3.17}$$

A possible way to value options at the initial time is to search for an admissible hedging strategy that does not allow loss of money for the company that supplies the service. In order to do that, the company could buy at time $t_0$ a fraction of the total bandwidth requested by the buyer along the direct route A-B for the delivery time interval $[T_d, T_f]$, and another fraction $\beta$ of the alternative route, A-C-B for the same period. The purchase price at time $t_0$ will be the market price, (the price of a forward contract with maturity $t = T_e$), that we assumed be $\mathbb{E}[S_i]$ with the parameter $p$ fixed. Indeed, this parameter can be connected with the price volatility and therefore estimated from the dataset. Thus, the value of such a portfolio at time $t_0$ can be written as follows

$$V_0 = (\alpha + 2\beta)\mathbb{E}[S_1] = (\alpha + 2\beta)[a(1-p) + bp] \quad .$$

On the other hand at time $T_e$, the company should sell the bandwidth fraction of the unused route and buy the missing part for the route that will be used for the connection, depending on the realizations of $Y_i(T_e)$. Below, we compute separately the different company gains, denoted $G_{H,M,L}$, at time $T_e$ deriving from these operations

- (Case $H$) The buyer will exercise the option buying at the strike price $K$ the connection that will be delivered along the direct route which has a market price $b$.

$$G_H = -(1-\alpha)b + 2\beta\mathbb{E}[S_2|(S_2 = a, S_3 = a)^c] = -(1-\alpha)b + 2\beta\frac{a(1-p)+b}{2-p}$$

- (Case $M$) The buyer will exercise the option buying at the strike price $K$ the connection that will be delivered along the direct route which has a market price $2a$.

$$G_M = -2(1-\beta)a + \alpha b$$

- (Case $L$) The buyer will not exercise the option and the company will sell the whole portfolio.

$$G_L = +\alpha a + 2\beta \mathbb{E}[S_2] = +\alpha a + 2\beta[a(1-p) + bp]$$

Let $C_0$ be the value that the buyer has to pay for the options at time $t_0$, therefore the request for the admissibility of the hedging strategy leads to the following inequalities

$$\begin{aligned} W_H &= C_0 - V_0 + G_H + K \geq 0 \\ W_M &= C_0 - V_0 + G_M + K \geq 0 \\ W_L &= C_0 - V_0 + G_L \geq 0 \end{aligned} \tag{3.18}$$

where $W$ denotes the total gain, *i.e.* the amount of money gained by the sell of $C_0$, minus the money spent to form the portfolio $V_0$, plus the gain due to the rebalancing strategy at time $T_e$.

At this point, one possible interpretation of the option fair price, is to consider the minimum among all prices, $C_0$, satisfying the inequality

$$\mathbb{E}[W] \geq 0\,, \tag{3.19}$$

where we choose the expected value of the total gain because for the moment we are interested in an average option price, ignoring its dependence on initial spot price. With the same idea in mind, we took the expected value of the initial spot price instead of that of its realization. However, in this simple case it is possible to prove that this condition changes the earlier inequalities system into the following system of equation:

$$\begin{cases} C_0 - V_0 + G_H + K = 0 \\ C_0 - V_0 + G_M + K = 0 \\ C_0 - V_0 + G_L = 0 \end{cases}$$

that yields the solutions

$$\begin{cases} \alpha = \frac{2a-K+p(b-2a)(2-p)}{b-a} \\ \beta = \frac{(b-2a)(2-p)}{2(b-a)} \\ C_0 = p[2a-K+p(b-2a)(2-p)] \end{cases}$$

where $C_0$ agrees with the numerical solution of system (3.18).

In figure 3.7, we report the analytical behaviour of $C_0$ as function of the parameter $p$, and the others parameters fixed. ($a = 15,\ K = 25,\ b = 40$).

As can be seen, the behaviour of $C_0$ is consistent with our model. In fact, $p = 0$ means

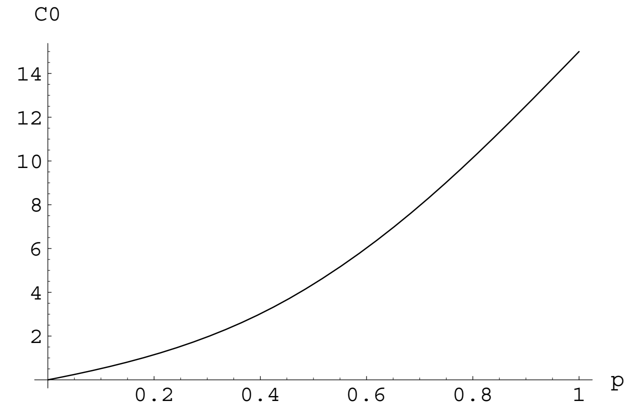



**Figure 3.7:** Bandwidth option price as function of $p$. ($a = 15,\ K = 25,\ b = 40$)

a constant spot price equal to $a$ and obviously no one would buy the option, $C(L) \equiv 0$ according to (3.17). On the other hand $p = 1$ means a constant spot price equal to $b$ that is $C_0 \equiv C(H) \equiv b - K$, again according to (3.17).

Note that, in this particular case where the system has a unique solution, finding the fair price of a European Call option via physical hedging is equivalent to compute the classical expectation

$$\begin{aligned} \mathbb{E}[(Y_{T_e} - K)^+] &= \sum_{i \in \{H,M,L\}} C(i)\mathbb{P}\{C(Y_{T_e} = C(i))\} \\ &= (b-K)p^2(2-p) + (2a-K)p(1-p)^2 \\ &= p[2a-K+p(b-2a)(2-p)] \end{aligned}$$

### 3.4.4 Options on two distinct routes with overlappings

Selling more options on the same network does not produce any new effect, since the different connections are independent, since we are not taking into account congestion problems, and the common hedging strategy is just a linear combination of the strategy for each option. However, things change on a more complicated network, because of partial overlappings among routes. To better analyse these option price properties, we consider a little more complicated network, the back-to-back triangle shown in figure 3.8.

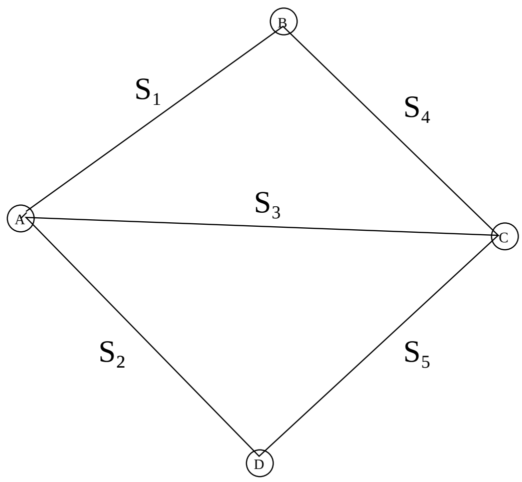



**Figure 3.8:** Back-to-Back Triangle Network

In such a topology, we want to investigate the effect of overlappings on our hedging strategy, and we do that by considering the case of a company that sells two bandwidth options on two distinct routes at two different buyers. Furthermore, we focus on a specific case, the sell of an option on the connection contract between the points A and B to a first buyer and of another option for the connection between A and D to a second buyer. Thus, both buyers share the A-C bandwidth for the alternative routes. We will show how a mixed hedging strategy for the seller, leads to a lower price for the single option. We apply the same conditions both for the parameters and the spot prices as we used before, and we introduce an index to distinct the two buyers, for instance $L_1 H_2$ means the low case for the first buyer and the high one for the second buyer.

In this way, we get 9 different case combinations, depending on all price realizations for the 5 direct routes:

$$L_1L_2\,,\,L_1M_2\,,\,L_1H_2\,,\,M_1L_2\,,\,M_1M_2\,,\,M_1H_2\,,\,H_1L_2\,,\,H_1M_2\,,\,H_1H_2$$

that corresponds to 9 inequalities for the hedging strategy. We choose to not use any simplification due to symmetry properties to avoid the risk of losing information. Following the same steps we did earlier, we denote with $\alpha$, $\beta$, $\gamma$, $\delta$, $\sigma$ the portions of bandwidth that the seller must buy at the initial time $t_0$ of the relative direct connections, $S_1$, $S_2$, $S_3$, $S_4$, $S_5$. Then, the new portfolio initial value can be written as follows:

$$V_0 = (\alpha + \beta + \gamma + \delta + \sigma)\mathbb{E}[S_1]\ .$$

Once more, we compute the different company gains for each case, omitting the obvious steps leading to these final formulas:

- (Case $LL$) Both buyers will not exercise the options.

$$G_{LL} = (\alpha + \beta)a + (\gamma + \delta + \sigma)\mathbb{E}[S_1]$$

- (Case $LM$) Only the second buyer will exercise his option, (alternative route).

$$G_{LM} = \alpha a + \beta b + \delta\mathbb{E}[S_1] - (1-\gamma)a - (1-\sigma)a + K$$

- (Case $LH$) Only the second buyer will exercise his option, (direct route).

$$G_{LH} = \alpha a - (1-\beta)b + \delta\mathbb{E}[S_1] + \frac{(\gamma+\sigma)[a(p(1-p)) + bp]}{p(2-p)} + K$$

- (Case $ML$) The option will be exercised only by the first buyer, (alternative route).

$$G_{ML} = \alpha b + \beta a + \sigma\mathbb{E}[S_1] - (1-\gamma)a - (1-\delta)a + K$$

- (Case $MM$) Both buyers will exercise the options, (both alternative routes), there will be an overlapping over $AC$ route.

$$G_{MM} = \alpha b + \beta b - (2-\gamma)a - (1-\delta)a - (1-\sigma)a + 2K$$

- (Case $MH$) Both buyers will exercise their options, the first one with alternative route, the second one with direct route.

$$G_{MH} = \alpha b + \sigma b - (1-\gamma)a - (1-\delta)a - (1-\beta)b + 2K$$

- (Case $HL$) Only the first buyer will exercise the option, (direct route).

$$G_{HL} = -(1-\alpha)b + \beta a + \sigma\mathbb{E}[S_1] + (\gamma+\delta)\frac{(a+b)(1-p)+bp}{(2-p)} + K$$

- (Case $HM$) Both buyer will exercise, the first one with direct route, and the second with the alternative one.

$$G_{HM} = \beta b + \delta b - (1-\alpha)b - (1-\gamma)a - (1-\sigma)a + 2K$$

- (Case $HH$) Both buyers will exercise the options (direct routes).

$$\begin{aligned} G_{HH} = -(1-\alpha)b - (1-\beta)b + \gamma\frac{(a+2b)p(1-p)+b(1-p)^2+bp^2}{1+p-p^2} + \\ +(\delta+\sigma)\frac{(a+2b)p(1-p)+a(1-p)^2+bp^2}{1+p-p^2} + 2K \end{aligned}$$

We now are able to write the inequalities system for the hedging strategy

$$W_{..} = 2C_0 - V_0 + G_{..} \geq 0$$

which for each case:

$$\begin{cases} 2C_0 - (\alpha+\beta)p(b-a) \geq 0 & (W_{LL}) \\ 2C_0 - (\alpha+\gamma+\sigma)p(b-a) + \beta(b-a)(1-p) + (k-2a) \geq 0 & (W_{LM}) \\ 2C_0 - \alpha p(b-a) + \beta(b-a)(1-p) + (\gamma+\sigma)\frac{(1-p)^2}{2-p} + (k-b) \geq 0 & (W_{LH}) \\ 2C_0 + \alpha(b-a)(1-p) - (\beta+\gamma+\delta)p(b-a) + (k-2a) \geq 0 & (W_{ML}) \\ 2C_0 + (\alpha+\beta)(b-a)(1-p) - (\delta+\sigma+\gamma)p(b-a) + 2(k-2a) \geq 0 & (W_{MM}) \\ 2C_0 + (\alpha+\beta+\sigma)(b-a)(1-p) - (\gamma+\delta)p(b-a) + (k-b) + (k-2a) \geq 0 & (W_{MH}) \\ 2C_0 + \alpha(b-a)(1-p) - \beta p(b-a) + (\gamma+\delta)(b-a)\frac{(1-p)^2}{2-p} + (k-b) \geq 0 & (W_{HL}) \\ 2C_0 + (\alpha+\beta+\delta)(b-a)(1-p) - (\gamma+\sigma)p(b-a) + (k-b) + (k-2a) \geq 0 & (W_{HM}) \\ 2C_0 + (\alpha+\beta)(b-a)(1-p) + \gamma\frac{(b-a)(1-p)^2(1+p)}{1+p-p^2} + (\delta+\sigma)\frac{(b-a)p(1-p)^2}{1+p-p^2} \geq 0 & (W_{HH}) \end{cases} \quad (3.20)$$

The single option fair price, according to our interpretation, is

$$C_0^* = \min_{\alpha,\beta,\gamma,\delta,\sigma \in \text{sol.(3.20)}} \left[ C_0 | \mathbb{E}[W_{..}] \geq 0 \right]. \tag{3.21}$$

In figure 3.9 we show the numerical solutions as functions of the probability parameter $p$, for two single option prices relative to the sell of one or two options by the same company in the presence of route overlappings ( respectively the option price found in (3.19) and the solution of (3.21)).

