Impact of time illiquidity in a mixed market without full observation

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March 20, 2015

Abstract

We study a problem of optimal investment/consumption over an infinite horizon in a market with two possibly correlated assets : one liquid and one illiquid. The liquid asset is observed and can be traded continuously, while the illiquid one can be traded only at discrete random times, corresponding to the jumps of a Poisson process with intensity λ, is observed at the trading dates, and is partially observed between two different trading dates. The problem is a nonstandard mixed discrete/continuous optimal control problem, which we solve by a dynamic programming approach. When the utility has a general form, we prove that the value function is the unique viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation and characterize the optimal allocation in the illiquid asset. In the case of power utility, we establish the regularity of the value function needed to prove the verification theorem, providing the complete theoretical solution of the problem. This enables us to perform numerical simulations, so as to analyze the impact of time illiquidity and how this impact is affected by the degree of observation.

Keywords: Investment-consumption problem, liquidity risk, optimal stochastic control, Hamilton-Jacobi-Bellman equation, viscosity solutions, regularity of viscosity solutions.

MSC 2010 Classification: 93E20, 91G80, 35D40, 35B65.
JEL Classification: C61, G11.

*This research was partially supported by the PRIN research project "Metodi deterministici e stocastici nello studio di problemi di evoluzione" of the Italian Minister of University and Research.

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1 Introduction

Following the seminal works of Merton on portfolio management, a classical assumption in mathematical finance is to suppose that assets can be continuously traded by the agents operating in the market. However, this assumption is unrealistic in practice, especially in the case of less liquid markets, where investors cannot buy and sell assets immediately, and have to wait before being able to unwind a position.

In the recent years, several articles have studied the impact of this type of illiquidity. Rogers and Zane [25], Matsumoto [21], Pham and Tankov [23] (see also [7, 24]) consider an investment model where the discrete trading times are given by the jump times of a Poisson process with constant intensity $\lambda > 0$. Bayraktar and Ludkovski [3] study a portfolio liquidation problem in a similar context.

The aforementioned papers focus on an agent investing exclusively in an illiquid asset. However, in practice it is common to have several correlated tradable assets with different liquidity. For instance, an index fund over some given financial market will usually be more liquid than the individual tracked assets, while sharing a positive correlation with those assets. An investor in this market will then have the possibility of hedging his exposure in the less liquid assets by investing in the index and rebalancing his position frequently.

To our knowledge few papers consider the case of a market with two (possibly correlated) assets, one liquid and one illiquid. This is the case of Longstaff [20], who analyzes a two agents portfolio problem in a market with a liquid asset and another asset that becomes non tradable for a given time period. Schwartz and Tebaldi [26] consider a market with a liquid asset that can be traded continuously, and an illiquid asset that cannot be traded and is liquidated at a terminal date. In a recent paper, Ang, Papanikolaou and Westerfield [1], in an infinite horizon framework with discounted power utility of consumption, take a less restrictive point of view on the tradability of the illiquid asset, assuming, as in [7, 14, 21, 23, 25], that it may be traded at discrete random times.

Following [1, 26], we also consider a market with a liquid asset and an illiquid one. In particular, as in [1], the illiquid asset can be traded at some discrete random dates. From the modeling side, the main novelty of our paper is that it treats the case of incomplete observation of the illiquid asset price between trading dates, modeled through an observation parameter interpolating the two extreme cases of full and no observation. This new feature leads us to follow a different methodology than [1], relying on the tool of viscosity solutions to study the associated HJB equation.

More precisely, we study a problem of optimal investment/consumption over an infinite horizon in a market consisting of a liquid and an illiquid asset. The liquid asset is continuously observed and can be continuously traded. The illiquid asset is correlated with the liquid one, with correlation parameter $\rho \in (-1, 1)$, and can be traded only at discrete random times, corresponding to the jumps of a Poisson process with intensity $\lambda > 0$. We assume that the illiquid asset can be observed at the trading dates (as in [7, 14, 21, 23]), but introduce a new feature in the model - with respect to the aforementioned literature - allowing the possibility of partial information between trading dates. We introduce a parameter, $\gamma \in [0, 1]$, measuring the degree of observation of the illiquid asset between two trading dates. The limit cases, $\gamma = 0$ and $\gamma = 1$, correspond, respectively, to the observation settings of [7, 23, 24] and [1, 25, 26].

The mathematical problem is a nonstandard mixed discrete/continuous optimal control problem. By means of a suitable use of Dynamic Programming, extending the idea of [23], we show that the stochastic control problem between trading times can be written as an infinite horizon stochastic time-inhomogeneous control problem. Then, we apply the usual machinery of DP for such problems and, using some results of [9],[1] characterize the value function $\widehat{V}$ of this auxiliary problem as the unique viscosity solution of a HJB equation. At this stage, the viscosity characterization only pertains to the optimal allocation in the illiquid asset.[2] In order to go further and characterize the optimal feedback allocation in the liquid asset and the optimal feedback consumption strategy, we need to prove a regularity result for $\widehat{V}$. This nonstandard regularity result and the related analysis of the optimal consumption and allocation in the liquid asset were left out of the analysis in [9]. Here we provide this result - Theorem 4.6, which is the main theoretical contribution of the paper - in the special case of power utility.[3] It gives a full theoretical solution to the problem. A numerical scheme is proposed for implementation and numerical results are then provided and discussed for different values of the relevant parameters γ, λ, ρ.

Section 2 describes the market model and formulates the investment/consumption problem. Section 3 shows how, by a suitable dynamic programming principle, the problem can be reduced to a standard continuous time stochastic control problem; it presents useful properties of the value functions - the original one and the auxiliary one - and characterizes them by means of viscosity solutions; finally, it characterizes the optimal investment in the illiquid asset. Section 4 solves the problem in the case of power utility and provides an iterative scheme. Finally, Section 5 is devoted to the discussion of the numerical results obtained.

2 Model and optimization problem

Consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, on which are defined:

- A Poisson process $(N_t)_{t\geq 0}$, with intensity $\lambda > 0$. We denote by $(\tau_k)_{k\geq 1}$ its jump times; moreover we set $\tau_0 = 0$.
- Two independent standard Brownian motions $(B_t)_{t\geq 0}$, $(W_t)_{t\geq 0}$, independent also of the Poisson process $(N_t)_{t\geq 0}$.

2.1 Market model

The market model consists of two risky assets with correlation $\rho \in (-1, 1)$:

[1] In [9], these results are proved for $\gamma = 0$. Their extension to the general case $\gamma \in [0, 1]$ is straightforward, see Subsection 3.2.

[2] The outcome of this part of the analysis is already in [9], in the special case $\gamma = 0$.

[3] Our assumption on the utility function covers only the case of positive power, unlike [1]. However the method can be modified to cover the case of negative power as well (see also Remark 2.5).

- A liquid risky asset that can be traded continuously; given $b_L, \sigma_L > 0$, its price L_t evolves according to

$$dL_t = L_t\,(b_L dt + \sigma_L dW_t). \tag{2.1}$$

- An illiquid risky asset that can only be traded at the trading times τ_k; given $b_I, \sigma_I > 0$, its price I_t evolves according to

$$dI_t = I_t\left(b_I dt + \sigma_I\,(\rho dW_t + \sqrt{1-\rho^2}dB_t)\right). \tag{2.2}$$

Without loss of generality, we assume $L_0 = I_0 = 1$. We also suppose the availability of a riskless asset with deterministic dynamics. For simplicity, we assume that the interest rate on this asset is constant and equal to 0.

Remark 2.1. *If the riskless interest rate is not* 0*, one needs to add an extra term in all the equations. In the special case of power utility in Section 4, the assumption that the rate is null is without loss of generality, as it can be eliminated through the discount factor of the objective functional (the constant β in* (2.10) *below) by a suitable change of variables (see in [16, p. 189, Remark 2]).*

2.2 Information

The information setting is the following.

- The liquid asset L is continuously observed.
- The illiquid asset I is observed at the trading random times $(\tau_k)_{k\in\mathbb{N}}$.
- The illiquid asset I is partially observed in the time interval (τ_k, τ_{k+1}).

To formalize the last issue, we suppose that the Brownian motion B_t can be split as

$$B_t = \gamma B_t^{(1)} + \sqrt{1-\gamma^2}B_t^{(2)}, \quad \gamma \in [0,1],$$

where $B^{(1)}, B^{(2)}$ are mutually independent Brownian motions, also independent of W, N, with $B^{(1)}$ observed and $B^{(2)}$ unobserved. Let $(\mathcal{N}_t)_{t\geq 0}$, $(\mathcal{W}_t)_{t\geq 0}$, $(\mathcal{B}_t^{(1)})_{t\geq 0}$ be the filtrations generated, respectively, by N, W, $B^{(1)}$. Define the σ-algebra $\mathcal{I}_t = \sigma\left(I_{\tau_k}\mathbf{1}_{\{\tau_k\leq t\}},\ k\in\mathbb{N}\right)$, $t \geq 0$, and the filtration

$$\mathbb{G}^0 := (\mathcal{G}_t^0)_{t\geq 0}; \qquad \mathcal{G}_t^0 = \mathcal{N}_t \vee \mathcal{I}_t \vee \mathcal{W}_t \vee \mathcal{B}_t^{(1)} = \sigma(\tau_k, I_{\tau_k}; \tau_k \leq t) \vee \mathcal{W}_t \vee \mathcal{B}_t^{(1)}.$$

The observation filtration is $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, where $\mathcal{G}_t = \mathcal{G}_t^0 \vee \sigma(\mathbb{P}\text{-null sets})$. This means that, at time t, the agent has:

- full information on the past of the liquid asset up to time t;
- full information on the trading dates of the illiquid asset realized before t and on the price of the illiquid asset at such trading dates;
- partial information (as described above) on the price of the illiquid asset at t.

The parameter γ measures how much information on I is available in the random interval (τ_k, τ_{k+1}). The limit cases are:

- $\gamma = 0$, corresponding to having no information on B in the interval (τ_k, τ_{k+1}) (as in the information setting of [24]);
- $\gamma = 1$, corresponding to full information and recovering the information setting of [1].

Remark 2.2. *In order to motivate this setting, note that the observation of* L *corresponds to the observation of the process* W*, while the "observation" of* $B^{(1)}$ *should be understood as follows. The price of the illiquid asset is observed at* $(\tau_k)_{k\in\mathbb{N}}$ *(as in [7, 23, 24]), while, at different times* $t \in (\tau_k, \tau_{k+1})$*, the agent observes* $I_t^{(1)}$*, evolving according to*

$$dI_t^{(1)} \;=\; I_t^{(1)}\big(b_I dt + \sigma_I(\rho\, dW_t + \sqrt{1-\rho^2}\,\gamma\, dB_t^{(1)})\big), \qquad I_{\tau_k}^{(1)} \;=\; I_{\tau_k}.$$

Then $I_t = I_t^{(1)} \cdot I_t^{(2)}$*, where* $dI_t^{(2)} = I_t^{(2)}\sigma_I\sqrt{1-\rho^2}\sqrt{1-\gamma^2}\, dB_t^{(2)}$, $I_{\tau_k}^{(2)} = 1$, *is an unobserved component of* I*. Between two trading dates, the price* I *is partially known: the factor* $I^{(1)}$ *is observed, but* $I^{(2)}$ *is not. Within the interval* (τ_k, τ_{k+1}) *the knowledge of* $(L, I^{(1)})$ *is equivalent to the knowledge of* $(W, B^{(1)})$*. In this sense* W *and* $B^{(1)}$ *are observed and the observation filtration is* $\mathbb{G}$*.*

2.3 Trading/consumption strategies and wealth dynamics

Define the set of admissible trading/consumption strategies as follows. Consider all the triplets (c, π, α) such that

(h1) $c = (c_t)_{t\geq 0}$ is a continuous-time nonnegative process, $(\mathcal{G}_t)_{t\geq 0}$-predictable, with locally integrable trajectories; c_t represents the consumption rate at time t;

(h2) $\pi = (\pi_t)_{t\geq 0}$ is a continuous-time process, $(\mathcal{G}_t)_{t\geq 0}$-predictable, with locally square integrable trajectories; π_t represents the amount of money invested in the liquid asset at time t;

(h3) $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, is a discrete process, where α_k is $\mathcal{G}_{\tau_k}$-measurable; α_k represents the amount of money invested in the illiquid asset in the interval $(\tau_k, \tau_{k+1}]$.