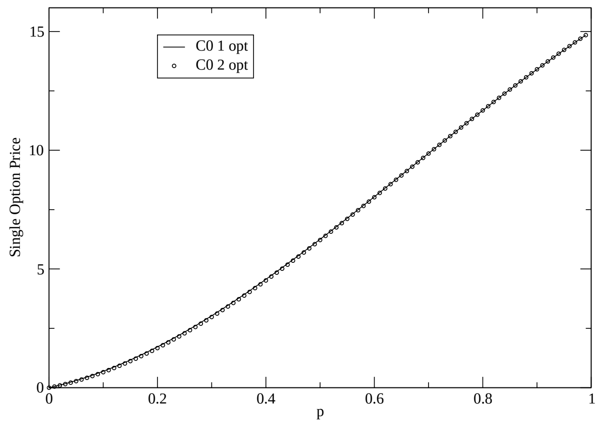



**Figure 3.9:** Single option prices as functions of the probability parameter $p$, with respect to the sell of one and two options at the same time by the same company to two different buyers. $b = 40$, $a = 15$, $k = 25$.

As can be seen in figure 3.9, both prices seemingly coincidence but with a more specific analysis we can point out the differences. We compute the difference of the value of the first case, to analyse it as a sort of relative error. The result can be seen in figure 3.10. Therefore, the price difference can reach the $5\%$ that means a huge saving of money for a company operating in the market. Moreover, this result shows how much research work is needed to develop fair tools for the bandwidth market. The rise of this new kind of arbitrage for selling companies is due especially to those cases in which one buyer does not exercise the option while the other exercises the option through the alternative route. In those cases, the seller can use the portion of common edge bought for hedging the buyer that does not exercise to deliver the service for the other buyer instead of selling it to the market at the lower price.

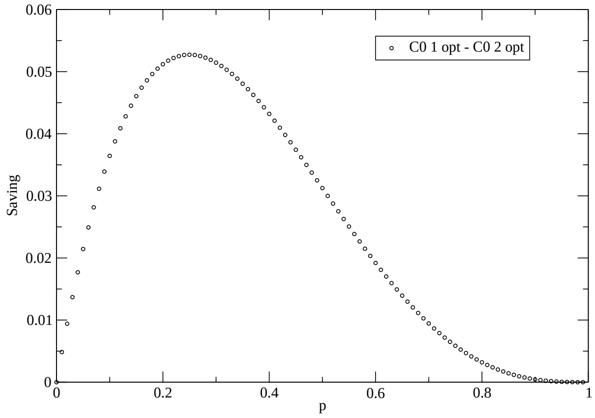



**Figure 3.10:** Absolute difference between the single option price in the simple case and the one in the case with overlappings. Numerical behaviour with parameters: $b = 40$, $a = 15$, $k = 25$.

To support this reasoning, we compute the probability of realization of $LM$ case that is one of those cases we mentioned.

The figure 3.11 shows the behaviour of the probability of $LM$ case, and this agrees with the

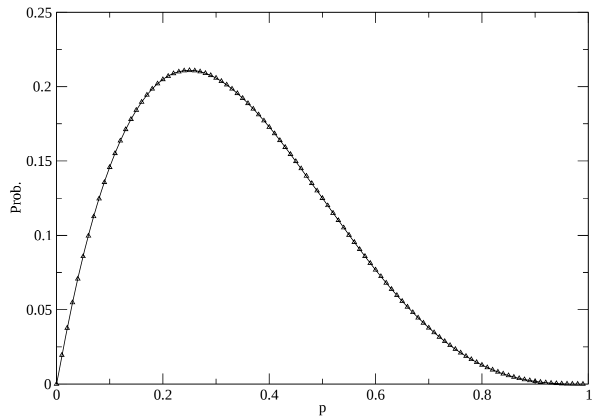



**Figure 3.11:** Probability of the $LM$ case

saving behaviour strengthening our thesis.

## 3.5 Underlying Continuous Time Markov Chain

In order to get closer to a realistic spot price process, we move from a Bernoullian r.v. to a Markov process, in particular to a Two-States continuous time Markov chain.
We consider a Markov chain $(S_t)_{t\geq 0}$ on $\mathbb{S} = \{a,\, b\}$ where $a, b \in \mathbb{R}^+$, $a < b$, describing the Bandwidth spot price of a point-to-point connection as in Figure 3.2. As done for the Bernullian case, we suppose that the price process of each edge of the triangle network follows such as Markov chain, so we get 3 *i.i.d* Markov process $(S_t^i)_{t\geq 0},\ i = 1, 2, 3$ with the same generator matrix $Q$ and the same initial distribution $\pi$

$$\pi = (\pi_1\,, \pi_2) \qquad , \quad Q = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a & \;\; b \end{array} \\ \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \end{array}$$

First of all, we compute the associated stochastic matrix $P(t)$, (the states space is finite) as the unique solution of the backward equation

$$\begin{cases} P'(t) = QP(t) \\ P(0) = I \end{cases} \Rightarrow \begin{cases} p'_{11}(t) = -\lambda p_{11}(t) + \lambda p_{21}(t) \\ p'_{12}(t) = -\lambda p_{12}(t) + \lambda p_{22}(t) \\ p'_{21}(t) = \mu p_{11}(t) - \mu p_{21}(t) \\ p'_{22}(t) = \mu p_{12}(t) - \mu p_{22}(t) \\ P(0) = I \end{cases}$$

As usual, we look for solutions having the form

$$p_{ij}(t) = c_{ij}^1 e^{\gamma_1 t} + c_{ij}^2 e^{\gamma_2 t}$$

where $\gamma_1,\ \gamma_2$ are the eigenvalues of $Q$ and $c_{ij}^k$ some constants depending on $i,\ j$. Then

$$\det(xI - Q) = 0 \Rightarrow x = \begin{cases} 0 \\ -(\mu + \lambda) \end{cases}$$

and the constants are determined solving the systems

$$\begin{cases} p_{ij}(0) = 1 = c_{ij}^1 + c_{ij}^2 \\ p'_{ij}(0) = q_{ij} \end{cases}$$

Finally, we write down the associated stochastic matrix

$$P(t) = \begin{pmatrix} \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\mu+\lambda)t} & \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}e^{-(\mu+\lambda)t} \\ \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu}e^{-(\mu+\lambda)t} & \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\mu+\lambda)t} \end{pmatrix}$$

Bearing in mind what we did for the Bernoullian case, Section 3.4.3, we calculate the probability that the chain is in the state $a$ or $b$ at time $t$

$$\begin{aligned} \mathbb{P}\{S_t = b\} &= \sum_{k\in\mathbb{S}} \mathbb{P}\{S_t = b|S_0 = k\}\mathbb{P}\{S_0 = k\} \\ &= p_{a,b}(t)\pi_1 + p_{b,b}(t)\pi_2 \end{aligned}$$

and obviously

$$\begin{aligned} \mathbb{P}\{S_t = a\} &= 1 - \mathbb{P}\{S_t = b\} \\ &= p_{a,a}(t)\pi_1 + p_{b,a}(t)\pi_2 \end{aligned}$$

Identifying $\mathbb{P}\{S_t = b\}$ with $p(t)$, recalling the $p$ probability parameter in the Bernoullian case, we can repeat all the calculations done to compute the fair price of a European call option with strike price $K$ on the bandwidth market spot price $Y_t^1 = \min\{S_t^1, S_t^2 + S_t^3\}$, obtaining

$$\begin{cases} \alpha = \frac{2a-K+p(t)(b-2a)(2-p(t))}{b-a} \\ \beta = \frac{(b-2a)(2-p(t))}{2(b-a)} \\ C_0 = V_0 + (2a-K) - a2z - by \end{cases}$$

where $\alpha, \beta \in [0,1]$ are portions of bandwidth bought at the initial time, $\alpha$ refers to direct route, and $\beta$ to the alternative one. $V_0$ is the value of the bandwidth portfolio at initial time, *i.e*

$$V_0 = (\alpha + 2\beta)(a\pi_1 + b\pi_2)$$

If we plot $C_0$ as function of $p(t)$ we obtain again the Figure 3.7.

## 3.6 Underlying Geometric Brownian Motion

Modelling bandwidth spot prices with a Bernullian random variable, as far as it helps in a first analysis, is not sufficient to investigate all possible issues on option pricing over a network. A better approximation, even if just of “zero order”, is modelling spot prices by a Geometric Brownian Motion (GBM) as in the classical Black-Scholes model.

As before, we consider a finite time horizon $[t_0, T]$, and all operations can be done only at the initial time $t_0$ and at the final time $T$, even if the evolution of prices is a continuous function of time. Henceforth, we will refer to an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the usual conventions, *i.e.* $\Omega$ represents the set of different scenarios, $\omega$, which can occur in the market, $\mathcal{F}$ the $\sigma$-algebra of subsets of $\Omega$, and $\mathbb{P}$ the probability measure on $(\Omega, \mathcal{F})$.

A generic GBM follows the linear Stochastic Differential Equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad S_{t=0} = s_0 > 0 \tag{3.22}$$

where the volatility $\sigma$ is positive and bounded, the drift part $\mu \in \mathbb{R}$ and $W_t$ is a standard Brownian motion started at $t = 0$. This equation has a strong unique non-exploding solution, and it is a simple application of Itô’s formula to show that the solution is given by

$$S_t = s_0 \exp\left( (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \right), \quad t \geq 0.$$

For simplicity, we assume that the drift part $\mu$ and the interest rate $r$ are equal to $0$ as done earlier. For instance, the equation (3.22), with the constant volatility set to equal the average of volatility square over the lifetime of a generic contract, has already been used with a non-storable underlying asset (see [53, 25, 26, 27, 28]).

### 3.6.1 Alternative Path Price Process Approximation

The alternative path price consists of a sum of the each single link which makes up the path. Generally, underlying prices are modelled as log-normal random variables as already outlined before. The troublesome problem with the path price modellization is that a sum of log-normal variables is obviously not log-normally distributed.

Since there exists no closed form distribution or characteristic function for the sum of two or more lognormal variables, many approximation methods have been developed in literature to approximate this, including:

- Considering Arithmetic Brownian Motion instead of Geometric Brownian Motion as suggested in [29]. Such a change clash with the most popular models for asset prices, requires additional restrictions to avoid negative prices. In other respects, it is suitable for basket or spread options, whose underlyings can be negatives as differences of positive quantities, and it allows for the inclusion of mean-reverting.

- Moment approximation, where the common feature of these lognormal sum approximations is that they model the sum distribution being lognormal as well. One of this approximation techniques is the Fenton-Wilkinson approximation [30] which we choose as a good compromise for the dimension of our problem. This technique simply derives the mean and the standard deviation of the resulting lognormal distribution by matching them directly with the mean and the standard deviation of the sum.

### *Fenton-Wilkinson approximation*

**Assumption 3.1.** *The alternative routing prices are given by the following stochastic differential equation*

$$dX(t,T) = dS_2(t,T) + dS_3(t,T) = \mu^* X_t dt + \sigma^* X_t dW_t \tag{3.23}$$

*where $\mu^*$ and $\sigma^*$ are derived by the Fenton-Wilkinson approximation [30]*

We use a simple analytical approximation for the sum of multiple lognormally distributed random variables. The Fenton-Wilkinson approximation is quite accurate in our case, due to the limited range of standard deviations. Its accuracy is tested with Monte Carlo simulations, yielding an approximation error below $0.1\%$.