Given an initial wealth $r \geq 0$ and a triplet (c, π, α) satisfying (h1)–(h3), the wealth process R is obtained by recursion on $k \in \mathbb{N}$:

$$R_0 \;=\; r, \tag{2.3}$$

$$R_t \;=\; R_{\tau_k} + \int_{\tau_k}^t \big(\pi_s(b_L ds + \sigma_L dW_s) - c_s ds\big) + \alpha_k\left(\frac{I_t}{I_{\tau_k}} - 1\right), \quad t \in (\tau_k, \tau_{k+1}]. \tag{2.4}$$

In general R is not $\mathbb{G}$-predictable (unless $\gamma = 1$), as I is not. Following [1, Sec. 3, p. 9] and [26, Sec. 2, p. 7], we split R into:

- a liquid part X (observable), containing the money held in the liquid asset, the money held in the bank account and the consumption;
- an illiquid part A_t (partially observable).

They are defined in the intervals $[\tau_k, \tau_{k+1})$, $k \in \mathbb{N}$, as

$$X_t \;=\; R_{\tau_k} - \alpha_k + \int_{\tau_k}^t \big(\pi_s(b_L ds + \sigma_L dW_s) - c_s ds\big), \tag{2.5}$$

$$A_t \;=\; \alpha_k \frac{I_t}{I_{\tau_k}}. \tag{2.6}$$

Obviously

$$R_t \;=\; X_t + A_t, \qquad \forall t \geq 0. \tag{2.7}$$

Observe that the process R is continuous, while the processes X, A are not, due to the rebalancing. Moreover, at time τ_k the process R does not depend on the value of α_k, whereas the processes X, A do.

The class of admissible controls is the set of triplets of processes (c, π, α) satisfying (h1)–(h3) and such that the corresponding wealth process R is nonnegative (no-bankruptcy constraint). The latter class depends on the initial wealth r. Denote it by $\mathcal{A}(r)$ and note that it is not empty for every $r \geq 0$, as the null strategy $(c, \pi, \alpha) = (0, 0, 0)$ belongs to it. As $\rho \in (-1, 1)$, the illiquid asset may become very large or small, independently of what happens to the liquid asset. Hence, having a short position in the illiquid asset or having a negative liquid wealth implies a positive probability of negative wealth. These facts suggest that requiring the positivity of R should be equivalent to requiring the positivity of both X and A.

Proposition 2.3. *Let $r \geq 0$. The following facts are equivalent:*

1. $(c, \pi, \alpha) \in \mathcal{A}(r)$*;*

2. $X_t \geq 0$*,* $A_t \geq 0$*, for every* $t \geq 0$*;*

3. (c, π, α) *fulfills (h1)–(h3),* $0 \leq \alpha_k \leq R_{\tau_k}$ *for every* $k \in \mathbb{N}$*, and*

$$-\int_{\tau_k}^{t} \big(\pi_s(b_L ds + \sigma_L dW_s) - c_s ds\big) \;\leq\; R_{\tau_k} - \alpha_k, \qquad \forall t \in [\tau_k, \tau_{k+1}),\ \forall k \in \mathbb{N}.$$

Proof. $3 \Leftrightarrow 2 \Rightarrow 1$ is straightforward, so it only remains to prove $1 \Rightarrow 2$. Fix $(c, \pi, \alpha) \in \mathcal{A}(r)$ and $t \geq 0$. Let $s > t$, $k \in \mathbb{N}$, and consider the non-negligible event $E_{s,k} := \{\tau_k \leq t < s < \tau_{k+1}\}$ and the probability $\mathbb{P}_{s,k}(\cdot) = \frac{\mathbb{P}(\cdot \cap E_{s,k})}{\mathbb{P}(E_{s,k})}$. As $\cup_{s>t,\, k\in\mathbb{N}} E_{s,k} = \Omega$, it suffices to show that, for each $s > t$ and $k \in \mathbb{N}$, we have $X_t \geq 0$ and $A_t \geq 0$, $\mathbb{P}_{s,k} -$ a.s.. So we work on the probability space $(E_{s,k},\ \mathcal{F} \cap E_{s,k},\ \mathbb{P}_{s,k})$ and consider, in the interval $[t, s]$, the filtration $\mathbb{H} := (\mathcal{H}_u)_{u\in[t,s]}$, with $\mathcal{H}_u := \mathcal{G}_u \vee \sigma(B_r;\ r \geq 0)$, where, with an abuse of notation, we still indicate by $\mathcal{G}_u$ the σ-algebra $\mathcal{G}_u$ restricted to $E_{s,k}$. The idea behind the use of the filtration $\mathbb{H}$ is that, as the fluctuations of I due to B cannot be hedged, the agent who wants to check at time t the admissibility of a strategy has to take into account all the possible scenarios of B. So, conditioning the future wealth with respect to (the present information $\mathcal{G}_t$ and) B, the agent must get an almost surely nonnegative random variable. In the rest of the proof, all the equalities and inequalities are intended $\mathbb{P}_{s,k}$-a.s..

As B, W, N are independent, W is still a Brownian motion under this filtration in the probability space defined above. By a Girsanov change of measure, if necessary, without loss of generality we can take $b_L = 0$. Then, letting $T_n := \inf\big\{u \in [t, s] \,|\, \int_t^u \pi_r \sigma_L dW_r \leq -n\big\}$, the process $\left(\int_t^{u\wedge T_n}\pi_r\sigma_L dW_r - c_r dr\right)_{t\le u\le s}$ is a $\mathbb{H}$-supermartingale, thus from (2.5) it follows $\mathbb{E}_{s,k}[X_{s\wedge T_n}|\mathcal{H}_t]\le X_t$. Hence, from (2.6)-(2.7), conditioning with respect to $\mathcal{H}_t$,

$$\mathbb{E}_{s,k}\left[R_{s\wedge T_n}\;\middle|\;\mathcal{H}_t\right] \;=\; \mathbb{E}_{s,k}\left[X_{s\wedge T_n}+A_{s\wedge T_n}\;\middle|\;\mathcal{H}_t\right] \;\le\; X_t+\alpha_k\frac{\mathbb{E}_{s,k}\left[I_{s\wedge T_n}\;\middle|\;\mathcal{H}_t\right]}{I_{\tau_k}}.$$

Letting $n\to\infty$, we can apply Fatou's lemma on the left hand side (by assumption $R\ge 0$) and dominated convergence on the right hand side, obtaining

$$\mathbb{E}_{s,k}\left[R_s\,\middle|\,\mathcal{H}_t\right] \;\le\; X_t+\alpha_k\frac{\mathbb{E}_{s,k}\left[I_s\,\middle|\,\mathcal{H}_t\right]}{I_{\tau_k}}.$$

As $(c,\pi,\alpha)\in\mathcal{A}(r)$, we obtain

$$0 \;\le\; X_t+\alpha_k\frac{\mathbb{E}_{s,k}\left[I_s\,\middle|\,\mathcal{H}_t\right]}{I_{\tau_k}}. \tag{2.8}$$

Let us exploit this inequality by looking at the conditional law of its right hand side given $\mathcal{G}_t$. On $E_{s,k}$ we can decompose $I_s=I_s^{(1)}I_s^{(2)}I_s^{(3)}$, where

$$I_s^{(1)}=e^{(b_I-\sigma_I^2/2)s+\sigma_I(\rho W_t+\sqrt{1-\rho^2}B_t)},\quad I_s^{(2)}=e^{\sigma_I\sqrt{1-\rho^2}(B_s-B_t)},\quad I_s^{(3)}=e^{\sigma_I\rho(W_s-W_t)}.$$

$I_s^{(i)}$, for $i=1,2,3$, are lognormal and, resp., $\mathcal{G}_t$-measurable, $\sigma(B_u-B_t,\ u\in[t,s])$-measurable, and $\sigma(W_u-W_t,\ t\le u\le s)$-measurable. As $W_\cdot-W_t$ is independent of $\mathcal{H}_t$, we have $\mathbb{E}_{s,k}\left[I_s\,\middle|\,\mathcal{H}_t\right]=I_s^{(1)}I_s^{(2)}\mathbb{E}[I_s^{(3)}]$ and (2.8) becomes

$$0 \;\le\; X_t+\frac{\alpha_k}{I_{\tau_k}}\,\mathbb{E}[I_s^{(3)}]I_s^{(1)}I_s^{(2)}. \tag{2.9}$$

Note that $I_s^{(1)},X_t,\alpha_k,I_{\tau_k}$ are $\mathcal{G}_t$-measurable. On the other hand, $I_s^{(2)}$ is independent of $\mathcal{G}_t$, hence the conditional law of $I_s^{(2)}$ given $\mathcal{G}_t$ is lognormal and nondegenerate, as $|\rho|<1$. In particular, it has full support in $(0,\infty)$. Then, taking into account that $I_{\tau_k}>0$, $I_s^{(1)}>0$, $\mathbb{E}[I^{(3)}]>0$, it is clear that, to have (2.9), it must be $X_t\ge 0$ and $\alpha_k\ge 0$. The latter is equivalent to $A_t\ge 0$ and we conclude. □

2.4 Optimization problem

The optimization problem consists in maximizing, over the set $\mathcal{A}(r)$, the expected discounted utility of consumption over an infinite horizon: given a utility function U and a discount factor $\beta>0$, the optimization problem is

$$\text{Maximize}\quad \mathbb{E}\left[\int_0^\infty e^{-\beta s}U(c_s)ds\right],\quad \text{over } (c,\pi,\alpha)\in\mathcal{A}(r). \tag{2.10}$$

Assumption 2.4. *The preferences of the agent are described by a utility function $U:\mathbb{R}_+\to\mathbb{R}$ continuous, nondecreasing, concave, such that $U(0)=0$, and the following growth condition holds: there exist $K_U>0$, $p\in(0,1)$ such that $U(c)\le K_U\,\frac{c^p}{p}$.*

Remark 2.5. *In applications one is often interested in power utility functions* $U(c) = \frac{c^p}{p}$, $p \in (-\infty, 1)$, *with the convention that* $U(c) = \log c$ *when* $p = 0$. *Assumption 2.4 includes only the case* $p \in (0,1)$. *The case of negative exponent is interesting as well, as it seems to capture agents' behavior (see [2]). We work with Assumption 2.4, but stress that the case* $p \le 0$ *can be treated similarly by suitable modifications, even if a bit more difficult to handle (see also Remark 2.6 in [7]). It is treated in [1] under full observation.*

Assumption 2.6. *The discount factor* β *is such that* $\beta > k_p$, *where*

$$k_p := \sup_{u_L \in \mathbb{R}, u_I \in [0,1]} \left\{ p(u_L b_L + u_I b_I) - \frac{p(1-p)}{2}(u_L^2 \sigma_L^2 + u_I^2 \sigma_I^2 + 2\rho u_L u_I \sigma_L \sigma_I) \right\}. \quad (2.11)$$

Remark 2.7. *The assumption on* β *is related to the investment/consumption problem in a liquid market. Let* $p \in (0,1)$ *and consider an agent with initial wealth* r, *consuming at rate* c_t, *investing in* L_t, I_t *continuously, with respective proportions* u_t^L, u_t^I, *and under the constraint that* $u_t^I \in [0,1]$. *Suppose that preferences are represented by the utility function* $U^{(p)}(c) = c^p/p$, *with* $p \in (0,1)$. *Denote by* $\mathcal{A}_{Mert}(r)$ *the set of strategies keeping wealth nonnegative and define*

$$V_{Mert}^{(p)}(r) \quad := \quad \sup_{(u^L, u^I, c) \in \mathcal{A}_{Mert}(r)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U^{(p)}(c_t) dt\right], \quad (2.12)$$