For the sake of simplicity, we consider a unit time interval, and we look at the two log-normal random variables

$$S_T^{(1)} = S_0^{(1)} e^{\epsilon_1} \quad \text{and} \quad S_T^{(2)} = S_0^{(2)} e^{\epsilon_2}$$

where $\epsilon_i \sim N(\mu_i, \sigma_i)$. Therefore, we have

$$X_T = S_T^{(1)} + S_T^{(2)} \cong X_0 e^Z \qquad , \quad Z \sim \mathcal{N}(\mu^*, \sigma^*) \tag{3.24}$$

The mean $\mu^*$ and the standard deviation $\sigma^*$ of $Z$ are derived by matching the first two moments of the both sides of (3.24). In our specific case we have, $\mu_i = 0$, $\sigma_i = \sigma$ and the $S_T^{(i)}$

uncorrelated and we get

$$\mu^* = \frac{3}{2}\ln 2 + \frac{\sigma^2}{2} - \frac{1}{2}\ln(e^{\sigma^2}+1)$$

$$(\sigma^*)^2 = \ln(e^{\sigma^2}+1) - \ln 2$$

Finally, the approximated probability density functions read

$$f(s) = \frac{1}{s\sqrt{2\pi\sigma^2}} e^{-(\ln s - \mu)^2/(2\sigma^2)} \quad , \quad g(u) = \frac{1}{u\sqrt{2\pi(\sigma^*)^2}} e^{-(\ln u - \mu^*)^2/(2(\sigma^*)^2)} , \quad u,\, s \geq 0\,.$$

### 3.6.2 Analytical Solution of Black-Scholes type pricing (Margrabe Style)

The computation of the option price when the underlying asset follows a Geometric Brownian Motion can be done in two ways. From one hand, we can directly solve the SDE of two assets, both correlated or uncorrelated, which has been solved in [12]. Conversely, from the other hand we can derive the price of a European Option on the minimum of two risky assets from a combination of well known analytical results of financial theory.
We adopt the second approach.
The aim of this section is to compute

$$C_0 = \mathbb{E}\left[\max(Y_{T_e} - K, 0)|\mathcal{F}_{t_0}\right] \equiv \mathbb{E}\left[\max(\min(X_{T_e}, S_{T_e}) - K, 0)|\mathcal{F}_{t_0}\right]$$

We denote a standard (one asset $S$) European Call with maturity, $T$ and strike price $K$, with $C(S, T, K)$
With the same notation, we define:

- a Call on the minimum of two asset with $C_{min}(S, X, K, T)$
- a Call on the maximum of three assets with $C_{max}(S, X, K, T)$
- and, finally a Call on the maximum of three assets with $f_{max}(S, X, K, T)$

. We look at the payoff of our Call Option and observe that

$$\text{PayOff}[C_{min}(S, X, T, K)] = \max(\min(X_T, S_T) - K, 0) =$$
$$= \max(S_T - K, 0) + \max(X_T - K, 0) - \max(\max(X_T, S_T) - K, 0)$$

Then for the value of the Call Option, we get

$$C_{min}(S, X, T, K) = C(S, K, T) + C(X, K, T) - C_{max}(S, X, K, T)$$

We consider the last summandum, and we rewrite in such a way

$$\text{PayOff}[C_{max}(S, X, T, K)] = \max(\max(X_T, S_T) - K, 0) = \max(S_T, X_T, K) - K$$

Then the Call on the maximum can be decomposed as

$$C_0 = \mathbb{E}\left[\max(\min(X_T, S_T) - K, 0)|\mathcal{F}_{t_0}\right] =$$
$$= \mathbb{E}[\max(S_T - K, 0) + \max(X_T - K, 0) - \max(S_T, X_T, K) + K|\mathcal{F}_{t_0}] =$$
$$= C(S, K, T) + C(X, K, T) - f_{max}(S, X, K, T) + Ke^{-rT} \tag{3.25}$$

The value of an option on the maximum of three assets can be derived following the same idea as in [31].

$$f_{max}(S, X, K, T) = s_0 N_2\left(d_{X/S}, d_{K/S}; \rho_{X/S,K/S}\right) + \tag{3.26}$$
$$+x_0 N_2\left(d_{S/K}, d_{X/K}; \rho_{S/K,X/K}\right) + Ke^{-rT} N_2\left(d_{X/S}, d_{K/S}; \rho_{X/S,K/S}\right) \tag{3.27}$$

where

$$\sigma_{i/j}^2 = \sigma_i^2 + \sigma_j^2 - \rho_{ij}\sigma_i\sigma_j$$
$$\rho_{i/l,j/l} = \frac{1}{\sigma_{i/l}\sigma_{j/l}}\left(\sigma_i\sigma_j\rho_{ij} - \sigma_i\sigma_l\rho_{il} - \sigma_j\sigma_l\rho_{jl} + \sigma_l^2\right)$$
$$d_{i/j} = \frac{\ln\frac{i_0}{j_0}\frac{1}{2}\sigma_{i/j}^2 T}{\sigma_{i/j}\sqrt{T}}$$
$$i, j \in \{S, X, K\} \quad \text{and} \quad K_0 = e^{-rT}K$$

and obviously $\sigma_K = 0$.

Substituting (3.26) into (3.25) we get the pricing formula for a European Call Option on the minimum of two risky assets.

A detailed proof of this formula follows Margrabre Formula for exchange options closely, and it is derived in [32].

## 3.7 Results

We have shown in this chapter the importance of network arbitrage in modelling “fair” bandwidth prices and especially contingent claims on them. First of all, we have academically built up a pricing model on a two link bandwidth network for a bernoullian price process for the underlying assets. We pointed out pricing and mostly hedging problem in such a topology. Hence we proposed a solution both for price and hedging strategy, proving its consistency with the standard pricing formulas.

Finally, we extend our results to the continuum, considering underlying assets following a Geometric Brownian Motion price process.

*Chapter 4*

# Loss Networks

In order to lay the foundations for future extensions of triangle based pricing results to a generalized bandwidth network , in this chapter we look at the mathematical properties of the graph modelling a real network.

We show how traffic activity on a certain communication link, which is directly related to prices of such a connection, does not influence traffic activity on a distant link. Such a result allows us to think a global network as a series of local networks, where results already found in the previous chapter apply.

We derive estimates of an upper bound for the exponential decay rate, both in space and time, of the correlation function between traffic activities on distant routes. In particular, we show how this decay depends on the Euclidean distance between such routes.

For the sake of simplicity, we restrict our analysis to an equilibrium dynamics, where the bandwidth capacity available on each link exceeds the average capacity reserved by the connections established on that link. In other words, we do not deal with congestion or failure issues at this first stage of analysis, referring for a review to [36, 41].

# 4.1 Bandwidth Graphs

As already pointed out in section (2.2), graphs are commonly used for modelling telecommunications networks. They have nodes (geographical locations) and edges (point-to-point connections). A sequence of edges that connect two nodes is called a path (route). Edge do not have a direction, but rather it is traversable in both directions, at the same cost and a fixed QoS (Quality Of Service). Therefore we only look at concerned with unoriented graphs.
One of the first attempt to describe a network as a graph can be found in [42], where a traditional arc-node model is developed. We now focus our attention on the covariance between traffic activities on distant routes.

## 4.1.1 A Mathematical Framework

Before introducing the model on which we prove our result, we need to introduce a rigorous description of telecommunications networks as graphs. We start with few definitions of graph theory.

**Definition 4.1.** (Simple Unoriented Graph) *A simple unoriented graph, $G$, is a pair $G = (V, E)$ where*

- $V$ *is a finite set, called the* vertices *of $G$, and*
- $E$ *is a subset of all two-element subsets of $V$,* i.e.

$$E \subset \{F : F \subset V, |F| = 2\}$$

*and it is called the edges of $G$*

For example, the triangle network graph of Fig. (3.2) is $G = K_3$, where the vertices are the cities and the edges are the point-to-point connections between pairs of cities and $K_3$ refers to a fully meshed graph.
Moreover, we define a path between two cities as a *route* $r$

**Definition 4.2.** (Route) *A route $r$ on $G$ from $u \in V$ to $v \in V$, with $u \neq v$, is a subset of $E$ such that:*

- $\exists!\ b_u \in r$ *such that* $u \in b_u$*;*
- $\exists!\ b_v \in r$ *such that* $v \in b_v$*;*
- *for any* $b \in \{r \setminus b_x \setminus b_y\}$ $\exists!\ a \neq c \in r$ *such that* $a \cap b \neq b \cap c$ *and* $|a \cap b| = |b \cap c| = 1$.

where the first two conditions assure that from each end edge of a route starts only a single link belonging to the route, while the third condition forces the route not to split. Therefore, the route connecting two points is a single path. We denote the length of a route $r$ with $|r|$.

Among all the graphs we are only interested in the subfamily of connected graphs, which means that for any two distinct elements $u$ and $v$ of $V$ there is a route $r$ from $u$ to $v$.

**Definition 4.3.** (Degree of a Vertex) *Let* $G = (V,\, E)$ *be a graph and* $v \in V$ *a vertex. The neighbourhood of* $v$ *is the set*

$$N_G(v) = \{u \in G | \{v, u\} \in E\}.$$

*We define the degree of* $v$*,* $\deg(v)$ *to be the number of its neighbours,* i.e.

$$\deg(v) = |N_G(v)|.$$

### 4.1.2 Generalized Bandwidth Graph Equilibrium Model

We now consider an increasing sequence of graphs, labelled by the index $n$, $G_n = (V_n, E_n)$, such that $\forall n \in \mathbb{N}$

1. $E_n \subset \{F : F \subset V_n, |F| = 2\}$
2. $|V_n| < +\infty$,
3. $G_n \subset G_{n+1}$,
4. $\sup_n \max_{v \in V_n} \deg(v) = D < +\infty$.

We also define $G_\infty = \lim_n G_n := (\cup_n V_n, \cup_n E_n)$.

We denote with $\mathcal{R}_n$ the set of routes $r$ of $G_n$ with finite length.

We assume that each connection requires the same capacity, which we take to be $1$. Moreover, we assume that each edge has a fixed integer capacity and consequently, we define a capacity function $C : E_\infty \to \mathbf{N} \cup \{0, +\infty\}$ fixing the maximum capacity of each link, *i.e.* $C(e)$ is the maximum number of unit connections allowed on edge $e$.

First, we define the space of configurations of the routes of finite length on $G_n$, as $\Omega_n := (\mathbb{N} \cup \{0\})^{\mathcal{R}_n}$, which means that if $\eta \in \Omega_n$ then $\eta(r)$ is the number of unit connections following the route $r$ at a fixed time.

We restrict our analysis to the subset of the space of all possible configurations denote by $\Omega_n^C := \{\eta \in \Omega_n : \sum_r \eta(r)\mathbf{1}(r) \leq C\}$, where $\mathbf{1}(r)$ refers to the indicator function that the configuration is in route $r$. Such a restriction assures that any initial configuration does not already include congested routes.

We introduce then a Markov process, the birth and death process, in which connections between two different cities could be established or dropped along a fixed route. For any $\eta \in \Omega_n^C$, $r \in \mathcal{R}_n$ we define

$$\eta^{r\pm} := \begin{cases} \eta \pm \mathbf{1}(r) & \text{if } \eta \pm \mathbf{1}(r) \in \Omega_n^C, \\ \eta & \text{otherwise.} \end{cases}$$

where, we use the notation $\eta^{r+}$ ($\eta^{r-}$) to denote the configuration where a connection is established (dropped) on route $r$.

For any given $f : \Omega_n^C \to \mathbb{R}$, and $r \in \mathcal{R}_n$ we let

$$f^{r\pm} : \Omega_n^C \to \mathbb{R} \qquad \text{as} \qquad f^{r\pm}(\eta) := f(\eta^{r\pm})$$
$$\text{and for the gradient} \quad \nabla_r^{\pm} f := f^{r\pm} - f.$$

We are now ready to introduce a stochastic dynamics on such a graph. We assume that connections on a route $r$ are established as a Poisson stream of rate $\nu$ defined as

$$\nu : \mathcal{R}_\infty^L \to [0, +\infty)$$

subjected to the conditions

1. $\|\nu\|_\infty < +\infty$

2. there exists $L \in \mathbb{N}$ such that $|r| > L \Rightarrow \nu(r) = 0$.

The second condition limits the route range, similarly to what is usually done in modelling communication network flows, introducing a kind of local behaviour.

Following the basic model of a loss network, as defined in [43], we say that the Markov process $\eta(r)$, *i.e.* the number of active connections on route $r$, has a unique stationary distribution whose invariant probability measure $\pi_n^C(\eta)$ is given by

$$\pi_n^C(\eta) = (Z_n^C)^{-1}\mathbf{1}(\eta \in \Omega_n^C) \prod_{r\in\mathcal{R}_n} \frac{\nu(r)^{\eta(r)}}{\eta(r)!} \quad \text{for every} \quad \eta \in \Omega_n \tag{4.1}$$

where $(Z_n^C)^{-1}$ is the appropriate normalization factor defined as

$$Z_n^C = \sum_{\eta\in\Omega_n^C} \prod_{r\in\mathcal{R}_n} \frac{\nu(r)^{\eta(r)}}{\eta(r)!} \, .$$

This probability measure arises from the classical theory developed for telephone calls on telecommunication networks, where calls between two points start according to a Poisson process of rate $\nu$ and occupy unit portions of the route for their holding periods (exponentially distributed if the process is a Markov process), in which case it is easy to verify that the detailed balance condition holds

$$\pi_n^C(\eta(r)) \cdot \nu(r) = \pi_n^C(\eta(r) + \mathbf{1}(r)) \cdot (\eta(r) + 1), \quad \eta, \eta + \mathbf{1}(r) \in \Omega_n^C$$

Therefore, for any $\eta \in \Omega_n^C$, the Markov generator of such a process is a well defined linear (unbounded, with dense domain $\mathcal{D}(\mathcal{L}_n^C)$) and it can be written as

$$(\mathcal{L}_n^C f)(\eta) := \sum_{r\in\mathcal{R}_n^L} \left[\eta(r)(\nabla_r^- f)(\eta) + \nu(r)\mathbf{1}(\eta^{r+} \in \Omega_n^C)(\nabla_r^+ f)(\eta)\right] \, , \quad f \in \mathcal{L}_n^C \, .$$

which is self adjoint in $L^2(\pi_n^C)$.

Furthermore, we define the Dirichlet form, *i.e.* a non-negative quadratic form on $\mathcal{D}(\mathcal{L}_n^C) \times \mathcal{D}(\mathcal{L}_n^C)$, associated with $\mathcal{L}_n^C$ given by

$$\mathcal{E}_n^C[f] := -\pi_n^C[f\mathcal{L}_n^C f] = \sum_r \pi_n^C \left[\eta(r)(\nabla_r^- f)^2\right] = \sum_r \nu(r)\pi_n^C \left[\mathbf{1}(\eta^{r+} \in \Omega_n^C)(\nabla_r^+ f)^2\right]$$

where $\mathcal{E}_n^C[f] := \mathcal{E}_n^C[f,\, f]$, and we use the notation $\pi_n^C[f]$ for $\sum f\pi_n^C$.

Finally, we define a positive preserving contraction semigroup, (*i.e.* a Markov semigroup) on $L^2(\pi_n^C)$ as $P_t^{C,n} := \exp[t\mathcal{L}_n^C]$, $t \geq 0$, which concludes the model dynamics.

## 4.2 Upper bound for two-point correlation function

In order to derive our main result about estimates for both time and space exponential decay rate of the covariance between communication traffic, *i.e.* number of connections, along distant links, we introduce a useful tool of functional analysis, the *Poincaré inequality*[44]

$$\pi_n^C[f;f] \le \frac{1}{k}\mathcal{E}_n^C[f] \quad \text{for every } f \in \mathcal{D}(\mathcal{L}_n^C)\,, \tag{4.2}$$

where $\pi_n^C[f;f]$ is the variance of $f$ with respect to the $\pi_n^C$ measure, $\mathrm{Var}(f) := \pi_n^C[(f - \pi_n^C[f])^2]$. It is well known that the Poincaré inequality is equivalent to exponential convergence to equilibrium in $L^2(\pi_n^C)$, which means $\|P_t^{C,n} f - \pi_n^C[f]\|_{L^2(\pi_n^C)} \le \mathrm{e}^{-kt}\|f - \pi_n^C[f]\|_{L^2(\pi_n^C)}$.

**Definition 4.4.** (Spectral gap) *The largest $k \ge 0$ for which* (4.2) *holds is the* spectral gap *of the generator $\mathcal{L}_n^C$ in $L^2(\pi_n^C)$ and it is denoted by $\mathrm{gap}(\mathcal{L}_n^C)$.*

In other words, the spectral gap is also defined as the absolute value of the second largest eigenvalue of the generator of the process, see e.g. [45].
We develop a method based on a Bochner-Bakry-Emery approach [46], to obtain estimates on the exponential rate of decay of the covariance between the traffic along two edges.
A first result is the following theorem:

**Theorem 4.1.** *For any $D \in \mathbb{R}^+$, $L \in \mathbb{N}$ there exists a constant $K(D,\, L)$ such that*

$$\mathrm{gap}(\mathcal{L}_n^C) \ge 1 - \|\nu\|_\infty K(D, L)\,.$$

This theorem gives a lower bound of the spectral gap of $\mathcal{L}_n^C$, and moreover it links such an estimate to the maximum intensity rate with which the connections are established on a certain route $r$, and to a constant $K(D, L)$ related in some way to the maximum number of neighbours.
We follow the approach as in [48]. Such an approach adapt to a class of Markov process with discontinuous trajectories an approach to functional inequalities originally developed in the context of Riemannian geometry, see Bochner [47], allowing to obtain lower bounds for the spectral gap of the Laplacian in Riemannian manifolds. Afterwards, Bakry and Emery

[46] extended those results in a more general context.

**Proof.** By Corollary 2.2 of [48] we have that for all $f \in \mathcal{D}(\mathcal{L}_n^C)$

$$\pi_n^C[(\mathcal{L}_n^C f)^2] \geq \sum_r \pi_n^C \left[\eta(r)(\nabla_r^- f)^2\right] + \\ + \sum_r \nu(r)^2 \pi_n^C \left[(\nabla_r^+ f)^2, \eta + \mathbf{1}(r) \in \Omega_n^C, \eta + 2\mathbf{1}(r) \notin \Omega_n^C\right] + \\ + \sum_{r \neq s} \nu(r)\nu(s)\pi_n^C \left[(\nabla_r^+ f)(\nabla_s^+ f), \eta + \mathbf{1}(r) \in \Omega_n^C, \eta + \mathbf{1}(r) + \mathbf{1}(s) \notin \Omega_n^C\right], \quad (4.3)$$

where we got an inequality by removing the last positive of the left hand side of the original corollary.

Using the inequality: $(\nabla_r^+ f)(\nabla_s^+ f) \geq -(1/2)[(\nabla_r^+ f)^2 + (\nabla_s^+ f)^2]$ on the last term, by a straightforward computation we get:

$$\pi_n^C[(\mathcal{L}_n^C f)^2] \geq \mathcal{E}_n^C[f] + \\ + \sum_r \nu(r)^2 \pi_n^C \left[(\nabla_r^+ f)^2, \eta + \mathbf{1}(r) \in \Omega_n^C, \eta + 2\mathbf{1}(r) \notin \Omega_n^C\right] + \\ - \sum_r \nu(r)\pi_n^C \left[\mathbf{1}(\eta + \mathbf{1}(r) \in \Omega_n^C)(\nabla_r^+ f)^2 \sum_{s: r \neq s} \nu(s)\mathbf{1}(\eta + \mathbf{1}(r) + \mathbf{1}(s) \notin \Omega_n^C)\right].$$

Now, observe that if $\eta + \mathbf{1}(r) \in \Omega_n^C$ and $\eta + \mathbf{1}(r) + \mathbf{1}(s) \notin \Omega_n^C$ hold simultaneously, than $r \cap s \neq \emptyset$ and $\eta + \mathbf{1}(r) \in \Omega_n^C$ and this allows us to bound the last term above, in such a way

$$\mathbf{1}(\eta + \mathbf{1}(r) \in \Omega_n^C) \sum_{s: r \neq s} \nu(s)\mathbf{1}(\eta + \mathbf{1}(r) + \mathbf{1}(s) \notin \Omega_n^C) \leq \\ \leq \mathbf{1}(\eta + \mathbf{1}(r) \in \Omega_n^C)\|\nu\|_\infty |\{s : |s \cap r| \neq \emptyset, |s| \leq L\}| \leq \\ \leq \|\nu\|_\infty K(D, L)\mathbf{1}(\eta + \mathbf{1}(r) \in \Omega_n^C),$$

where we introduce a constant, $K(D, L)$, such that

$$|\{s : |s \cap r| \neq \emptyset, |s| \leq L\}| \leq K(D, L) < +\infty\,.$$

The dependence of the constant $K(D, L)$ on the maximum degree of the graph derives from the natural relationship between such a degree, *i.e.* number of neighbours and the cardinality of the set $\{s : |s \cap r| \neq \emptyset\}$. This leads to the following inequality

$$- \sum_r \nu(r)\pi_n^C \left[ \mathbf{1}(\eta + \mathbf{1}(r) \in \Omega_n^C)(\nabla_r^+ f)^2 \sum_{s:r \neq s} \nu(s)\mathbf{1}(\eta + \mathbf{1}(r) + \mathbf{1}(s) \notin \Omega_n^C) \right] \geq$$
$$\geq -\|\nu\|_\infty K(D,L)\mathcal{E}_n^C[f].$$

Finally, we are able to rewrite the starting inequality, (4.3), in a simplified form

$$\pi_n^C[(\mathcal{L}_n^C f)^2] \geq \mathcal{E}_n^C[f](1 - \|\nu\|_\infty K(D,L)\mathcal{E}_n^C[f])$$

which is exactly the Proposition 1.1 of [48] that implies the conclusion. ■

This theorem leads to a better estimation of the decay rate in temporal dimension for the covariance between traffic activities on different links. In order to obtain an exponential decay in distance, the space dimension, we adapt a result derived for a Glauber-type dynamics in a volume with boundary condition. Precisely, we adapt Theorem 4.1 of [49] to our case, *i.e.* a discrete birth and death process on a network graph, proving that the uniform positivity of the spectral gap implies the exponential decay of the covariance of two local functions.
For brevity, we first redefine the notation to avoid heavy expressions:
Let $\pi := \pi_n^C$, $P_t := P_t^{C,n}$, $\| \cdot \|_2$ the $L^2(\pi)$-norm, $\mathcal{L} := \mathcal{L}_n^C := \sum_\gamma c(\cdot,\gamma)\nabla_\gamma$, $\mathcal{E}[\cdot] := \mathcal{E}_n^C[\cdot]$, $\Omega := \Omega_n^C$, $\nabla^{r+} := \nabla_r^+$, where we defined $c(\eta, r+) := \nu(r+)\mathbf{1}(\eta + \mathbf{1}(r) \in \Omega^C)$ and $c(\eta, r-) := \eta(r)$.
We assume $\pi[f] = \pi[g] = 0$ and by invariance of measure $\pi[P_t f] = \pi[P_t g] = 0$ for any $t \geq 0$.

The aim of our computation is to estimate the absolute value of the covariance, *i.e.* bounding it from above. In order to do that, we start with a first useful bound for such a covariance with respect to the $\pi$-measure, let $f,\, g \in \mathcal{D}(\mathcal{L})$ be two generic functions, we have

$$|\pi[f,g]| = |\pi[P_t(fg)]| = |\pi[P_t f P_t g] + \pi[P_t(fg) - P_t f P_t g]| \leq$$
$$\leq \|P_t f\|_2 \|P_t g\|_2 + |\pi[P_t(fg) - P_t f P_t g]| \qquad (4.4)$$

where we used the invariance of $\pi$ for $P_t$ and the Schwarz inequality. Now, we focus our analysis on the last summandum of the right hand side of the above inequality, since the first term is easily bounded by the "gap theorem" with the Dirichlet form. In fact for the "$f$ term", and respectively for the $g$ one, we get

$$\|P_t f\|_2 \leq \mathrm{e}^{-t/k}\|f\|_2$$

recalling that $k$ is the spectral gap of $\mathcal{L}$.
We rewrite then the covariance $|\pi[P_t(fg) - P_t f P_t g]|$ in a form that can be easily bounded. Using a general identity for self-adjoint Markov semigroups, we have

$$\begin{aligned}\pi[P_t(fg) - P_t f P_t g] = 2\int_0^t ds\; \mathcal{E}(P_s f, P_s g) = \\ = 2\int_0^t ds\; \sum_r \nu(r)\pi\left[\mathbf{1}(\eta^{r+} \in \Omega)\nabla^{r+}P_s f \nabla^{r+} P_s g\right]\end{aligned} \tag{4.5}$$

Now, the estimate of the covariance turns to the estimate of the two integrand terms $|\nabla^{r+}P_s f|$ and $|\nabla^{r+}P_s g|$. Since they are equivalent, we focus only on the first term, and this leads then in the following Proposition:

**Proposition 4.2.** *For all $f \in \mathcal{D}(\mathcal{L})$, there exist a constant $R < +\infty$ such that*

$$|\nabla^{r+}P_t f| \leq \|\nabla^{r+}f\|_\infty + \int_0^t \left(\sum_{\rho:\rho\cap r\neq\emptyset} \nu(\rho)\|\nabla^{\rho+}P_s f\|_\infty + R\|\nabla^{r+}P_s f\|_\infty\right) ds\,.$$

*holds true.*

**Proof.** In order to prove it, we proceed as follows: we first compute the derivative of $\nabla^{r+}P_t f$, we then estimate such a derivative, and finally we integrate the estimate to get the thesis. We have

$$\frac{d}{dt}\nabla^{r+}P_t f = \mathcal{L}\nabla^{r+}P_t f + [\nabla^{r+}, \mathcal{L}]P_t f = \mathcal{L}\nabla^{r+}P_t f + (\nabla^{r+}\mathcal{L}P_t f - \mathcal{L}\nabla^{r+}P_t f)\,, \tag{4.6}$$

where $[\nabla^{r+}, \mathcal{L}]$ is the commutator.
To bound last equation, we first look at the last summandum of the right hand side, changing the notation for simplicity $g \equiv P_t f$, and we observe that for any $g \in \mathcal{D}(\mathcal{L})$, by definition we get

$$\nabla^{r+}\mathcal{L}g - \mathcal{L}\nabla^{r+}g = (\mathcal{L}g)^{r+} - \mathcal{L}g - \mathcal{L}g^{r+} + \mathcal{L}g = (\mathcal{L}g)^{r+} - \mathcal{L}g^{r+}.$$

and by definition of $\mathcal{L}$ and $c(\eta, r+)$ we have

$$(\mathcal{L}g)^{r+} = \sum_{\gamma} c^{r+}(\cdot,\gamma)[(g^{\gamma})^{r+} - g^{r+}],$$

and

$$\mathcal{L}g^{r+} = \sum_{\gamma} c(\cdot,\gamma)[(g^{r+})^{\gamma} - g^{r+}].$$

Thus, putting all these passages together we finally obtain

$$\begin{aligned}(\mathcal{L}g)^{r+} - \mathcal{L}g^{r+} &= \sum_{\gamma}\{c^{r+}(\cdot,\gamma)[(g^{\gamma})^{r+} - g^{r+}] - c(\cdot,\gamma)[(g^{r+})^{\gamma} - g^{r+}]\} = \\ &= \sum_{\gamma}\{[c^{r+}(\cdot,\gamma)(g^{\gamma})^{r+} - c(\cdot,\gamma)(g^{r+})^{\gamma}] + [c(\cdot,\gamma) - c^{r+}(\cdot,\gamma)]g^{r+}\} = \\ &= \sum_{\gamma}\{[c^{r+}(\cdot,\gamma) - c(\cdot,\gamma)](g^{r+})^{\gamma} + c^{r+}(\cdot,\gamma)[(g^{\gamma})^{r+} - (g^{r+})^{\gamma}] + [c(\cdot,\gamma) - c^{r+}(\cdot,\gamma)]g^{r+}\} = \\ &= \sum_{\gamma} \nabla^{r+}c(\cdot,\gamma)\nabla^{\gamma}(g^{r+}) + \sum_{\gamma} c^{r+}(\cdot,\gamma)[(g^{\gamma})^{r+} - (g^{r+})^{\gamma}].\end{aligned}$$

where sums range over all limited routes $\gamma \in \Omega$.