This is a constrained Merton problem which dominates our problem, in the sense that $V_{Mert}^{(p)}(r)$ *is higher than the optimal value of our problem, up to the multiplicative constant* K_U *of Assumption 2.4. One can see (for instance solving the HJB equation) that* $V_{Mert}^{(p)}$ *is finite if and only if Assumption 2.6 is satisfied and that, in this case,* $V_{Mert}^{(p)}(r) = \left(\frac{1-p}{\beta - k_p}\right)^{1-p} r^p$. *Therefore, Assumption 2.6 guarantees, together with the growth condition of Assumption 2.4, finiteness for our problem too.*

Further note that the constrained liquid investment/consumption problem described above can always be reduced to the case where the two assets are independent, because

$$dX_t \quad = \quad X_t\Big(u_t^L \frac{dL_t}{L_t} + u_t^I \frac{dI_t}{I_t}\Big) \quad = \quad X_t\left(\Big(u_t^L + \frac{\rho b_L \sigma_I}{\sigma_L} u_t^I\Big)\frac{dL_t}{L_t} + u_t^I \frac{dJ_t}{J_t}\right),$$

where J *is the process defined below in* (3.7) *(taking* $\gamma = 0$*), and the problem is equivalent to an agent investing in* L *and* J, *with the same constraint for the proportion invested in* I. *However, this reduction does not work for the illiquid problem that we consider: neither the observation constraint (the integrand in* L *being* $\mathbb{G}$*-adapted), nor the trading constraint (the amount held in the illiquid asset being constant between* τ_k *and* τ_{k+1}*) are preserved by this transformation.*

From now on Assumptions 2.4 and 2.6 will be standing assumptions.

3 Dynamic Programming

We denote the value function of the optimal stochastic control problem (2.10) by V:

$$V(r) \quad := \quad \sup_{(c,\pi,\alpha) \in \mathcal{A}(r)} \mathbb{E}\left[\int_0^\infty e^{-\beta s} U(c_s) ds\right], \quad r \ge 0. \quad (3.1)$$

Proposition 3.1. *V is everywhere finite, concave, p-Hölder continuous and nondecreasing. Moreover, there exists $K_V > 0$ such that*

$$V(r) \quad \leq \quad K_V r^p, \qquad r \geq 0. \tag{3.2}$$

Proof. As observed in Remark 2.7, finiteness and (3.2) follow from the growth condition of Assumption 2.4 and Assumption 2.6, by comparing with a constrained Merton problem. Concavity of V follows, by standards arguments, from concavity of U and linearity of the state equation. Monotonicity follows, by standard arguments, from monotonicity of U. Finally, p-Hölder continuity follows from concavity, monotonicity of V, and (3.2). □

Following [23], we state a Dynamic Programming Principle (DPP) to reduce our mixed discrete/continuous problem to a standard one between two trading times.

Proposition 3.2 (DPP)**.** *We have*

$$V(r) \quad = \quad \sup_{(c,\pi,\alpha)\in\mathcal{A}(r)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s} U(c_s)ds + e^{-\beta\tau_1} V\left(R_{\tau_1}\right)\right], \quad \forall\, r \geq 0. \tag{3.3}$$

Proof. The proof parallels [24]. We only indicate modifications. The main difference is that, in our setting, there is some additional information between τ_n and τ_{n+1} brought by $(\mathcal{W}_\cdot \vee \mathcal{B}_\cdot^{(1)})$, so that the processes are no longer deterministic on $(\tau_n, \tau_{n+1}]$ given $\mathcal{G}_{\tau_n}$, but only predictable with respect to $(\mathcal{W}_\cdot \vee \mathcal{B}_\cdot^{(1)})$. Then, one has to use the fact that a process $(\xi_t)_{t\geq 0}$ is $\mathbb{G}$-predictable if and only if it admits a decomposition (see, e.g., Lemma 2.1 in [22]),

$$\xi_t(\cdot) \quad = \quad f_0(s,\cdot)\mathbf{1}_{\{t\leq\tau_1\}} + \sum_{n\geq 1} f_n(s,\cdot,\tau_1, I_{\tau_1},\ldots,\tau_n, I_{\tau_n})\mathbf{1}_{\{\tau_n<t\leq\tau_{n+1}\}},$$

where each f_n is $\mathcal{P}^{W,B^{(1)}} \otimes \mathcal{B}(\mathbb{R}^{2n})$-measurable, $\mathcal{P}^{W,B^{(1)}}$ being the predictable σ-algebra corresponding to $(\mathcal{W}_\cdot \vee \mathcal{B}_\cdot^{(1)})$. Then, one proceeds as in [24], by considering conditional controls and using a countable selection (one needs, in addition, a technical result similar to Lemma 3.2 in [27] for the shifting procedure). □

Remark 3.3. *Our control problem is similar to the one in [22] (see also [5]), so that a similar approach seems possible. However, it does not perfectly fit that setting for several reasons. First, our controls α_k are measurable with respect to $\mathcal{G}_{\tau_k}$, whereas, in [22], they are measurable with respect to $\mathcal{G}_{\tau_k^-}$. Second, we have an infinite number of trading times τ_k, whereas [22] considers a finite number. Third, we consider an infinite horizon, so the backward recursive approach cannot be employed.*

We use DPP to relate our original problem to a continuous-time control problem. For each $x \geq 0$, let $\mathcal{A}_0(x)$ be the set of couples of stochastic processes $(c_s, \pi_s)_{s\geq 0}$ such that

- $(c_s)_{s\geq 0}$ is $(\mathcal{W}_s \vee \mathcal{B}_s^{(1)})_{s\geq 0}$-predictable, nonnegative, and has locally integrable trajectories;
- $(\pi_s)_{s\geq 0}$ is $(\mathcal{W}_s \vee \mathcal{B}_s^{(1)})_{s\geq 0}$-predictable, and has locally square-integrable trajectories;

- $x+\int_0^{\cdot}(-c_s ds+\pi_s(b_L ds+\sigma_L dW_s))\geq 0$.

By Lemma A.1, (3.3) becomes

$$V(r) \;=\; \sup_{0\leq a\leq r}\ \sup_{(c,\pi)\in\mathcal{A}_0(r-a)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds+e^{-\beta\tau_1}V\left(R_{\tau_1}\right)\right]. \tag{3.4}$$

We rewrite the inner optimization problem in (3.4), i.e.

$$\sup_{(c,\pi)\in\mathcal{A}_0(r-a)} \mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds+e^{-\beta\tau_1}V\left(R_{\tau_1}\right)\right]. \tag{3.5}$$

Define (see Remark 3.4(i) for explanations)

$$b_Y \;:=\; \gamma^2 b_I+(1-\gamma^2)\frac{\rho b_L\sigma_I}{\sigma_L}, \qquad b_J \;:=\; (1-\gamma^2)\big(b_I-\frac{\rho b_L\sigma_I}{\sigma_L}\big), \tag{3.6}$$

and, given $x,y\geq 0$ and $(c,\pi)\in\mathcal{A}_0(x)$, define $J,\tilde{X}^{x,c,\pi},\tilde{Y}^y$ as solutions to

$$dJ_t \;=\; J_t\left(b_J dt+\sigma_I\sqrt{1-\rho^2}\sqrt{1-\gamma^2}dB_t^{(2)}\right), \quad J_0=1, \tag{3.7}$$

$$d\tilde{X}_t \;=\; -c_t ds+\pi_t(b_L dt+\sigma_L dW_t), \quad \tilde{X}_0=x, \tag{3.8}$$

$$d\tilde{Y}_t \;=\; \tilde{Y}_t\left(b_Y dt+\sigma_I\big(\rho dW_t+\sqrt{1-\rho^2}\ \gamma dB_t^{(1)}\big)\right), \quad \tilde{Y}_0=y. \tag{3.9}$$

Then, for each $t\in[0,\tau_1)$, we have $X_t=\tilde{X}_t^{r-\alpha_0,c,\pi}$, $A_t=\tilde{Y}_t^{\alpha_0}\cdot J_t$. Set $\mathcal{W}_\infty:=\bigvee_{t\geq 0}\mathcal{W}_t$, $\mathcal{B}_\infty^{(1)}:=\bigvee_{t\geq 0}\mathcal{B}_t^{(1)}$, $\mathcal{B}_\infty^{(2)}:=\bigvee_{t\geq 0}\mathcal{B}_t^{(2)}$. As τ_1 is independent of $\mathcal{W}_\infty\vee\mathcal{B}_\infty^{(1)}\vee\mathcal{B}_\infty^{(2)}$ and has distribution $\mathcal{E}(\lambda)$, whereas c, J, $\tilde{X}^{x,c,\pi},\tilde{Y}^y$ are $(\mathcal{W}_\infty\vee\mathcal{B}_\infty^{(1)}\vee\mathcal{B}_\infty^{(2)})$-measurable, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds+e^{-\beta\tau_1}V(R_{\tau_1})\;\Big|\;\mathcal{W}_\infty\vee\mathcal{B}_\infty^{(1)}\vee\mathcal{B}_\infty^{(2)}\right]\\
=\;&\int_0^\infty \lambda e^{-\lambda t}\left(\int_0^t e^{-\beta s}U(c_s)ds+e^{-\beta t}V(\tilde{X}_t^{r-a,c,\pi}+J_t\cdot\tilde{Y}_t^a)\right)dt\\
=\;&\int_0^\infty e^{-\beta s}U(c_s)\int_s^\infty \lambda e^{-\lambda t}dt\,ds+\int_0^\infty \lambda e^{-(\lambda+\beta)t}V(\tilde{X}_t^{r-a,c,\pi}+J_t\cdot\tilde{Y}_t^a)dt\\
=\;&\int_0^\infty e^{-(\beta+\lambda)t}\left(U(c_t)+\lambda V(\tilde{X}_t^{r-a,c,\pi}+J_t\cdot\tilde{Y}_t^a)\right)dt,
\end{aligned}$$

where, in the second equality, we used Fubini's Theorem. On the other hand, as J is independent of $\mathcal{W}_\infty\vee\mathcal{B}_\infty^{(1)}$, whereas c, $\tilde{X}^{x,c,\pi},\tilde{Y}^y$ are $(\mathcal{W}_\infty\vee\mathcal{B}_\infty^{(1)})$-measurable, conditioning the equality above with respect to $\mathcal{W}_\infty\vee\mathcal{B}_\infty^{(1)}$, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau_1} e^{-\beta s}U(c_s)ds+e^{-\beta\tau_1}V(R_{\tau_1})\;\Big|\;\mathcal{W}_\infty\vee\mathcal{B}_\infty^{(1)}\right]\\
=\;&\mathbb{E}\left[\int_0^\infty e^{-(\beta+\lambda)t}\left(U(c_t)+\lambda V(\tilde{X}_t^{r-a,c,\pi}+J_t\cdot\tilde{Y}_t^a)\right)dt\;\Big|\;\mathcal{W}_\infty\vee\mathcal{B}_\infty^{(1)}\right],\\
=\;&\int_0^\infty e^{-(\beta+\lambda)t}\left(U(c_t)+\lambda G_\gamma[V](t,\tilde{X}_t^{r-a,c,\pi},\tilde{Y}_t^a)\right)dt,
\end{aligned}$$

where $G_\gamma[V](t,x,y) := \mathbb{E}\left[V(x+yJ_t)\right]$ (the dependence on γ coming from the dependence on γ of J). Then, we can rewrite (3.5) as

$$\sup_{(c,\pi)\in\mathcal{A}_0(r-a)} \mathbb{E}\left[\int_0^\infty e^{-(\beta+\lambda)t}\left(U(c_t)+\lambda G_\gamma[V](t,\tilde{X}_t^{r-a,c,\pi},\tilde{Y}_t^a)\right)dt\right]. \tag{3.10}$$

It is useful to define G_γ as a linear operator from the space $\mathcal{M}_1(\mathbb{R}_+;\mathbb{R})$ of measurable functions with at most linear growth to the space of measurable functions $\mathcal{M}(\mathbb{R}_+^3;\mathbb{R})$:

$$\begin{aligned} G_\gamma:\ \mathcal{M}_1(\mathbb{R}_+;\mathbb{R}) &\longrightarrow \mathcal{M}(\mathbb{R}_+^3;\mathbb{R}) \\ \psi &\longmapsto G_\gamma[\psi](t,x,y) := \mathbb{E}\left[\psi(x+yJ_t)\right]. \end{aligned} \tag{3.11}$$

Useful properties of G_γ are listed in Proposition A.2.