Now, we focus on the last term of the last identity, and we notice that if $\eta + \mathbf{1}(r) \notin \Omega \Rightarrow \nabla^{r+}P_t f(\eta) = 0$, by definition, so we are interested in

$$\sum_{\gamma} c^{r+}(\eta,\gamma)[(g^{\gamma})^{r+}(\eta) - (g^{r+})^{\gamma}(\eta)]$$

when $\eta + \mathbf{1}(r) \in \Omega$, *i.e.* when the increment of the route still belong to the set $\Omega$.

In order to simplify the computation, we split the sum above into two part, the one when a new connection is established, $c^{r+}(\eta, \rho+)$, on a fixed route, and the other when a connection is dropped, $c^{r+}(\eta, \rho-)$.

- **Case** $\gamma = \rho+$

  Since $\eta + \mathbf{1}(r) \in \Omega$ we have by definition $c^{r+}(\eta,\gamma) = \nu(\rho)\mathbf{1}(\eta^{r+} + \mathbf{1}(\rho) \in \Omega) = \nu(\rho)\mathbf{1}(\eta + \mathbf{1}(r) + \mathbf{1}(\rho) \in \Omega)$. But if $\eta + \mathbf{1}(r) + \mathbf{1}(\rho) \in \Omega$

  $$(g^{\rho+})^{r+}(\eta) = g(\eta + \mathbf{1}(r) + \mathbf{1}(\rho)) = (g^{r+})^{\rho+}(\eta)\,,$$

  which implies:

  $$c^{r+}(\eta,\rho+)[(g^{\rho+})^{r+}(\eta) - (g^{r+}(\eta))^{\rho+}] = 0\,.$$

- **Case** $\gamma = \rho-$

  This case can be splitted once more in two sub cases, in fact we have

$$c^{r+}(\eta, \rho-) = c(\eta^{r+}, \rho-) = c(\eta + \mathbf{1}(r), \rho-) = \begin{cases} \eta(\rho) & \text{if } \rho \neq r \\ \eta(r) + 1 & \text{if } \rho = r. \end{cases}$$

  This means that $c^{r+}(\eta, \rho-) = \eta(\rho) + \mathbf{1}(\rho = r)$, still if $\eta + \mathbf{1}(r) \in \Omega$. Now, we compute $c^{r+}(\eta, \rho-)[(g^{\rho-})^{r+}(\eta) - (g^{r+})^{\rho-}(\eta)]$ when $\eta + \mathbf{1}(r) \in \Omega$.

  - **If** $\rho \neq r$ **Then**

$$c^{r+}(\eta, \rho-)[(g^{\rho-})^{r+}(\eta) - (g^{r+})^{\rho-}(\eta)] = \\ = \eta(\rho)[(g(\eta + \mathbf{1}(r) - \mathbf{1}(\rho)) - g(\eta + \mathbf{1}(\rho) - \mathbf{1}(r))] = 0.$$

  - **If** $\rho = r$ **and** $\eta(r) > 0$ **Then**

$$c^{r+}(\eta, \rho-)[(g^{\rho-})^{r+}(\eta) - (g^{r+})^{\rho-}(\eta)] = \\ = (\eta(r) + 1)[(g(\eta + \mathbf{1}(r) - \mathbf{1}(r)) - g(\eta + \mathbf{1}(r) - \mathbf{1}(r))] = 0.$$

  - **If** $\rho = r$ **and** $\eta(r) = 0$ **Then**

$$c^{r+}(\eta, \rho-)[(g^{\rho-})^{r+}(\eta) - (g^{r+})^{\rho-}(\eta)] = \\ = (\eta(r) + 1)[(g(\eta + \mathbf{1}(r) - \mathbf{1}(r)) - g(\eta + \mathbf{1}(r))] = -\nabla^{r+} g(\eta).$$

We then proved that if $\eta + \mathbf{1}(r) \in \Omega$

$$(\mathcal{L}g)^{r+}(\eta) - \mathcal{L}g^{r+}(\eta) = \sum_{\gamma} \nabla^{r+} c(\cdot, \gamma) \nabla^{\gamma}(g^{r+}) - \mathbf{1}(\eta(r) = 0) \nabla^{r+} g(\eta).$$

To complete the computation of the first derivative, we remark that $\nabla^{r+} c(\eta, \rho-) = \mathbf{1}(r = \rho)$, while, by definition

$$\nabla^{r-}(g^{r+})(\eta) = \begin{cases} 0 & \text{if } \eta(r) = 0 \\ g(\eta) - g(\eta + \mathbf{1}(r)) & \text{if } \eta(r) > 0 \end{cases}$$

This implies:

$$\sum_{\rho} \nabla^{r+} c(\eta, \rho-) \nabla^{\gamma}(g^{r+})(\eta) = -\mathbf{1}(\eta(r) > 0) \nabla^{r+} g(\eta),$$

which leads us to write

$$(\mathcal{L}g)^{r+}(\eta) - \mathcal{L}g^{r+}(\eta) = \sum_{\rho+} \nabla^{r+}c(\cdot,\rho+)\nabla^{\gamma}(g^{r+}) - \nabla^{r+}g(\eta).$$

Finally, we substitute all these computations into (4.6):

$$\frac{d}{dt}\nabla^{r+}P_tf = \mathcal{L}\nabla^{r+}P_tf + \sum_{\rho}\nu(\rho)\nabla^{r+}\mathbf{1}(\cdot+\mathbf{1}(\rho)\in\Omega)\nabla^{\rho+}((P_tf)^{r+}) - \nabla^{r+}P_tf. \quad (4.7)$$

As a countercheck, we notice that LHS $= 0$ if $\eta + \mathbf{1}(r) \notin \Omega$ as it should be. At this point, we are allowed to integrate the derivative to go on our estimate, and by semigroup properties and (4.7), we get

$$\begin{aligned}\nabla^{r+}P_tf - P_t\nabla^{r+}f &= \int_0^t \frac{d}{ds}P_{t-s}\nabla^{r+}P_{t-s}f\ ds = \\ &= \int_0^t \left(-\mathcal{L}P_{t-s}\nabla^{r+}P_sf + P_{t-s}\frac{d}{ds}\nabla^{r+}P_sf\right)\ ds = \\ &= \int_0^t P_{t-s}\left(\sum_{\rho}\nu(\rho)\nabla^{r+}\mathbf{1}(\cdot+\mathbf{1}(\rho)\in\Omega)\nabla^{\rho+}((P_sf)^{r+}) - \nabla^{r+}P_sf\right)\ ds.\end{aligned}$$

that can be rewritten as

$$\begin{aligned}&\nabla^{r+}P_tf = \\ &= P_t\nabla^{r+}f + \int_0^t P_{t-s}\left(\sum_{\rho}\nu(\rho)\nabla^{r+}\mathbf{1}(\cdot+\mathbf{1}(\rho)\in\Omega)\nabla^{\rho+}((P_sf)^{r+}) - \nabla^{r+}P_sf\right)\ ds.\end{aligned}$$

This is an integro-differential equation, whose core is almost equal to zero. In other words, we observe that $\nabla^{r+}\mathbf{1}(\eta+\mathbf{1}(\rho)\in\Omega)\nabla^{\rho+}(P_sf)^{r+}(\eta) = 0$, unless

- $\eta + \mathbf{1}(r) \in \Omega$ otherwise $\nabla^{r+} \equiv 0$
- $\eta + \mathbf{1}(\rho) \in \Omega$ otherwise $\nabla^{\rho+} \equiv 0$
- $\eta + \mathbf{1}(r) + \mathbf{1}(\rho) \notin \Omega$ otherwise $\nabla^{r+}\mathbf{1}(\eta+\mathbf{1}(\rho)\in\Omega) = 0$.

This implies that $r \cap \rho \neq \emptyset$, *i.e.* there exist overlapping routes, and in this case

$$\begin{aligned}&-\nabla^{\rho+}((P_sf)^{r+})(\eta) = -(P_sf)^{r+}(\eta+\mathbf{1}(\rho)) + (P_sf)^{r+}(\eta) = \\ &\quad = -P_sf(\eta+\mathbf{1}(\rho)) + P_sf(\eta+\mathbf{1}(r)) = -\nabla^{\rho+}(P_sf)^{r+}(\eta) = \nabla^{r+}(P_sf)^{\rho+}(\eta) = \\ &\qquad = \nabla^{r+}P_sf - \nabla^{\rho+}P_sf\end{aligned}$$

With these restrictions in mind, we rewrite the equation (4.7) in a more suitable way,

$$\nabla^{r+} P_t f = P_t \nabla^{r+} f +$$
$$+ \int_0^t P_{t-s} \Bigg( \sum_{\rho : \rho \cap r \neq \emptyset} \nu(\rho) \mathbf{1}(\cdot + \mathbf{1}(r), \cdot + \mathbf{1}(\rho) \in \Omega, \cdot + \mathbf{1}(r) + \mathbf{1}(\rho) \notin \Omega) \times$$
$$\times \left( (P_s f)^{r+} - (P_s f)^{\rho+} \right) - \nabla^{r+} P_s f \Bigg) \, ds \, .$$

The inequality (4.2) that concludes the proof, arises from removing the indicator function, since $\mathbf{1}(\cdot + \mathbf{1}(r), \cdot + \mathbf{1}(\rho) \in \Omega, \cdot + \mathbf{1}(r) + \mathbf{1}(\rho) \notin \Omega) \leq 1$, applying the absolute values to both sides, and finally noticing that $|P_t \nabla^{r+} f| \leq \| P_t \nabla^{r+} f \|_\infty \leq \| \nabla^{r+} f \|_\infty$.

$$|\nabla^{r+} P_t f| \leq |P_t \nabla^{r+} f| + \int_0^t P_{t-s} \Bigg( \sum_{\rho : \rho \cap r \neq \emptyset} \nu(\rho) \left( |\nabla^{r+} P_s f| + |\nabla^{\rho+} P_s f| \right) + |\nabla^{r+} P_s f| \Bigg) \, ds \leq$$
$$\leq \|\nabla^{r+} f\|_\infty + \int_0^t \Bigg( \sum_{\rho : \rho \cap r \neq \emptyset} \nu(\rho) \left( \|\nabla^{r+} P_s f\|_\infty + \|\nabla^{\rho+} P_s f\|_\infty \right) + \|\nabla^{r+} P_s f\|_\infty \Bigg) \, ds \leq$$
$$\leq \|\nabla^{r+} f\|_\infty + \int_0^t \Bigg( \sum_{\rho : \rho \cap r \neq \emptyset} \nu(\rho) \|\nabla^{\rho+} P_s f\|_\infty + R \|\nabla^{r+} P_s f\|_\infty \Bigg) \, ds$$

■

Before we continue our initial computation, we dwell on the above inequality defining $F(t, \rho) := \|\nabla^{\rho+} P_t f\|_\infty$, so we have

$$F(t, r) \leq F(0, r) + \int_0^t \Bigg( \sum_{\rho : \rho \cap r \neq \emptyset} \nu(\rho) F(s, \rho) + R(L) F(s, r) \Bigg) \, ds.$$

By equation (4.5) of [49], we claim there are two constants $\delta = \delta(\nu, L) > 0$ and $M = M(\nu, L) < +\infty$ such that

$$F(t, r) \leq M \mathrm{e}^{Mt} \sum_{\rho} \mathrm{e}^{-\delta d(\rho, r)} F(0, \rho) \tag{4.8}$$

where $d(\cdot, \cdot)$ is the Euclidean distance.