3.1 The auxiliary control problem

The optimization problem (3.10) is a continuous, non autonomous, stochastic control problem over an infinite horizon that we call auxiliary problem. One can apply the dynamic programming approach to this problem defining the same problem for generic initial data. For each $t\geq 0$, consider the couples of stochastic processes (c,π) such that

(h1)′ $(c_s)_{s\geq t}$ is $(\mathcal{W}_s\vee\mathcal{B}_s^{(1)})_{s\geq t}$-predictable, nonnegative, and has locally integrable trajectories,

(h2)′ $(\pi_s)_{s\geq t}$ is $(\mathcal{W}_s\vee\mathcal{B}_s^{(1)})_{s\geq t}$-predictable, and has locally square-integrable trajectories,

and define, for $x\geq 0$,

$$\mathcal{A}_t(x) := \left\{(c,\pi)\text{ fulfilling (h1)}'-\text{(h2)}'\ \Big|\ x+\int_t^\cdot(-c_s ds+\pi_s(b_L ds+\sigma_L dW_s))\geq 0\right\}.$$

Let $(t,x,y)\in\mathbb{R}_+^3$. Given $(c,\pi)\in\mathcal{A}_t(x)$, let $(\tilde{X}_s^{t,x,c,\pi})_{s\geq 0},(\tilde{Y}_s^{t,y})_{s\geq 0}$ be the solutions to

$$d\tilde{X}_s = -c_s ds+\pi_s(b_L ds+\sigma_L dW_s),\quad \tilde{X}_t = x, \tag{3.12}$$

$$d\tilde{Y}_s = \tilde{Y}_s\left(b_Y ds+\sigma_I\left(\rho dW_s+\sqrt{1-\rho^2}\ \gamma dB_s^{(1)}\right)\right),\quad \tilde{Y}_t = y. \tag{3.13}$$

By definition of $\mathcal{A}_t(x)$, we have $\tilde{X}^{t,x,c,\pi}\geq 0$. Moreover, $\tilde{Y}^{t,y}\geq 0$. Define the (auxiliary) value function

$$\widehat{V}(t,x,y) := \sup_{(c,\pi)\in\mathcal{A}_t(x)}\mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}\left(U(c_s)+\lambda G_\gamma[V](s,\tilde{X}_s^{t,x,c,\pi},\tilde{Y}_s^{t,y}\right)ds\right]. \tag{3.14}$$

Associating to every locally bounded function $\hat{v}$ on $\mathbb{R}_+^3$, the function, defined for $r\geq 0$, $[\mathcal{H}\hat{v}](r) := \sup_{0\leq a\leq r}\hat{v}(0,r-a,a)$, by (3.4) and (3.14), we get

$$V(r) = [\mathcal{H}\widehat{V}](r),\quad \forall r\geq 0. \tag{3.15}$$

The problems (3.14)-(3.15) are coupled: $\widehat{V}$ is defined in terms of V in (3.14) and V is expressed in terms of $\widehat{V}$ in (3.15).

Remark 3.4. ***(i)*** *The choices for the drifts* b_Y *and* b_J *in* (3.6) *are motivated by the fact that we need a couple of processes* $(\tilde{Y}, J)$ *such that: (i)* $\tilde{Y} \cdot J = A$ *on* $[0, \tau_1)$*, where* A *is defined in* (2.6)*; (ii)* $\tilde{Y}$ *is* $\mathbb{G}$*-adapted; (iii)* J *is independent of* $\mathbb{G}$*. Therefore, it is natural to consider the processes* (3.7) *and* (3.9)*, with* b_Y, b_J *that can be chosen under the constraint* $b_J + b_Y = b_I$*. Define the constants*

$$k_{L,Y,p} := \sup_{u_L \in \mathbb{R}, u_Y \in [0,1]} \Big\{ p(u_L b_L + u_Y b_Y) - \frac{p(1-p)}{2}(u_L^2 \sigma_L^2 + u_Y^2 \sigma_I^2(\rho^2 + \gamma^2(1-\rho^2)) + 2\rho u_L u_Y \sigma_L \sigma_I)\Big\}, \quad (3.16)$$

$$k_{J,p} := \sup_{u_J \in [0,1]} \left\{ p b_J u_J - \frac{p(1-p)}{2} \sigma_I^2 (1-\rho^2)(1-\gamma^2) u_J^2 \right\}. \quad (3.17)$$

These constants naturally appear, respectively, in Lemma A.3 and Proposition A.2-(v). Combining these two results with (3.2)*, one gets an estimate on the growth of* $\widehat{V}$ *(see* (3.20) *below) under the condition that* $\beta > k_{L,Y,p} + k_{J,p}$*. In Lemma A.4, it is proved that, for our choice of* b_Y, b_J*,*

$$k_{L,Y,p} + k_{J,p} = k_p, \quad (3.18)$$

which is the minimum possible value of $k_{L,Y,p} + k_{J,p}$*. So our drift choices enable us to treat the auxiliary problem without restrictions on* β *other than Assumption 2.6.*

(ii) *The auxiliary problem* (3.14) *is not autonomous, due to the dependence of* $G_\gamma[V]$ *on time (in general). In the case of full observation* ($\gamma = 1$)*, one has* $J \equiv 1$ *and* $G_1[V](t,x,y) = V(x+y)$*, hence, consistent with [1], we get an autonomous problem:*[4]

$$\sup_{(c,\pi) \in \mathcal{A}_0(x)} \mathbb{E}\left[\int_0^\infty e^{-(\beta+\lambda)s} \left(U(c_s) + \lambda V\left(\tilde{X}_s^{0,x,\pi,c} + \tilde{Y}_s^{0,y} \right) \right) ds \right].$$

Therefore, time inhomogeneity of our auxiliary problem is due to the lack of full information. *We also note that we take, in* (3.14)*, the discount* $e^{-(\beta+\lambda)(s-t)}$ *in place of the usual* $e^{-(\beta+\lambda)s}$ *with this choice, we can get rid of the exponential terms in the HJB equation.*

(iii) *The value function* $\widehat{V}$ *at time* $t = 0$ *is the analogue of the value function of [1], where the agent starts with* given *initial liquid endowment* x *and initial illiquid endowment* y *(in [1],* $t = 0$ *is not a rebalancing time, so, unlike our case, the agent is not allowed to split the total endowment* $r = x + y$ *in a different proportion at* $t = 0$*). In that regard, note that there is no loss of generality in assuming that* $t = 0$ *is a trading time of the illiquid asset: to treat the problem where* $t = 0$ *is not a trading date for the illiquid asset and the initial endowment in liquid and illiquid are respectively* x *and* y*, it suffices to not perform the first static optimization* (3.15)*.*

[4] Our dynamic programming approach differs from [1]. Our approach seems to be the only one possible to deal with partial information ($\gamma < 1$). Hence, the differential problem we get is different from the one in [1]; this difference remains when our control problem coincides with one of [1] ($\gamma = 1$ and power utility). Nevertheless, as intuitively expected, our problem is autonomous like the one in [1], when the two control problems coincide.

3.2 HJB equation and viscosity characterization of $\widehat{V}$

This section characterizes $\widehat{V}$ as unique constrained viscosity solution of an associated HJB equation. First, we state some qualitative properties of $\widehat{V}$. We omit the proof, which parallels [9], where only the case $\gamma = 0$ is considered.

Proposition 3.5. *For every $t \geq 0$, $\widehat{V}(t,\cdot,\cdot)$ is concave with respect to (x,y) and nondecreasing with respect to x and y. Moreover, it satisfies the boundary condition*

$$\widehat{V}(t,0,y) \;=\; \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}\lambda G_\gamma[V](s,0,\tilde{Y}_s^{t,y})ds\right], \quad \forall t \geq 0,\ \forall y \geq 0. \tag{3.19}$$

Finally, $\widehat{V}$ is continuous on $\mathbb{R}^3_+$ and satisfies, for some $K_{\widehat{V}} > 0$, the growth condition

$$0 \;\leq\; \widehat{V}(t,x,y) \;\leq\; K_{\widehat{V}} e^{k_{J,p}t}(x+y)^p, \qquad \forall (t,x,y) \in \mathbb{R}^3_+. \tag{3.20}$$

Let $\mathcal{S}_2$ denote the space of real symmetric 2×2 matrices. Standard arguments of stochastic control (see, e.g., [27, Ch. 4]) associate to $\widehat{V}$ the HJB equation[5]

$$-\hat{v}_t + (\beta+\lambda)\hat{v} - \lambda G_\gamma[\mathcal{H}\hat{v}] - \sup_{c\geq 0,\pi\in\mathbb{R}} H_{cv}(y, D_{(x,y)}\hat{v}, D^2_{(x,y)}\hat{v};\ c,\pi) \;=\; 0, \tag{3.21}$$

where, for $(y,q,Q) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathcal{S}_2$, $c \geq 0, \pi \in \mathbb{R}$, the function H_{cv} is defined as

$$\begin{aligned} &H_{cv}(y,q,Q;c,\pi) \\ :=\;& U(c) + (\pi b_L - c)q_1 + b_Y y q_2 + \frac{\sigma_L^2\pi^2}{2}Q_{11} + \pi\rho\sigma_I\sigma_L y Q_{12} + (\rho^2 + \gamma^2(1-\rho^2))\frac{\sigma_I^2}{2}y^2 Q_{22}. \end{aligned}$$

Remark 3.6. *Equation* (3.21) *has two nonlocal terms: $\mathcal{H}$ and $G_\gamma[\cdot]$. The first is due to illiquidity, the second to partial observation and it disappears in the case of full observation (see Remark 3.4(ii)).*

Definition 3.7. **(1)** An upper-semicontinuous (resp., lower-semicontinuous) function v is a *viscosity subsolution* (resp., *supersolution*) to (3.21) at $(t,x,y) \in \mathbb{R}^3_+$, if

$$\begin{aligned} &-\varphi_t(t,x,y) + (\beta+\lambda)\varphi(t,x,y) - \lambda G_\gamma[\mathcal{H}v](t,x,y) \\ &- \sup_{c\geq 0,\pi\in\mathbb{R}} H_{cv}(y, D_{(x,y)}\varphi(t,x,y), D^2_{(x,y)}\varphi(t,x,y);\ c,\pi) \;\leq\; 0 \quad (\text{resp., } \geq 0), \end{aligned}$$

for each $\varphi \in C^{1,2}(\mathbb{R}^3_+;\mathbb{R})$ such that $\varphi(t,x,y) = v(t,x,y)$ and $\varphi \geq v$ (resp., $\leq$) on $[t,t+\varepsilon)\times\mathcal{O}$, for some neighborhood $\mathcal{O} \subset \mathbb{R}^2_+$ of (x,y) and some $\varepsilon > 0$.