Therefore, we collected all the ingredients to rewrite the covariance from eq. (4.5). In order to point out the temporal and space exponential decay, for any $f \in \mathcal{D}(\mathcal{L})$ we define

$\Lambda_f := \{\rho : \nabla^{\rho+} f \not\equiv 0\}$ and we call $f$ *local* if $|\Lambda_f| < +\infty$. Furthermore, we consider $f$ and $g$, local functions, with $\Lambda_f \cap \Lambda_g = \emptyset$, then by eq. (4.5), we get

$$\pi[P_t(fg) - P_t f P_t g] \leq 2\|\nu\|_\infty \int_0^t ds \sum_r F(s,r)G(s,r) \leq$$
$$\leq 2M^2 \mathrm{e}^{2Mt} \|\nu\|_\infty \sum_\rho F(0,\rho) \sum_{\rho'} G(0,\rho') \sum_r \mathrm{e}^{-\delta(d(\rho,r)+d(\rho',r))}$$

At this point, since $F(0,\rho) = \|\nabla^{\rho+} f\|_\infty = 0$ unless $\rho \in \Lambda_f$, last term can be bounded recalling the inequality (4.8) and Lemma 4.2 of [49]

$$2M^2 \mathrm{e}^{2Mt} \|\nu\|_\infty \sum_\rho \|\nabla^{\rho+} f\|_\infty \sum_{\rho'} \|\nabla^{\rho'+} g\|_\infty (|\Lambda_f| \wedge |\Lambda_g|) \mathrm{e}^{-\frac{\delta}{2} d(\Lambda_f,\Lambda_g)} \leq$$
$$\leq K \|\nu\|_\infty \sum_\rho \|\nabla^{\rho+} f\|_\infty \sum_{\rho'} \|\nabla^{\rho'+} g\|_\infty \mathrm{e}^{Mt - m d(\Lambda_f,\Lambda_g)}$$

if $|\Lambda_f| \wedge |\Lambda_g| \leq \exp(m d(\Lambda_f, \Lambda_g))$, where $m > 0$ and $K > 0$ are suitable constants and $a \wedge b$ stands for the minimum between $a$ and $b$.

Finally, the main result comes straightforward by putting together all the above passages and rewriting covariance of eq. (4.4) in such a way

$$|\pi[f,g]| \leq \|P_t f\|_2 \|P_t g\|_2 + |\pi[P_t(fg) - P_t f P_t g]| \leq$$
$$\leq K \|\nu\|_\infty \sum_\rho \|\nabla^{\rho+} f\|_\infty \sum_{\rho'} \|\nabla^{\rho'+} g\|_\infty \mathrm{e}^{Mt - m d(\Lambda_f,\Lambda_g)} + \mathrm{e}^{-4t\,\mathrm{gap}(\mathcal{L})} \|f\|_2 \|g\|_2 \leq$$
$$\leq K' \left( \|\nu\|_\infty \sum_\rho \|\nabla^{\rho+} f\|_\infty \sum_{\rho'} \|\nabla^{\rho'+} g\|_\infty + \|f\|_2 \|g\|_2 \right) \mathrm{e}^{-\frac{m}{2} d(\Lambda_f,\Lambda_g)} \quad (4.9)$$

By choosing $t$ appropriately, *i.e.* $t = md(\Lambda_f, \Lambda_g)/(2M)$.

Inequality (4.9) point out how the rate is directly proportional to the Euclidean distances between two “local” networks.

## 4.3 Conclusion

The description of a bandwidth network as a graph leads to an deeper mathematical analysis of its topological properties, through tools of loss network theory. In this chapter, we have shown how a connection established on a fixed route between two cities does not influence the traffic activity on a distant route. More precisely, we first derived an estimate for a lower bound for the spectral gap of the generator of connections dynamics. Subsequently, we use this result to deduce exponential decay of two-point correlations both in time and space whose rate is proportional to the Euclidean distance between such points, *i.e. routes*.

This result open the way for further research, where pricing and hedging solutions found in the previous chapter in a triangle network framework, could be extended to a global network thought as a series of local networks.

# Chapter 5

# Market Data Analysis

In this chapter we analyse a real price dataset. The data we have contains monthly sales offers on the direct route London - New York covering the time interval 1999-2002. Since such a dataset contains prices of forward contracts with different maturities, we develop a model in order to link the dynamics of those prices with the dynamics of price process for a forward contract with fixed maturity on the same route. Moreover we propose a simple model for the price process consisting of a GBM with Poissonian spikes. We also estimate parameter values from real data for a GBM process, using a truncated increments variation technique to detect and remove spike prices. We simulate then the model in order to mimic the prices data.

## 5.1 From Real Data to Underlying Data

In this section we develop a model in order to link the dynamics of forward prices with different maturities to the dynamics of price process for a forward contract with fixed maturity on the same route.

We summarize our data

- Available Data (Internet/TeleGeography) → Prices of Forward Contracts with different $t_0$ and $T_e$, same delivery period

Assumption: same time interval between $t_0$ and $T_e$ →$T_e - t_0 = D$

$$F(t_0 + \Delta, T_e + \Delta) \equiv f_{t_0,T_e}(\Delta)$$

- Underlying Data Needed: Prices of the same Forward Contract at different times.

$$\widetilde{F}(t_0, T_e, t) \equiv \widetilde{f}_{t_0,T_e}(t)$$

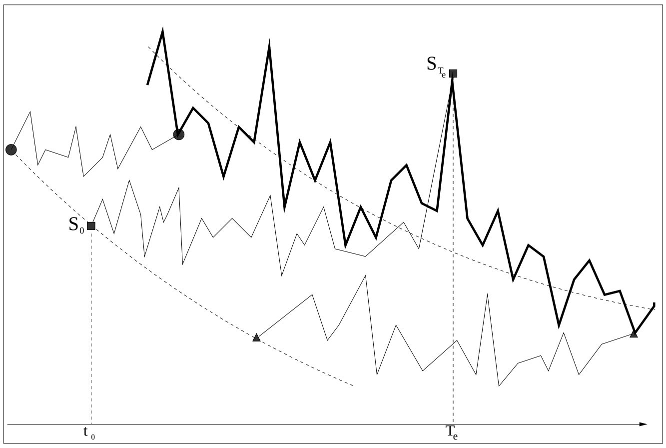



**Figure 5.1:** Underlying data model

What we need now is to show that prices of different forward contracts at delivery time, built according to our model, started at different time but belonging to the same class, *i.e.* same delivery period, same QoS constraints, same bandwidth capacity and same time interval between the initial time and the delivery time, yield a stochastic process consistent with the data we have.

Figure 5.2 depicts what we are looking for. The bold line represents the data we got from the market, the broken lines symbolize the long-term trend (technological innovations, increase of bandwidth capacity, etc..), and the continuous lines represent the evolution of forward contract prices according to our model.

1. **GBM case with deterministic $s_0$ prices**

   Suppose the price of a forward contract follows a Geometric Brownian Motion

$$F(t_0, T_e, t) \equiv S(t) = s_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0) + \sigma(B_t - B_{t_0})\right]$$

with the usual meaning of the parameters and $t \in [t_o, T_e]$.

If we fix the time, $t \equiv T_e$, we get a random variable (Log-Normal Distributed):

$$S(T_e) = s_0(t_0) \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) D + \sigma\sqrt{D}\mathcal{N}(0,1)\right]$$

where $s_0(t_0)$ is for the initial price $s_0$ at time $t_0$, just to make ourselves free to identify $s_0$ with the initial price of a forward contract independent of the time it was drawn up. Suppose the prices, $s_0(t')$, of forward contracts at initial time evolve according to a deterministic function $g_0$

$$s_0(t') = g_0(t' + D)\,.$$

where $t'$ is related to $t$ by the obvious eq. $t' = t - D$, in such a way as to obtain $t' = t_0$ if we set $t = T_e$. In other words, $t$ is the time of exercise of the forward contract and $t'$ is the time when the contract was drawn up.

Then, we can rewrite the equation for the r.v. $S(T_e)$ in such a way

$$S_{T_e}(t) = g_0(t) \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) D + \sigma\sqrt{D}\mathcal{N}(0,1)\right]\,,$$

which represents a continuous stochastic process, a Log-Normal random variables series, with mean and variance respectively

$$\mathbb{E}[S_{T_e}(t)] = g_0(t)e^{\mu D} \quad \text{and} \quad Var[S_{T_e}(t)] = g_0^2(t)\left(e^{2\left(\mu + \frac{\sigma^2}{2}\right)D} - e^{2\mu D}\right)$$

Alternatively, we could look at the logarithmic of the $S_{T_e}(t)$, to reach a more tractable form of the stochastic process

$$\ln\left(\frac{S_{T_e}(t)}{g_0(t)}\right) = \left(\mu - \frac{1}{2}\sigma^2\right) D + \sigma\sqrt{D}\mathcal{N}(0,1)\,,$$

that is well known distributed according to a Normal distribution,

$$\ln\left(\frac{S_{T_e}(t)}{g_0(t)}\right) \sim \mathcal{N}\left(\left(\mu - \frac{1}{2}\sigma^2\right) D, \sigma^2 D\right)$$

2. **GBM case with stochastic $s_0$ prices**

More realistically, we introduce a randomness component even in the evolution of the initial prices, $s_0(t')$. A natural way to do that, is to model the initial price with another GBM, such as

$$s_0(t') = A \exp\left[\left(\mu_* - \frac{1}{2}\sigma_*^2\right) t' + \sigma_*(\widetilde{B}_{t'} - \widetilde{B}_0)\right] \tag{5.1}$$

where $A$ is some given constant larger than zero, standing for the initial price at time $t = 0$ and $\widetilde{B}_t$ is another Brownian Motion independent of $B_t$.

From now on, we denote with $G_0(t)$ the stochastic process $s_0(t')$, defined in 5.1, yielding the initial prices, and for simplicity, we impose the condition $t > D$ to avoid tedious and useless discussions.

The above assumptions lead to the following equation for the forward contract exercise prices

$$S_{T_e}(t) = G_0(t) \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) D + \sigma\sqrt{D}\mathcal{N}(0,1)\right] . \tag{5.2}$$

which is another continuous stochastic process, but let's look at with more detail.

We can rewrite the equation (5.2) replacing $G_0(t)$ by its explicit form, getting

$$S_{T_e}(t) = A \exp\left[\left(\mu_* - \frac{1}{2}\sigma_*^2\right)(t - D) + \sigma_*(\widetilde{B}_t - \widetilde{B}_D)\right] \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) D + \sigma\sqrt{D}\mathcal{N}(0,1)\right]$$

where $\widetilde{B}_t - \widetilde{B}_D$ stands for $\widetilde{B}_{t-D} - \widetilde{B}_0$.

After few simple steps, we get for the process $S_{T_e}(t)$ the following suitable form

$$S_{T_e}(t) = K_0 \exp\left[\left(\mu_* - \frac{1}{2}\sigma_*^2\right) t + \sqrt{\sigma_*^2 t + D(\sigma^2 - \sigma_*^2)}\mathcal{N}(0,1)\right]$$

where

$$K_0 = A \exp\left\{\left[(\mu - \mu_*) - \frac{1}{2}(\sigma^2 - \sigma_*^2)\right] D\right\}$$

To obtain this form we made use of the independence of the two Brownian Motions, $\widetilde{B}_t$ and $B_t$. Therefore, the sum of the two normal random variables deriving from them, *i.e.* $\widetilde{x} \sim (\widetilde{B}_t - \widetilde{B}_D) \sim \mathcal{N}(0, \sigma_*^2(t - D))$ and $x \sim (B_D - B_0) \sim \mathcal{N}(0, \sigma^2 D)$, yields to another normal random variable with mean equal to the sum of the means and variance equal to the sum of the respective variances

$\widetilde{x} + x = \bar{x} \sim \mathcal{N}(0, \sigma_*^2(t - D) + \sigma^2 D)$.

If we now consider the reasonable assumption that the variance of the stochastic process describing the evolution of the initial prices of a forward contract is not so different from the one of the process driving the evolution of the same contract, *i.e.* $\sigma_* \approx \sigma$, and also we take $t > D$, (we have a dataset of years compared with $D$ that usually refers to months), we can approximate the stochastic process for $S_{T_e}(t)$ in such a way

$$S_{T_e}(t) \approx K_0 \exp\left[\left(\mu_* - \frac{1}{2}\sigma_*^2\right) t + \mathcal{N}(0, \sigma_*^2 t)\right]$$

which is a GBM with the same trend as the initial prices process but with different mean and variance. The fact that the two processes have the same trend can be justified by thinking that, on average, the decreasing rate in time of the initial forward contract price, due to technological innovations or increase of available bandwidth capacities, reflects on exercise prices in the same measure.