(2) A continuous function v is a *constrained viscosity solution* to (3.21), if it is a subsolution at all $(t,x,y) \in \mathbb{R}^3_+$, a supersolution at all $(t,x,y) \in \mathbb{R}_+ \times (0,+\infty) \times \mathbb{R}_+$, and satisfies the boundary condition

$$\hat{v}(t,0,y) \;=\; \mathbb{E}\Big[\int_t^\infty e^{-(\beta+\lambda)(s-t)}\lambda G_\gamma[\mathcal{H}\hat{v}](s,0,\tilde{Y}_s^{t,y})ds\Big], \quad \forall t \geq 0,\ \forall y \geq 0. \tag{3.22}$$

[5] In the standard derivation of the HJB equation, one gets $-\lambda G_\gamma[V]$ in place of the third term. Therefore, the HJB equation is coupled with (3.15), as in [23, 24]. We insert the nonlocal term (3.15) directly in the equation.

Remark 3.8. ***(i)*** *To simplify proofs, we use a definition of viscosity solution that differs from the usual one. In the standard definition (see [6]), the test functions φ stay above v in a neighborhood of (t,x,y), whereas the property in our case is required only for $s \geq t$. Our definition is more restrictive, as we enlarge the set of test functions. Nevertheless, the two definitions are equivalent if the comparison principle for viscosity solutions holds for the standard definition (see [15]).*

(ii) *The notion of* constrained *viscosity solution is specific to our stochastic control problem. The boundaries $\mathcal{B}_1 := \{(x,y) \in \mathbb{R}^2_+ \,:\, x = 0,\ y > 0\}$ and $\mathcal{B}_2 := \{(x,y) \in \mathbb{R}^2_+ \,:\, x > 0,\ y = 0\}$ are both absorbing (that is, state trajectories staring on $\mathcal{B}_i$ remain on $\mathcal{B}_i$), although differently. When the initial state is on $\mathcal{B}_2$, the control problem degenerates to a one dimensional control problem whose HJB equation is* (3.21) *restricted to $\mathcal{B}_2$. For this reason, the value function must satisfy viscosity sub- and supersolution properties. When the initial state is on $\mathcal{B}_1$, the control problem vanishes, as $\mathcal{A}_t(0) = \{(0,0)\}$, and the natural condition to impose is of Dirichlet type. However, the viscosity subsolution property holds on $\mathcal{B}_1$: for this reason, we require it in our definition, even if it is redundant for the purpose of establishing a comparison property.*

Theorem 3.9. *$\widehat{V}$ is the unique constrained viscosity solution to* (3.21) *satisfying* (3.20).

Proof. The proof that $\widehat{V}$ is a viscosity subsolution on $\mathbb{R}^3_+$ and a viscosity supersolution on $\mathbb{R}_+ \times (0,+\infty)^2$ parallels, e.g., [27, Ch. 4, Th. 5.2] (see also [4] for an approach via the stochastic Perron method, which requires only a viscosity comparison property and no Dynamic Programming Principle). The Dirichlet boundary condition (3.22) follows from (3.19) and (3.15). The growth condition (3.20) was proven. When $y = 0$, as noted in Remark 3.8(ii), the control problem is one-dimensional and $\widehat{V}$ is a viscosity supersolution by standard arguments.

Uniqueness is a consequence of the comparison principle (Proposition 3.10), whose proof parallels [9], where only the case $\gamma = 0$ is treated. □

Proposition 3.10. *Let v_1 (resp., v_2) be a viscosity subsolution (resp., supersolution) to* (3.21) *on $\mathbb{R}_+ \times (0,+\infty) \times \mathbb{R}_+$. Assume that v_1, v_2 satisfy the growth condition* (3.20) *and*

$$v_1(t,0,y) \quad \leq \quad \mathbb{E}\Big[\int_t^\infty e^{-(\beta+\lambda)(s-t)} \lambda G_\gamma[\mathcal{H}v_1](s,0,Y_s^{t,y})ds\Big] \tag{3.23}$$

(resp., $\geq$ for v_2). Then $v_1 \leq v_2$ on $\mathbb{R}^3_+$.

3.3 Optimal policy in the illiquid asset

The results obtained above characterize the optimal allocation policy in the illiquid asset. If $\widehat{V}$ can be computed numerically as a viscosity solution of (3.21) (see Section 4.6), then the optimal allocation policy $(\alpha_k^*)_{k\in\mathbb{N}}$ in the illiquid asset can be derived. At time $t = 0$, (3.15) implies that α_0^* is an optimal allocation in the illiquid asset if and only if $\alpha_0^* \in \operatorname{argmax}_{0\leq a\leq r} \widehat{V}(0, r-a, a)$. Note that the case $\alpha_0^* = 0$ cannot be ruled out at this stage. The property can be generalized to the random trading dates τ_k : α_k^* is an optimal allocation in the illiquid asset if and only if

$$\alpha_k^* \quad \in \quad \operatorname{argmax}_{0\leq a\leq R_{\tau_k}} \widehat{V}(0, R_{\tau_k} - a, a), \qquad k \in \mathbb{N}, \tag{3.24}$$

as a consequence of the Markov property. We omit the proof for brevity.

4 Power utility

To characterize optimal policies, smoothness of $\widehat{V}$ is needed. Unfortunately, the HJB equation (3.21) is degenerate, as the control problem has two state variables and a one dimensional Brownian motion, and the regularity theory for PDEs does not cover that case. Hence, to proceed, we assume that the utility function U is of power type, so that, as in [1, 26], the problem reduces to one spatial dimension.

Assumption 4.1. $U(c) = \frac{c^p}{p}, \; p \in (0,1)$.

Assumption 4.1 holds from here on.

Remark 4.2. *When $p \le 0$, the problem is investigated in [1] assuming full observation of the illiquid asset, which corresponds to our case $\gamma = 1$.*

4.1 Reduction to one spatial variable

Proposition 4.3. *There exists $K_V > 0$ such that*

$$V(r) = K_V r^p, \quad \forall r \ge 0. \tag{4.1}$$

Hence

$$G_\gamma[V](t, \xi x, \xi y) = \xi^p G_\gamma[V](t,x,y), \quad \forall t \ge 0, \; \forall (x,y) \in \mathbb{R}_+^2, \; \forall \xi \ge 0, \tag{4.2}$$

and

$$\widehat{V}(t, \xi x, \xi y) = \xi^p \, \widehat{V}(t,x,y), \quad \forall \xi \ge 0, \; \forall x, y \ge 0. \tag{4.3}$$

Proof. By linearity of the state equations, $\mathcal{A}(\xi r) = \xi \mathcal{A}(r)$ for every $\xi \ge 0$ and $r \ge 0$. By homogeneity of U, we get (4.1); then (4.2) follows from (A.1). Again by linearity, $\mathcal{A}_t(\xi x) = \xi \mathcal{A}_t(x)$ for every $\xi \ge 0$ and $x \ge 0$. Then (4.3) follows from (4.1) and (4.2). □

Under Assumption 4.1, (4.3) becomes

$$\widehat{V}(t,x,y) = \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)} \left(\frac{c_s^p}{p} + \lambda G_\gamma[V](s, \tilde{X}_s^{t,x,c,\pi}, \tilde{Y}_s^{t,y})\right) ds\right]. \tag{4.4}$$

If $y = 0$, then $\tilde{Y}_s^{t,y} \equiv 0$. So, by Proposition 4.3,

$$\widehat{V}(t,x,0) = \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)} \left(\frac{c_s^p}{p} + \lambda K_V [\tilde{X}_s^{t,x,c,\pi}]^p\right) ds\right]. \tag{4.5}$$

This is a standard homogeneous Merton type problem, for which $\widehat{V}(t,x,0) = K_0 x^p$ for some $K_0 > 0$. We omit to treat this case.[6]

[6] A necessary and sufficient condition excluding the case $y = 0$ is given in Proposition 4.12 below.

If $y > 0$, then $\tilde{Y}_s^{t,y} > 0$ for every $s \geq t$. In view of (4.3), we consider a new state process Z, defined as the ratio of $\tilde{X}$ and $\tilde{Y}$. More precisely, let $(c, \pi) \in \mathcal{A}_t(x)$, $\tilde{X} := \tilde{X}^{t,x,c,\pi}$, $\tilde{Y} := \tilde{Y}^{t,y}$, $Z_s := \frac{\tilde{X}_s}{\tilde{Y}_s}$. Itô's formula yields

$$dZ_s \;=\; -\hat{c}_s ds + \hat{\pi}_s(\widehat{K}_1 ds + \sigma_L dW_s) + Z_s(\widehat{K}_2 ds + K_\gamma dB_s^{(1)}), \tag{4.6}$$

where

$$\begin{cases} \hat{c}_s := \frac{c_s}{\tilde{Y}_s}, \\ \hat{\pi}_s := \frac{\pi_s}{\tilde{Y}_s} - \frac{\rho\sigma_I}{\sigma_L}\frac{\tilde{X}_s}{\tilde{Y}_s}, \\ \widehat{K}_1 := b_L - \rho\sigma_I\sigma_L, \\ \widehat{K}_2 := \gamma^2\left(-b_I + \frac{\rho b_L \sigma_I}{\sigma_L} + (1-\rho^2)\sigma_I^2\right), \\ K_\gamma := -\sigma_I\gamma\sqrt{1-\rho^2}. \end{cases} \tag{4.7}$$

Then, by Proposition 4.3, and for $f_\gamma(t,z) := G_\gamma[\xi \mapsto \xi^p](t,z,1)$, (4.4) becomes

$$\widehat{V}(t,x,y) \;=\; \sup_{(c,\pi)\in\mathcal{A}_t(x)} \mathbb{E}\left[\int_t^\infty e^{-(\beta+\lambda)(s-t)}(\tilde{Y}_s)^p\left(\frac{\hat{c}_s^p}{p} + \lambda K_V f_\gamma(s, Z_s)\right) ds\right]. \tag{4.8}$$

In order to simplify (4.8), we change probability measure. Let $\widehat{\mathbb{P}}$ be the measure with density process $\frac{(\tilde{Y}_s)^p}{\mathbb{E}[(\tilde{Y}_s)^p]}$. Under $\widehat{\mathbb{P}}$, the processes $\widehat{W}_s := W_s - p\rho\sigma_I s$ and $\widehat{B}_s^{(1)} := B_s^{(1)} - \gamma\sqrt{1-\rho^2}\, s$ are Brownian motions and the dynamics of Z can be written as

$$dZ_s \;=\; -\hat{c}_s ds + \hat{\pi}_s(K_1 ds + \sigma_L d\widehat{W}_s) + Z_s(K_2 ds + K_\gamma d\widehat{B}_s^{(1)}), \tag{4.9}$$

where

$$K_1 \;:=\; b_L - \rho\sigma_I\sigma_L(1-p), \qquad K_2 \;:=\; \gamma^2\left(-b_I + \frac{\rho b_L\sigma_I}{\sigma_L} + (1-\rho^2)(1-p)\sigma_I^2\right). \tag{4.10}$$

Moreover, (4.8) becomes

$$\widehat{V}(t,x,y) \;=\; y^p \cdot \sup_{(c,\pi)\in\mathcal{A}_t(x)} \widehat{\mathbb{E}}\left[\int_t^\infty e^{-K_\lambda(s-t)}\left(\frac{\hat{c}_s^p}{p} + \lambda K_V f^\gamma(s, Z_s)\right) ds\right], \tag{4.11}$$

where $\widehat{\mathbb{E}}$ is the expectation under $\widehat{\mathbb{P}}$ and

$$K_\lambda \;:=\; \beta + \lambda + \frac{(\rho^2 + \gamma^2(1-\rho^2))\sigma_I^2}{2}p(1-p) - p\rho\frac{b_L\sigma_I}{\sigma_L} - \gamma^2 p\left(b_I - \frac{\rho b_L\sigma_I}{\sigma_L}\right).$$

Given $z \geq 0$, we consider (4.9) with initial datum z as a controlled equation with controls $(\hat{c}, \hat{\pi})$, where[7]

(h1)″ $(\hat{c}_s)_{s\geq t}$ is $(\mathcal{W}_s \vee \mathcal{B}_s^{(1)})_{s\geq t}$-predictable, nonnegative, and has locally integrable trajectories;

(h2)″ $(\hat{\pi}_s)_{s\geq t}$ is $(\mathcal{W}_s \vee \mathcal{B}_s^{(1)})_{s\geq t}$-predictable, and has locally square-integrable trajectories.

[7] The filtration generated by $(\widehat{W}, \widehat{B}^{(1)})$ is the same as the filtration generated by $(W, B^{(1)})$.

Let $Z^{t,z,\hat{c},\hat{\pi}}$ be the solution to (4.9), starting from z at time t and under a control $(\hat{c}, \hat{\pi})$ fulfilling (h1)$''$ – (h2)$''$. Set

$$\hat{\mathcal{A}}_t(z) := \{(\hat{c}, \hat{\pi}) \text{ fulfilling (h1)}'' - \text{(h2)}'' \mid Z^{t,z,\hat{c},\hat{\pi}} \geq 0\},$$

and consider the one-dimensional stochastic control problem

$$\Phi(t,z) := \sup_{(\hat{c},\hat{\pi})\in\hat{\mathcal{A}}_t(z)} \widehat{\mathbb{E}}\left[\int_t^\infty e^{-K_\lambda(s-t)}\left(\frac{\hat{c}_s^p}{p} + \lambda K_V f^\gamma(s, Z_s^{t,z,\hat{c},\hat{\pi}})\right) ds\right]. \tag{4.12}$$

Proposition 4.4. *Let $x \geq 0$, $y > 0$ and $z := x/y$.*

1. *$(c,\pi) \in \mathcal{A}_t(x)$ if and only if $(\hat{c},\hat{\pi}) \in \hat{\mathcal{A}}_t(z)$, where $\hat{c}_s = \frac{c_s}{\tilde{Y}_s^{t,y}}$, $\hat{\pi}_s = \frac{\pi_s}{\tilde{Y}_s^{t,y}} - \frac{\rho\sigma_I}{\sigma_L}\frac{\tilde{X}_s^{t,x,c,\pi}}{\tilde{Y}_s^{t,y}}$.*
2. *$(c^*,\pi^*) \in \mathcal{A}_t(x)$ is optimal for (4.4) if and only if $(\hat{c}^*,\hat{\pi}^*) \in \hat{\mathcal{A}}_t(z)$ is optimal for (4.12), where $\hat{c}_s^* = \frac{c_s^*}{\tilde{Y}_s^{t,y}}$ $\hat{\pi}_s^* = \frac{\pi_s^*}{\tilde{Y}_s^{t,y}} - \frac{\rho\sigma_I}{\sigma_L}\frac{\tilde{X}_s^{t,x,c^*,\pi^*}}{\tilde{Y}_s^{t,y}}$.*
3. *$\widehat{V}(t,x,y) = y^p\ \Phi(t,z)$, for every $t \geq 0$.*

Proof. All the claims follow from the arguments above. □

In view of Proposition 4.4, from here on we study the optimization problem (4.12). Denote by $\mathcal{M}_p(\mathbb{R}_+^2, \mathbb{R})$ the space of measurable functions ψ such that $|\psi(t,z)| \leq C_0(1+|z|)^p$, and consider the nonlinear functional $\mathcal{H}_0 : \mathcal{M}_p(\mathbb{R}_+^2, \mathbb{R}) \to \mathbb{R}$, $\psi \mapsto \mathcal{H}_0[\psi] := \sup_{z\geq 0} \frac{\psi(0,z)}{(1+z)^p}$. Then $V(r) = \mathcal{H}[\widehat{V}](r) = \mathcal{H}_0[\Phi]r^p$, so that

$$K_V = \mathcal{H}_0[\Phi], \tag{4.13}$$

where K_V is the constant in (4.1). As $\hat{\mathcal{A}}_t(0) = \{(c,\pi) \equiv (0,0)\}$, we get the boundary condition for Φ

$$\Phi(t,0) \;=\; K_V \int_t^\infty e^{-K_\lambda(s-t)}\lambda f_\gamma(s,0)ds. \tag{4.14}$$

By (3.20) and Proposition (4.4)(3),[8] we get the growth condition for Φ

$$\Phi(t,z) \;\leq\; K_{\widehat{V}} e^{k_{J,p}t}(1+z)^p. \tag{4.15}$$

The HJB equation associated to (4.12) is

$$-\varphi_t + K_\lambda\varphi - \lambda K_V f_\gamma(t,z) - \sup_{\hat{c}\geq 0,\,\hat{\pi}\in\mathbb{R}} H_{cv}^0(z,\varphi,\varphi_z,\varphi_{zz}) \;=\; 0, \tag{4.16}$$

where

$$H_{cv}^0(z,\varphi,\varphi_z,\varphi_{zz};\hat{c},\hat{\pi}) \;=\; \frac{\hat{c}^p}{p} - \hat{c}\varphi_z + K_1\hat{\pi}\varphi_z + \frac{1}{2}\sigma_L^2\hat{\pi}^2\varphi_{zz} + K_2 z\varphi_z + \frac{1}{2}K_\gamma^2 z^2\varphi_{zz}.$$

[8] It could also be proven by dealing directly with the control problem (4.12).

By (4.13), we can replace K_V with $\mathcal{H}_0[\varphi]$ in (4.16) and also consider the equation with a nonlocal term

$$-\varphi_t + K_\lambda \varphi - K_2 z \varphi_z - \lambda f_\gamma(t,z)\mathcal{H}_0[\varphi] - \sup_{\hat{c}\geq 0,\, \hat{\pi}\in\mathbb{R}} H^0_{cv}(z,\varphi,\varphi_z,\varphi_{zz}) \;=\; 0. \tag{4.17}$$

Similarly, we can replace K_V with $\mathcal{H}_0[\varphi]$ in (4.14) and get an implicit nonlocal boundary condition :

$$\Phi(t,0) \;=\; \mathcal{H}_0[\Phi] \int_t^\infty e^{-K_\lambda(s-t)} \lambda f_\gamma(s,0) ds. \tag{4.18}$$

Proposition 4.5. *The function Φ is the unique continuous viscosity solution* [9] *over $\mathbb{R}_+ \times (0,+\infty)$ to* (4.16) *fulfilling* (4.14) *and* (4.15)*. Equivalently, Φ is the unique continuous viscosity solution over $\mathbb{R}_+ \times (0,+\infty)$ to* (4.17) *fulfilling* (4.18) *and* (4.15)*.*

Proof. The first fact follows from Proposition 4.4(3) and Theorem 3.9. The equivalence between the equations follows from uniqueness and (4.13). □

4.2 Smoothness of the value function

This subsection shows that the value function Φ is smooth. As the classification of the HJB equation (4.16) is sensitive to γ, we need to distinguish, from the point of view of PDE theory, the cases $\gamma = 0$ and $\gamma \neq 0$. Due to the presence of the term $\frac{K_\gamma^2}{2} z^2 \varphi_{zz}$, in the case $\gamma \neq 0$ ($K_\gamma > 0$), the PDE is a fully nonlinear (locally) nondegenerate parabolic equation, whereas, in the case $\gamma = 0$ ($K_\gamma = 0$), it is degenerate. In both cases, we prove that the solution is sufficiently smooth to construct optimal policies in feedback form. The degenerate case, $\gamma = 0$, is investigated by means of the dual problem illustrated in Remark 4.7 and discussed in detail in [10, Sec. 6]. For the nondegenerate case, $\gamma \neq 0$, we first localize the equation by restricting the set of controls to a compact one and then apply a result by Krylov (see Appendix). Hereafter, we denote by $C^{1,k}$ the class of functions which are once differentiable with respect to the time variable and k-times differentiable with respect to spatial variable, with continuous derivatives.

Theorem 4.6. *$\Phi \in C^{1,3}(\mathbb{R}_+ \times (0,+\infty);\mathbb{R})$ and $\Phi_z \in C^{1,2}(\mathbb{R}_+ \times (0,+\infty);\mathbb{R})$, with $\Phi_z > 0$ and $\Phi_{zz} < 0$ over $\mathbb{R}_+ \times (0,+\infty)$.*

Proof. We prove the claim in the case $\gamma \neq 0$, referring to [10] and Remark 4.7 below for the case $\gamma = 0$. Given $(\bar{t},\bar{z}) \in \mathbb{R}_+ \times (0,+\infty)$ and $\varepsilon \in (0,\bar{z})$, consider $D^\varepsilon(\bar{t},\bar{z})$, defined in (A.9), and let $\mathcal{P}(D_\varepsilon(\bar{t},\bar{z}))$ be the parabolic boundary of $D_\varepsilon(\bar{t},\bar{z})$:

$$\mathcal{P}(D_\varepsilon(\bar{t},\bar{z})) \;:=\; \{\bar{t}+\varepsilon\} \times [\bar{z}-\varepsilon, \bar{z}+\varepsilon] \cup [\bar{t},\bar{t}+\varepsilon] \times \{\bar{z}-\varepsilon,\bar{z}+\varepsilon\}.$$

By Propositions 4.5 and A.6, and by standard comparison results for viscosity solutions (see e.g. [6, 12]), Φ is the unique continuous viscosity solution on $D_\varepsilon(\bar{t},\bar{z})$ to the HJB

[9] The definition of viscosity solution is analogous to Definition 3.7(1).

equation (A.19) - the same as (4.16), but with constraints on the set of the variables $\hat{c}, \hat{\pi}$ - with Dirichlet continuous boundary condition

$$\varphi \ = \ \Phi, \qquad \text{on} \quad \mathcal{P}(D_\varepsilon(\bar{t}, \bar{z})). \tag{4.19}$$

On the other hand, by Theorem A.8, there exists a solution $C^{1,2}(D_\varepsilon(\bar{t}, \bar{z}); \mathbb{R})$ to (A.19) with boundary condition (4.19). As such a solution needs to be a viscosity solution,

$$\Phi \ \in \ C^{1,2}(D_\varepsilon(\bar{t}, \bar{z}); \mathbb{R}). \tag{4.20}$$

Moreover, by Lemma A.5, we have for $(t, z) \in D_\varepsilon(\bar{t}, \bar{z})$

$$\Phi_z(t, z) \ \geq \ m_\varepsilon \ > \ 0, \qquad \Phi_{zz}(t, z) \ \leq \ -\delta_\varepsilon \ < \ 0. \tag{4.21}$$

Due to (4.20)-(4.21) and Proposition 4.5, and by arbitrariness of $(\bar{t}, \bar{z}) \in \mathbb{R}_+ \times (0, +\infty)$, the function Φ is a classical solution to (4.16) and $\Phi_z > 0$, $\Phi_{zz} < 0$ in $\mathbb{R}_+ \times (0, +\infty)$. As a consequence the supremum in (4.16) can be made explicit, so that Φ satisfies, in the classical sense,

$$-\Phi_t + K_\lambda \Phi - K_2 z \Phi_z - \lambda K_V f_\gamma(t, z) - \widetilde{U}(\Phi_z) + \frac{1}{2} \frac{K_1^2}{\sigma_L^2} \frac{\Phi_z^2}{\Phi_{zz}} - \frac{K_\gamma^2}{2} z^2 \Phi_{zz} \ = \ 0, \tag{4.22}$$

where $\widetilde{U}(w) := \frac{1-p}{p} \, w^{-\frac{p}{1-p}}$, $w > 0$, is the Legendre transform of U. By Lemma A.7, we differentiate (4.22) and deduce that Φ_z is a viscosity solution to

$$\begin{aligned} -g_t + \Big(K_\lambda + K_2 + \frac{K_1^2}{\sigma_L^2}\Big) g + (K_2 - K_\gamma^2) z g_z + \tilde{U}'(g) g_z \\ -\Big(\frac{K_\gamma^2}{2} z^2 + \frac{K_1^2}{2\sigma_L^2} \frac{g^2}{g_z^2}\Big) g_{zz} + \lambda K_V (f_\gamma)_z \ = \ 0, \end{aligned} \tag{4.23}$$

with Dirichlet continuous boundary condition $g = \Phi_z$ on $\mathcal{P}(D_\varepsilon(\bar{t}, \bar{z}))$. Again, by standard comparison results for viscosity solutions, the function Φ_z is the unique viscosity solution to this problem. On the other hand, Theorem A.8[10] implies that this problem admits a $C^{1,2}(D_\varepsilon(\bar{t}, \bar{z}); \mathbb{R})$ solution. As before, we deduce $\Phi_z \in C^{1,2}(D_\varepsilon(\bar{t}, \bar{z}); \mathbb{R})$, hence $\Phi \in C^{1,3}(D_\varepsilon(\bar{t}, \bar{z}); \mathbb{R})$. We finally get the claim by arbitrariness of $(\bar{t}, \bar{z}) \in \mathbb{R}_+ \times (0, +\infty)$. □