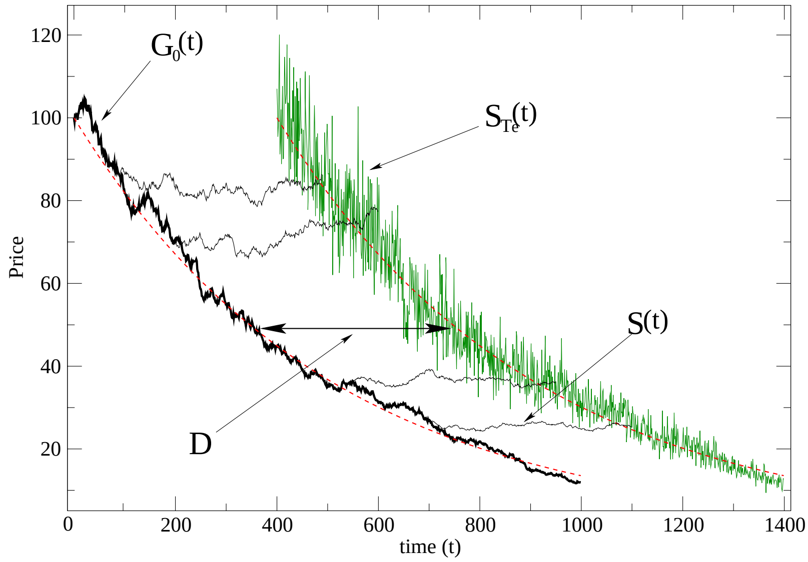



**Figure 5.2:** Numerical simulation of the GBM forward price evolution, $S(t)$, with a stochastic process for the initial price $s_0(t)$. Parameters for $s_0(t)$: $\mu_* = -2$, $\sigma_* = 0.3$; parameters for $S(t)$: $\mu = 0$, $\sigma = 0.1$

## 5.2 Real Data

A very important aspect of building a theoretical model for Bandwidth Markets is the statistical time series analysis of the underlying, *i.e.* the prices of the different contracts sold in a real market.

Getting free data regarding financial markets is always hard, and even more if all the markets are just private markets such as bandwidth markets. We used a free demo of `www.telegeography.com`, one of the online bandwidth markets, to get the price time series of three distinct contracts traded in the period $1999 - 2002$ for the direct route *London*

- *New York*.

The differences among these contracts just depend on the distinct bandwidths traded, respectively **E-1** $= 2.0 Mbps$, **DS-3** $= 45.0 Mbps$ e **STM-1** $= 155.5 Mbps$ and the prices are monthly amount of money that the buyer has to pay to make use of the connection.

Moreover, due to the lack of a global bandwidth market, the data we use only refer to those transactions which haven taken place through this particular web site, so we have partial information, irregular time intervals, etc. Notwithstanding, we consider this dataset rich enough to work out at least the main statistical features of bandwidth time series.

We display below, fig. 5.3-5.4-5.5, the three time series compared with an exponential function to figure out the drift component.

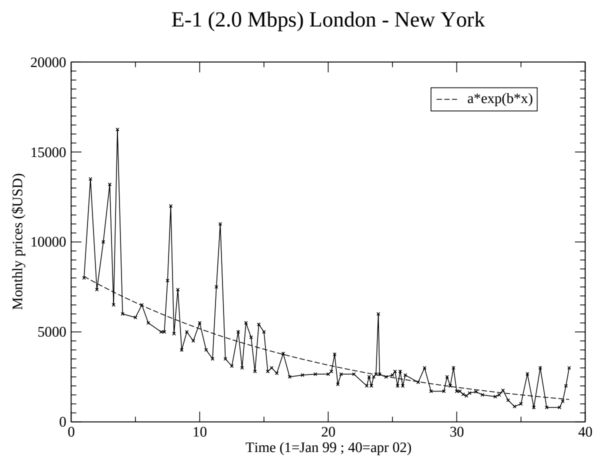



**Figure 5.3:** Exponential parameters: $a = 8495.76$ e $b = 0.0495632$

As can be seen, the time series have common features like decreasing exponential trends and huge positive spikes. The presence of a decreasing exponential trend agrees with the earlier works [36, 37, 38, 39, 40]. In fact, the decreasing behaviour is mainly due to the continuous supply of new technology in data transfer and dark fiber manufacture.

Furthermore, it is very interesting to note the presence of spikes exclusively positive, with short length and remarkable intensity with respect of the most increments, usually observed in all bandwidth markets. Spikes are a first analogy between price dynamic in Bandwidth and Electricity markets. In this last market, price spikes have been observed with magnitudes up to several hundred times normal prices [50]. These are cases of temporary market failure,

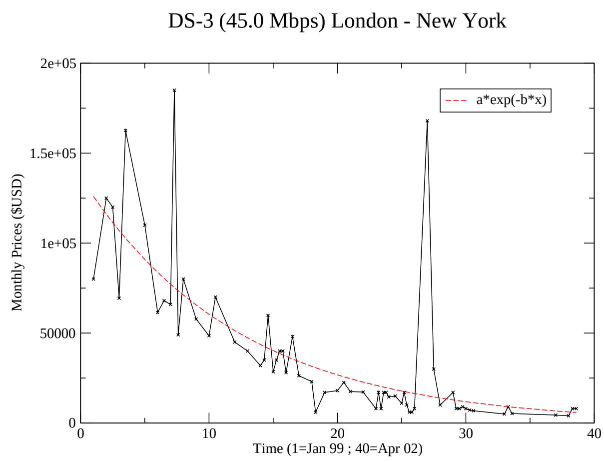



**Figure 5.4:** Exponential parameters: $a = 136592$ e $b = 0.0814904$

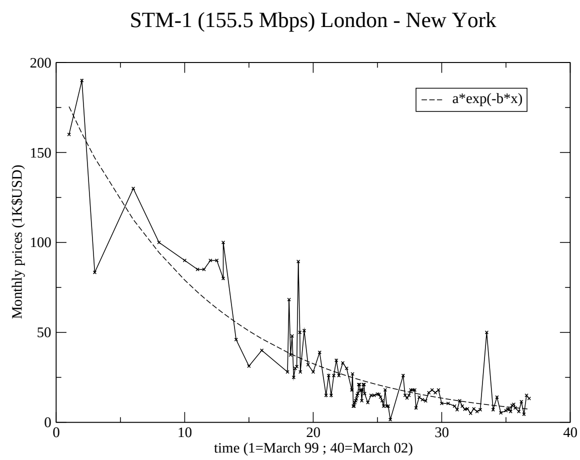



**Figure 5.5:** Exponential parameters: $a = 191.677$ e $b = 0.0885404$

where the price of a good that is in shortage presents a significant upward spike and will return to a more reasonable level only after the cause of the failure has been fixed. During such failures price spikes are inevitable when the underlying is not storable, which is the case of electricity (besides rare cases) and even more for bandwidth. Spot market models and option pricing incorporating discontinuous events have a long history [51], but have only recently been applied to electricity and bandwidth markets.

Mainly, these market failures can be classified into two principal layers. The first is a physical layer: a natural disaster, or most commonly construction works, infrastructure failure or temporary breakdown of a server. The second is a dynamic layer, a correlated and sudden increment of demand with consequent rise of congestion. Anyway, the authors of [36], in lack of a spike statistics, consider a Poisson process for the arrival time of spikes and a Gaussian distribution with zero mean and variance $0.4$ for the magnitude. In particular, this choice is more conservative than the ones commonly used in the electricity markets but easier to handle (in fact, the same authors changed their first choice with a less conservative [38], the Gamma distribution). Therefore, in our model we will use a Pareto distribution to fit the spike magnitude histograms we have, as in models commonly used for electricity.

### 5.2.1 Jumps/Spikes Detection

The problem of determining whether and when a process has jumps is becoming an increasingly important issue. In the case where a large jump occurs, a simple look at the dataset might be sufficient to decide this issue. But on the other hand, a visual inspection of most time series in practice does not provide a clear evidence for either the presence or the absence of small and medium sized jumps. Due to our poor availability of data, we just focus the attention on large sized jumps. Even if they are quite visible in our time series, we choose to adopt an econometric technique to detect such a jump, the truncated increments variation [52].

Actually, this technique has been developed to compute the variance of the continuous part of a process with jumps by using

$$\sum_i (X_{t_i} - X_{t_{i-1}})^2 \mathbb{1}_{\{|X_{t_i} - X_{t_{i-1}}| \leq c(\Delta)\}} \tag{5.3}$$

where $X_{t_i}$ is the discrete stochastic process (log-return of prices $S_t$), $\Delta$ is the time-span and $c(\Delta)$ is a deterministic function such that

$$c(\Delta) \xrightarrow{\Delta \to 0} 0 \qquad \text{and} \qquad \frac{\sqrt{\Delta \log \frac{1}{\Delta}}}{c(\Delta)} \xrightarrow{\Delta \to 0} 0$$

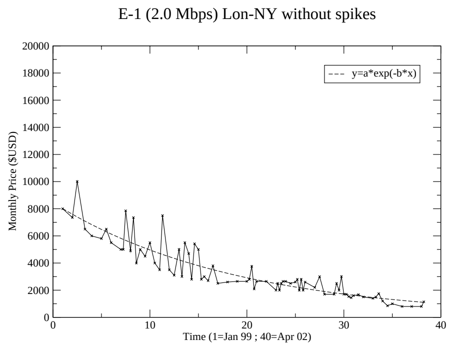
usually $c(\Delta) \sim 5\sigma$ for a given $\Delta$, with $\sigma$ being the volatility of the process.

Furthermore, if we assume that the process has a finite-activity jumps as we have in real data, the formula (5.3) leads to the following statement

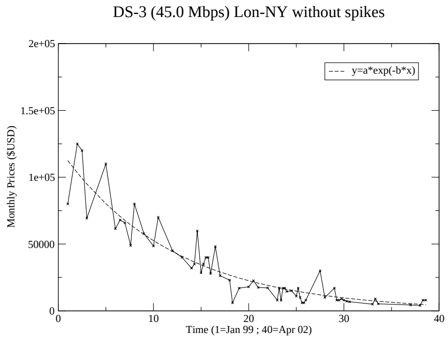
$$\text{There was a jump on} \quad (t_{i-1}, t_i] \iff |X_{t_i} - X_{t_{i-1}}| \geq c(\Delta) \quad .$$

Thus, we adopted this criteria to take the spikes off the three time series and we report the spikeless results in Figure 5.6.

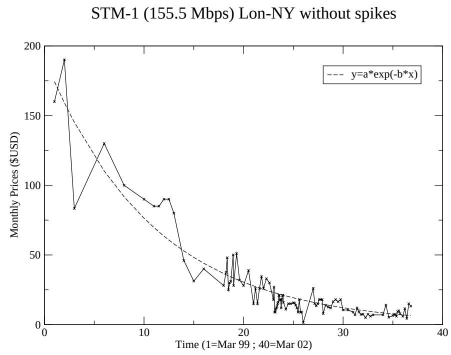
At this point we estimated again the parameters of the exponential fit getting more precise




(a) a=8495.76,b=0.0495632




(b) a=136592,b=0.0814904




(c) a=191.677,b=0.0885404

**Figure 5.6:** Time series without spikes.

results. Due to the lack of data, we used those better fits to compute the relative magnitude of spikes with respect to the trend function. In this way for each spike magnitude we have a number telling us how much its variation was with respect to the mean function (the trend) at that time. Then we grouped all those spikes together in the same histogram. We did that because of the exiguous number of events but even so, the total number of spikes does not allow us to fit the histogram with any probability distribution for the magnitude. We just report the histogram which still gives us information about the spike magnitude distribution, fig. 5.7.

Another important characteristic to analyse is the distribution of waiting time between two consecutive spikes. According to the earlier works both in bandwidth and electricity

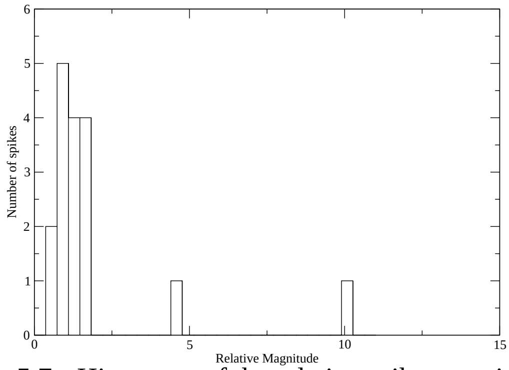



**Figure 5.7:** Histogram of the relative spike magnitudes

markets, we suppose also that spikes occur randomly then they should follow a Poisson process. In a Poisson process the probability distribution of the time intervals between consecutive spikes is an exponential distribution. Therefore, we try to fit the waiting time histogram coming out of data with that distribution. As can be seen in figure 5.8, the histogram we get has too few data to run a hypothesis test for any distribution, but as a first approximation, it looks like an exponential.