Remark 4.7. *The equation* (4.16)*, when* $\gamma = 0$*, is degenerate, as the second order term can vanish, but* Φ *is still a viscosity solution of it. Motivated by the fact that* $\Phi_z > 0$ *and* $\Phi_{zz} < 0$*, one rewrites the equation as*

$$-\varphi_t + K_\lambda \varphi - K_2 z \varphi_z - \lambda K_V f_0(t, z) - \widetilde{U}(\varphi_z) + \frac{1}{2} \frac{K_1^2}{\sigma_L^2} \frac{\varphi_z^2}{\varphi_{zz}} \ = \ 0. \tag{4.24}$$

[10] Note that $\tilde{U}'(g) g_z$ and g^2/g_z^2 are not well-defined for $g \leq 0$ or $g_z = 0$. We can use (4.21) to replace these terms by functions of (g, g_z) everywhere defined, with as much smoothness as needed, coinciding with (g, g_z) whenever $m_\varepsilon \leq g \leq M_\varepsilon$, $g_z \leq -\delta_\varepsilon$, and satisfying the assumptions of Theorem A.8.

Define the dual function $\psi(t,w) := \sup_{z\geq 0}\{\varphi(t,z) - wz\}$, $w>0$. From (4.24), one formally gets the equation for ψ

$$-\psi_t + K_\lambda \psi + (K_2 - K_\lambda) w\psi_w - \frac{K_1^2}{2\sigma_L^2} w^2 \psi_{ww} - \widetilde{U}(w) - \lambda K_V f_0(t, -\psi_w) = 0 \tag{4.25}$$

(boundary and growth conditions for φ also have a dual counterpart for ψ). The equation (4.25) is semilinear and (locally) nondegenerate, so the PDE theory for classical solutions can be used. Moreover, (4.25) is still of HJB type, itself associated with another (dual) control problem. Once the existence of a sufficiently smooth solution Ψ to (4.25) is derived, one may try to show that $\widetilde{\Psi}(t,z) := \inf_{w>0}\{\Psi(t,w) + zw\}$ is a classical solution to (4.24), and identify it with Φ. All these steps are nontrivial (see [10]).

4.3 Closed loop equation

The candidate optimal feedback maps provided by maximization in the HJB equation (4.16), for $z>0$, and by the fact that $\hat{\mathcal{A}}_t(0) = \{(0,0)\}$, for $z=0$, are

$$\widehat{C}^*(s,z) \;=\; \begin{cases} (U')^{-1}\left(\Phi_z(s,z)\right), & \text{if } z>0, \\ 0, & \text{if } z=0, \end{cases} \qquad \widehat{\Pi}^*(s,z) \;=\; \begin{cases} -\frac{K_1 \Phi_z(s,z)}{\sigma_L \Phi_{zz}(s,z)}, & \text{if } z>0, \\ 0, & \text{if } z=0. \end{cases} \tag{4.26}$$

These maps are measurable. Moreover, due to Theorem 4.6, $\widehat{C}^*(s,\cdot)$ and $\widehat{\Pi}^*(s,\cdot)$ are locally Lipschitz continuous in $(0,+\infty)$, uniformly in $s\in[0,T]$, for all $T>0$. The associated closed loop equation is

$$\begin{cases} dZ_s \;=\; -\widehat{C}^*(s,Z_s)ds + \widehat{\Pi}^*(s,Z_s)\left(K_1 ds + \sigma_L d\widehat{W}_s\right) + Z_s(K_2 ds + K_\gamma d\widehat{B}^{(1)}_s), \\ Z_t \;=\; z. \end{cases} \tag{4.27}$$

Proposition 4.8. *For every $(t,z)\in\mathbb{R}^2_+$, there exists a unique nonnegative strong solution $Z^{t,z,*}$ to (4.27) in $(\Omega,\mathcal{F},\widehat{\mathbb{P}})$.*

Proof. *Existence.* If $z=0$, the claim follows by setting $Z^{t,z,*}\equiv 0$. Let $z>0$ and $T>0$. The local Lipschitz continuity of $\widehat{C}^*(s,\cdot),\widehat{\Pi}^*(s,\cdot)$ and standard SDEs theory (see, e.g., [18, Ch. 5, Th. 2.9]) give, for each $\varepsilon\in(0,z)$, the existence of a unique solution $Z^{t,z,\varepsilon}\in[\varepsilon,\varepsilon^{-1}]$ in the stochastic interval $[t,\tau^T_\varepsilon)$, where τ^T_ε is implicitly defined as $\tau^T_\varepsilon := \inf\{s\in[t,T] \mid Z^{t,z,\varepsilon}_s \leq \varepsilon \text{ or } Z^{t,z,\varepsilon}_s \geq \varepsilon^{-1}\}$, with the convention $\inf\emptyset = T$. If $\varepsilon<\varepsilon'$, we have $\tau^T_\varepsilon > \tau^T_{\varepsilon'}$ and

$$Z^{t,z,\varepsilon}_s \;\equiv\; Z^{t,z,\varepsilon'}_s \;\text{ on }\; [t,\tau^T_{\varepsilon'}), \qquad \forall\, 0<\varepsilon<\varepsilon'. \tag{4.28}$$

Set $\tau^T := \lim_{\varepsilon\downarrow 0}\tau^T_\varepsilon$. By (4.28), there exists a unique solution $Z^{t,z,*}\geq 0$ in the interval $[t,\tau^T)$. We show that $Z^{t,z,*}$ can be extended to $[t,T]$, which implies the claim, by arbitrariness of T. By a Girsanov transformation, there exists a probability $\widehat{\mathbb{Q}}^T$, locally equivalent to $\widehat{\mathbb{P}}$, and $\widehat{\mathbb{Q}}^T$-Brownian motions $\widehat{W}^{\widehat{\mathbb{Q}}^T}$ and $\widehat{B}^{(1),\widehat{\mathbb{Q}}^T}$ such that (4.27) is rewritten in $[0,T]$ as

$$dZ_s \;=\; -\widehat{C}^*(s,Z_s)ds + \sigma_L\widehat{\Pi}^*(s,Z_s)d\widehat{W}^{\widehat{\mathbb{Q}}^T}_s + K_\gamma Z_s d\widehat{B}^{(1),\widehat{\mathbb{Q}}^T}_s.$$

By nonnegativity of $\widehat{C}^*$ and $Z^{t,z,*}$, the process $Z^{t,z,*}$ is a nonnegative $\widehat{\mathbb{Q}}^T$-supermartingale on $[t, \tau^T)$. It can be extended to a $\widehat{\mathbb{Q}}^T$-supermartingale ($L^1(\widehat{\mathbb{Q}}^T)$-bounded) on $[t,T]$ by setting $Z^{t,z,*} \equiv 0$ in $[\tau^T, T]$. Hence, by Doob's convergence Theorem (the usual proof for deterministic intervals - see e.g. Theorem 6.18 in [17] - can be adapted to our stochastic interval $[t, \tau^T)$), there exists a finite random variable $Z^{t,z,*}_{\tau^T}$ such that $\lim_{s \nearrow \tau^T} Z^{t,z,*}_s = Z^{t,z,*}_{\tau^T}$, $\widehat{\mathbb{Q}}^T$-a.s.. As $\widehat{\mathbb{Q}}^T \sim \widehat{\mathbb{P}}$, we also have

$$\lim_{s \nearrow \tau^T} Z^{t,z,*}_s \;=\; Z^{t,z,*}_{\tau^T}, \qquad \widehat{\mathbb{P}}\text{-a.s.}. \tag{4.29}$$

Hence, (4.29) yields the desired extension on $\{\tau^T = T\}$. Consider now the set $\{\tau^T < T\}$. In this set, $Z^{t,z,*}_{\tau^T_\varepsilon} \in \{\varepsilon, \varepsilon^{-1}\}$, so, by (4.29), necessarily $Z^{t,z,*}_{\tau^T} = 0$ $\mathbb{P}$-a.s.. This clearly implies $\lim_{s\nearrow\tau} Z^{t,z,*}_s = 0$, $\widehat{\mathbb{P}}-$a.s. on $\{\tau^T < T\}$. Hence, we can extend $Z^{t,z,*}$ to a solution defined over $[t,T]$ on $\{\tau^T < T\}$, by setting $Z^{t,z,*}_s \equiv 0$ for $s \in [\tau^T, T]$.

Uniqueness. The solution is unique on the stochastic interval $[t, \tau^T)$, defined in the existence part. On the set $\{\tau^T < T\}$, when it reaches 0, it must stay there, as it is a nonnegative $\widehat{\mathbb{Q}}^T$-supermartingale. Therefore, we have uniqueness on $[t,T]$ for all $T > t$, hence on $[t, +\infty)$. □

4.4 Verification theorem

Theorem 4.9. *Let $Z^{t,z,*}$ be the unique nonnegative solution to* (4.27) *and let $\widehat{C}^*, \widehat{\Pi}^*$ be the feedback maps defined in* (4.26)*. Define the feedback strategies*

$$\hat{c}^*_s \;:=\; \widehat{C}^*(s, Z^{t,z,*}_s), \quad \hat{\pi}^*_s \;:=\; \Pi^*(s, Z^{t,z,*}_s), \quad s \geq t. \tag{4.30}$$

Then $(\hat{c}^*, \hat{\pi}^*) \in \hat{\mathcal{A}}_t(z)$, and it is the unique optimal control for (4.12).

Proof. *Admissibility.* As (4.27) has a well-defined solution, then $\hat{c}^*$ and $\hat{\pi}^*$ satisfy the required integrability conditions $(\text{h1})'' - (\text{h2})''$. On the other hand, by uniqueness of solutions to (4.27), we must have $Z^{t,x,\hat{c}^*,\hat{\pi}^*} = Z^{t,z,*}$. As $Z^{t,z,*} \geq 0$, we conclude $(\hat{c}^*, \hat{\pi}^*) \in \hat{\mathcal{A}}_t(z)$.

Optimality. To prove optimality we distinguish two cases: $z = 0$ and $z > 0$.

(i) Case $z = 0$. Then $Z^{t,z,*} \equiv 0$, so also $(\hat{c}^*, \hat{\pi}^*) \equiv (0,0)$. On the other hand, $\hat{\mathcal{A}}_t(0) = \{(0,0)\}$, hence we conclude.