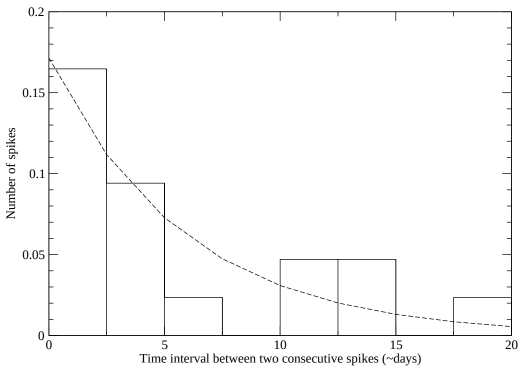



**Figure 5.8:** Waiting time distribution of the spikes

To better strengthen this hypothesis we compute the empirical mean and variance and we get for the mean

$$\hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{1}{\lambda} = 5.82647$$

where $n = 17$ is the number of data points that we have.

Secondly we compute the variance

$$\hat{\sigma}^2 = \sum_{i=1} n \frac{(X_i - \hat{\mu})^2}{n} = \frac{1}{\lambda^2} = 35.54437$$

Note as $\sqrt{\hat{\sigma}^2} = 5.96 \sim \hat{\mu} = 5.82$, in good agreement with the exponential distribution for which $\sigma = \mu = 1/\lambda$.

### 5.2.2 Parameters estimation for GBM part

Once we got the price processes without spikes, we consider the cleaned data as the continuous part of the whole process *i.e.* the part following a geometric Brownian motion. In order to estimate the parameters, mean and variance, of the GBM part of the price process, we compute the log-returns of data for each time series

$$r_i = \ln\left(\frac{S_{i+1}}{S_i}\right) \quad .$$

We report just an example in figure 5.9 where the fluctuations around the zero value strengthen our assumption of a pure decreasing exponential trend.

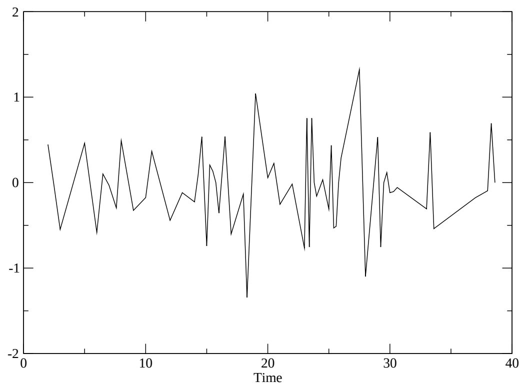



**Figure 5.9:** Log-returns time series of DS-3 prices

Thus, according to the assumption of a geometric Brownian Motion dynamics we get

$$r_i = \ln\left(\frac{S_{i+1}}{S_i}\right) = \mu\Delta t + \sigma(B(i+1) - B(i))$$

where $\Delta t$ is the time interval between data and we fix it at $1 \sim$ (one day) and $B(i)$ is the standard Brownian motion. The earlier equation leads to

$$r_i \sim \mathcal{N}(\mu, \sigma^2)$$

and $r_{i+1}$ is independent of $r_i$ because of the properties of the Brownian motion so that we get a set of *i.i.d.* random variables and we can estimate the mean

$$\hat{\mu} = \frac{1}{N-1} \sum_{i=1}^{N-1} r_i$$

and the variance

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum_{i=1}^{N-1} (r_i - \hat{\mu})^2 \quad .$$

The results for each time series are displayed in the table (5.1). The negative values of the

**Table 5.1:** Table of the estimated mean ($\hat{\mu}$) and standard deviation ($\hat{\sigma}$)

| | $\hat{\mu}$ | $\hat{\sigma}$ |
|---|---|---|
| E-1 | $-0.025$ | $0.555$ |
| DS-3 | $-0.038$ | $0.8024$ |
| STM-1 | $-0.0237$ | $1.1327$ |

means refer again to the decreasing feature of the exponential trend in original data.

Further, we normalize the log-returns in order to increase our statistics. We do that by subtracting the respective mean and dividing by the respective standard deviation from each log-return and for each time series.

$$\widetilde{r}_i^j = \frac{r_i^j - \hat{\mu}^j}{\hat{\sigma}^j}$$

where the index $j = 1, 2, 3$ refers to the time series into examination. Finally we compute the normalized histogram fitted with a normal distribution and we display in figure 5.10.

The high peak, in figure 5.10, refers to the zero value which means no variation of price. In our particular case, the high peak is due to the fact that we put the same value of the previous price recorded whenever we found a lack in the dataset.

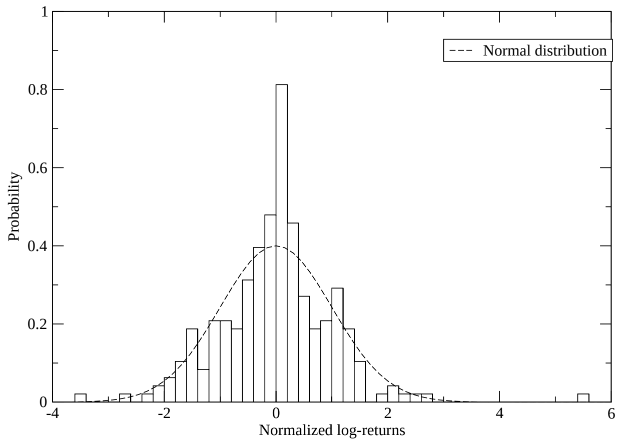



**Figure 5.10:** Probability distribution of the normalized log-return

### 5.2.3 Simulation

We have now collected all the ingredients to run a first simulation in order to reproduce these empirical processes. Contrary to classical models, most stochastic processes introduced to analyse Electricity and Bandwidth prices, are processes with jumps, as first deviation from classical models. Actually bandwidth time series exhibit spikes rather then jumps, due to short and temporary market failures. Treating spikes is a very delicate issue from the theoretical point of view. For this reason, standard approaches in the literature either enforce the recovery of the process' spikeless mean value within a few steps after a jump occurs [53], or are regime-switching models [53, 54]. Most regime-switching models distinguish between two regimes, one "normal" and one "jump" regime. We delay a deeper discussion about these models to future works, because at this moment we believe it is worthless without a sufficient amount of data.

On the contrary, a strong theoretical background has been developed in the last few decades to treat stochastic processes with jumps. Essentially, financial models with jumps can be divided into two categories. In the first category, named jump-diffusion models, the evolution of prices is given by a diffusion process, punctuated by jumps at random intervals. For instance, such an evolution can be represented by modelling the log price as a Lèvy process with a nonzero Gaussian component and a jump part, which is a compound Poisson process with finitely many jumps in every time interval. In these models, the dynamical structure of the process is easy to understand and describe, since the distribution of jump sizes is known. However they rarely lead to closed-form densities: statistical estimation and computation of moments may be quite difficult.

A typical example of a Lèvy process of jump-diffusion type has the following form

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

where $(N_t)_{t\geq 0}$ is the Poisson process counting the jumps of $X$ and $Y_i$ are jump sizes, *i.i.d.* random variables, following a known probability distribution $f$. The second category consists of models with infinite number of jumps in every interval, which are called infinite activity models. In these models, one does not need to introduce a Brownian component since the dynamics of jumps is already rich enough to generate non trivial small time behaviour [55] and it has been argued [56, 57] that such models give a more realistic description of the price process at various time scales. Following most works on Electricity and Bandwidth markets we prefer to make use of models belonging to the first category. We suggest a model that, while capturing the essential characteristics of spot prices to a reasonable degree, is simple enough to yield closed form expressions for bandwidth futures contracts and other derivatives.

As we did in the parameters estimation, we build up a process composed by a GBM with an additional jump term. Since we first just want to simulate this kind of process, we just give the discrete time form

$$\begin{cases} S_{i+1} = S_i \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\epsilon_i\right] \\ \widetilde{S}_{i+1} = S_{i+1} + \theta_{i+1} S_{i+1} Y_{i+1} \end{cases}$$

where $\widetilde{S}_i$ is the composed process, $\epsilon$ is a standard normal random variable from the Wiener process $W_t$, $\theta_i$ is a flag indicator settled in this way

$$\theta_i = \begin{cases} 0 & \text{if} \quad N_{i\Delta t} = N_{(i-1)\Delta t} \\ 1 & \text{if} \quad N_{i\Delta t} \neq N_{(i-1)\Delta t} \end{cases}$$

with $N_t$ being an homogeneous Poisson process with intensity $\lambda$, which is independent of $W_t$. This indicator tell us if there was at least one jump within the time interval $((i-1)\Delta t, i\Delta t)$.

Finally, $Y_i$ is the exponential jump sizes $Y_i \sim \exp(\lambda^\star)$[1] are *i.i.d.* random variables which are independent of both $N_t$ and $W_t$.

From the simulation point of view, we build up a time series of a stochastic process, $S_{t_i}$, following a GBM with mean and variance parameters we estimated from the dataset. Later, we extract from an exponential distribution, with intensity $\lambda$ the waiting times between two consecutive spikes so as to have a vector of the spike occurrence times $t_i^*$. We denote the total number of spikes between time $0$ and $T$ with $N_T$. At this point, we build up a second time series $\widetilde{S}_{t_i}$ in the following way

$$\widetilde{S}_{t_i} = \begin{cases} S_{t_i} & \text{if} \quad t_i \neq t_i^* \\ S_{t_i} \cdot Y_{t_i^*} & \text{if} \quad t_i = t_i^* \end{cases}$$

where $Y_{t_i^*}$ are *i.i.d.* random variables distributed according an exponential distribution with intensity parameter $\lambda^\star$.

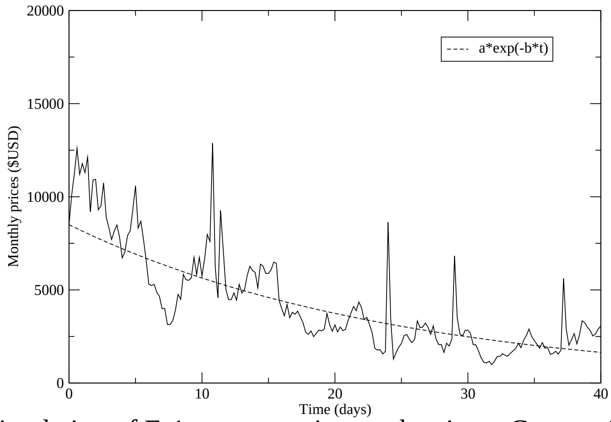



**Figure 5.11:** Simulation of E-1 contract prices path using a Geometric Brownian Motion with Spikes

In Figure 5.11, we show the result of a simulation of a daily prices path using the parameters estimated for the E-1 contract prices with the same time horizon but a higher discretization level.

The simulated path seems to capture well, at least visually, the essential features exhibited by

[1]Here $\lambda^\star$ is the intensity parameter of the hypothetical exponential distribution fitting the relative intensity of spikes. See Figure 5.7.

the bandwidth spot price sample path, like e.g. decreasing exponential trend, huge positive spikes.

# *Chapter 6*

# Conclusions

In this thesis we analysed pricing and modelling issues related to Bandwidth Markets. We introduced in chapter two such unique and emerging markets. A comparison to other perishable commodities has been done. Moreover we dwell on those peculiar features leading to new forms of arbitrage opportunities, such as network and temporal arbitrage. In chapter three we have shown the importance of network arbitrage in modelling “fair” bandwidth prices and especially contingent claims on them. First of all, we have academically build up a pricing model on a two link bandwidth network for a Bernoullian price process for the underlying asset. We pointed out pricing and mostly hedging problems in such a topology. Hence, we proposed a solution both for price and hedging strategy, proving its consistency with the standard pricing formulas.

Finally, we extend our results to the continuum, considering underlying assets following a Geometric Brownian Motion price process.

In chapter four we made use of tools of loss network theory to explore the topological properties of a bandwidth network as a graph. In Particular, we have shown how a connection established on a fixed route between two cities does not influence the traffic activity on a distant route. Therefore, we first derived an estimate for a lower bound for the spectral gap of the generator of connections dynamics. Subsequently, we use this result to deduce expo-

nential decay of two-point correlations both in time and space whose rate is proportional to the Euclidean distance between such points, *i.e. routes*.
This result open the way for further research, where pricing and hedging solutions found in the previous chapter in a triangle network framework, could be extended to a global network thought as a series of local networks. Lastly, in chapter five we looked at real price data. We have developed a model to link such data referring to leasing contracts with different maturities to the price evolution of a forward contract with fixed maturity. We proposed a simple model for the price process consisting of a GBM with Poissonian spikes. Furthermore, we have estimated the parameters for the model adopting a a truncated increments variation technique to detect and remove spike prices. Finally, we simulated such a model in order to mimic the prices data.

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