(ii) Case $z > 0$. Let $\tau := \inf\{s \geq t \,|\, Z^{t,z,\hat{c}^*,\hat{\pi}^*}_s = 0\}$, with the convention $\inf\emptyset = +\infty$. Due to Theorem 4.6, we can apply Dynkin's formula to $s \mapsto e^{-K_\lambda(s-t)}\Phi(s, Z^{t,z,\hat{c}^*,\hat{\pi}^*}_s)$ in $[t, \tau \wedge T)$, for all $T > t$. By Proposition 4.5 and Theorem 4.6, Φ solves (4.16) in classical sense. Hence, using the definition of $(\hat{c}^*, \hat{\pi}^*)$ and arguing as in standard verification theorems we get

$$\Phi(t,z) - \widehat{\mathbb{E}}\left[e^{-K_\lambda((\tau\wedge T)-t)}\Phi(\tau\wedge T, Z^{t,z,\hat{c}^*,\hat{\pi}^*}_{\tau\wedge T})\right] = \widehat{\mathbb{E}}\left[\int_t^{\tau\wedge T} e^{-K_\lambda(s-t)}\left(\frac{(\hat{c}^*_s)^p}{p} + \lambda K_V f_\gamma(s, Z^{t,z,\hat{c}^*,\hat{\pi}^*}_s)\right) ds\right].$$

Splitting on the sets $A_T = \{\tau < T\}$ and $A_T^c = \{\tau \geq T\}$, we write

$$\begin{aligned}\Phi(t,z) - \widehat{\mathbb{E}}\left[\mathbf{1}_{A_T}\ e^{-K_\lambda(\tau-t)}\Phi(\tau, Z_\tau^{t,z,\hat c^*,\hat\pi^*}) + \mathbf{1}_{A_T^c}\ e^{-K_\lambda(T-t)}\Phi(T, Z_T^{t,z,\hat c^*,\hat\pi^*})\right] \\ = \widehat{\mathbb{E}}\Bigg[\mathbf{1}_{A_T}\int_t^\tau e^{-K_\lambda(s-t)}\left(\frac{(\hat c_s^*)^p}{p} + \lambda K_V f_\gamma(s, Z_s^{t,z,\hat c^*,\hat\pi^*})\right)ds \\ + \mathbf{1}_{A_T^c}\int_t^T e^{-K_\lambda(s-t)}\left(\frac{(\hat c_s^*)^p}{p} + \lambda K_V f_\gamma(s, Z_s^{t,z,\hat c^*,\hat\pi^*})\right)ds\Bigg]. \quad (4.31)\end{aligned}$$

Noting that $Z^{t,z,\hat c^*,\hat\pi^*} \equiv 0$ and $\hat c^* \equiv 0$, from τ on, using (4.14) we get

$$\begin{aligned}\widehat{\mathbb{E}}\left[\mathbf{1}_{A_T}\ \left(e^{-K_\lambda(\tau-t)}\Phi(\tau, Z_\tau^{t,z,\hat c^*,\hat\pi^*}) + \int_t^\tau e^{-K_\lambda(s-t)}\left(\frac{(\hat c_s^*)^p}{p} + \lambda K_V f_\gamma(s, Z_s^{t,z,\hat c^*,\hat\pi^*})\right)ds\right)\right] \\ = \widehat{\mathbb{E}}\Bigg[\mathbf{1}_{A_T}\ \int_t^\infty e^{-K_\lambda(s-t)}\left(\frac{(\hat c_s^*)^p}{p} + \lambda K_V f_\gamma(s, Z_s^{t,z,\hat c^*,\hat\pi^*})\right)ds\Bigg].\end{aligned}$$

Hence, moving the term corresponding to $\mathbf{1}_{A_T}$ to the right hand side in (4.31), and adding and subtracting $\widehat{\mathbb{E}}\left[\mathbf{1}_{A_T^c}\int_T^{+\infty} e^{-K_\lambda(s-t)}\left(\frac{(\hat c_s^*)^p}{p} + \lambda K_V f_\gamma(s, Z_s^{t,z,\hat c^*,\hat\pi^*})\right)ds\right]$, we get

$$\begin{aligned}\Phi(t,z) - \widehat{\mathbb{E}}\left[\mathbf{1}_{A_T^c}\ e^{-K_\lambda(T-t)}\Phi(T, Z_T^{t,z,\hat c^*,\hat\pi^*})\right] \\ = \widehat{\mathbb{E}}\left[\int_t^\infty e^{-K_\lambda(s-t)}\left(\frac{(\hat c_s^*)^p}{p} + \lambda K_V f_\gamma(s, Z_s^{t,z,\hat c^*,\hat\pi^*})\right)ds\right] \\ - \widehat{\mathbb{E}}\Bigg[\mathbf{1}_{A_T^c}\int_T^{+\infty} e^{-K_\lambda(s-t)}\left(\frac{(\hat c_s^*)^p}{p} + \lambda K_V f_\gamma(s, Z_s^{t,z,\hat c^*,\hat\pi^*})\right)ds\Bigg]. \quad (4.32)\end{aligned}$$

Take $T \to \infty$ in (4.32). The second term of the left hand side goes to 0 by dominated convergence, due to Assumption 2.6, (3.18), (4.15); the second term of the right hand side goes to 0 by monotone convergence. So we conclude.

Uniqueness. As V is strictly concave (it is of power form), $f_\gamma(s,\cdot)$ is also strictly concave. Let $(\hat c_1^*, \hat\pi_1^*)$ and $(\hat c_2^*, \hat\pi_2^*)$ be optimal for (4.12). Strict concavity of $c \mapsto c^p/p$ yields $\hat c_1^* = \hat c_2^*$. Strict concavity of $f_\gamma(s,\cdot)$, for every $s \geq t$, yields $Z^{t,z,\hat c_1^*,\hat\pi_1^*} = Z^{t,z,\hat c_2^*,\hat\pi_2^*}$, from which $\hat\pi_1^* = \hat\pi_2^*$ follows. □

4.5 Optimal policies for the original problem

First, we characterize the optimal policy in the illiquid asset.

Proposition 4.10. *The optimal allocation policy α_k^* in the illiquid asset at time τ_k is $\alpha_k^* = a^*(R_{\tau_k})$, where $a^*(r)$ is the unique maximizer over $[0,r]$ of the function $\mapsto g(a;r) = a^p\Phi(0, \frac{r}{a} - 1)$.*

Proof. As $\Phi \geq 0$, $\Phi_z > 0$, and $\Phi_{zz} < 0$, a computation of $\partial^2 g/\partial a^2$ shows that $g(\cdot;r)$ is strictly concave in $(0,r)$, so it admits a unique maximizer in $[0,r]$. The claim follows from

Proposition 4.4(3) and (3.24). □

We prove two important properties of the optimal allocation in the illiquid asset.

Proposition 4.11. *The optimal rebalancing proportion* $\frac{X^*_{\tau_k}}{\alpha^*_k}$ *at the trading times of the illiquid asset* $(\tau_k)_{k\in\mathbb{N}}$ *is constant:*

$$\frac{X^*_{\tau_k}}{\alpha^*_k} = \frac{R^*_{\tau_k} - \alpha^*_k}{\alpha^*_k} = z^* := \mathit{argmax}_{z\geq 0} \frac{\Phi(0,z)}{(1+z)^p}, \qquad \forall k \in \mathbb{N},$$

where the value z^* *above is well defined, under the convention that* $z^* = \infty$ *if the supremum of* $\frac{\Phi(0,z)}{(1+z)^p}$ *is not attained (in this case, there is no investment in the illiquid asset).*

Consequently, at $(\tau_k)_{k\in\mathbb{N}}$*, the optimal allocation proportions in liquid and illiquid assets over total wealth are also constant:* $\frac{\alpha^*_k}{R^*_{\tau_k}} = \frac{1}{1+z^*}$, $\frac{X^*_{\tau_k}}{R^*_{\tau_k}} = \frac{z^*}{1+z^*}$, *for every* $k \in \mathbb{N}$, *with the conventions* $\frac{1}{\infty} = 0$, $\frac{\infty}{\infty} = 1$.

Proof. Consider $h(z) := \frac{\Phi(z)}{(1+z)^p}$, $z \geq 0$. We have $h(\frac{r}{a}-1) = g(a;r)$ for $a \in (0,r]$, where g is defined in Proposition 4.10. As g is continuous, $\lim_{z\to+\infty} h(z)$ exists and is equal to $g(0)$. So, setting $h(+\infty) := g(0)$, we consider the diffeomorphism $[0,r] \to [0,+\infty]$, $a \mapsto \frac{r}{a} - 1$, and note that a maximizes g over $[0,r]$ if and only if $z = \frac{r}{a} - 1$ maximizes h over $[0,+\infty]$. The maximizer of g over $[0,r]$ is unique, hence the maximizer of h over $[0,+\infty]$ is also unique. Calling it z^*, from the correspondence above, we get $z^* = \frac{r-a^*(r)}{a^*(r)}$, where $a^*(r)$ is defined in Proposition 4.10. □

Proposition 4.12.

1. $\alpha^*_0 < r$ *(if and only if* $r > 0$*).*
2. $\alpha^*_0 > 0$ *if and only if* $\frac{b_I}{\sigma_I} > \frac{\rho b_L}{\sigma_L}$.[11]

Proof. 1. If $r = 0$, then $\alpha^*_0 = 0$, due to the state constraint. Let $r > 0$ and assume, by contradiction, that $\alpha^*_0 = r$. This would yield $z^* = 0$ in Proposition 4.11, hence $\alpha^*_k = R_{,\tau_k}$ for all $k \in \mathbb{N}$. We should conclude, by the state constraint, that $c^*_t \equiv 0$. But this strategy cannot be optimal, as $V(r) > 0$.

2. *Necessity.* Consider the Merton problem described in Remark 2.7 and call its value function $V^{M,2}$. The optimal investment proportions in L and I for this problem are

$$(u^*_L, u^*_I) = \underset{u_L\in\mathbb{R},\, u_I\in[0,1]}{\operatorname{argmax}} \left\{ p(u_L b_L + u_I b_I) - \frac{p(1-p)}{2}(u_L^2\sigma_L^2 + u_I^2\sigma_I^2 + 2\rho u_L u_I \sigma_L \sigma_I) \right\}$$

Taking first the supremum on u_L, one sees that $u^*_I = 0$ if and only if $b_I \leq \frac{\rho b_L \sigma_I}{\sigma_L}$. In this case, denoting by $V^{M,1}$ the value function for an agent investing only in L, we have $V^{M,2} = V^{M,1}$. As $V^{M,1} \leq V \leq V^{M,2}$, we obtain $V = V^{M,1}$, and the optimal strategy for our original problem never invests in the illiquid asset I.

[11] This condition is the same as in the Merton (liquid) problem with two assets. The same result is obtained in [1] in the case of full observation.

Sufficiency. Assume $\frac{b_I}{\sigma_I} > \frac{\rho b_L}{\sigma_L}$ and set $h(a) := \widehat{V}(0, 1-a, a)$, $a \in [0,1]$. By (4.3), it suffices to show that $h'(0^+) > 0$. We have

$$\begin{aligned}
h'(0^+) &= \lim_{\eta\to 0} \frac{\widehat{V}(0,1-\eta,\eta) - \widehat{V}(0,1,0)}{\eta} \\
&= \lim_{\eta\to 0} \frac{(1-\eta)^p}{\eta}\Big(\widehat{V}(0,1,\frac{\eta}{1-\eta}) - \widehat{V}(0,1+\frac{\eta}{1-\eta},0)\Big) \\
&= \Big(\lim_{\eta\to 0} (1-\eta)^{p-1}\Big) \ \Big(\lim_{\delta\to 0} \frac{1}{\delta}\big(\widehat{V}(0,1,\delta) - \widehat{V}(0,1+\delta,0)\big)\Big) \\
&= \widehat{V}_y(0,1,0^+) - \widehat{V}_x(0,1^+,0).
\end{aligned}$$

We will show that the latter is strictly positive. Consider the auxiliary problem with initial data $(t,x,y) = (0,1,0)$. In this case the problem is the Merton type problem (4.5), with value function $\widehat{V}(0,x,0) = K_0 x^p$, so $\widehat{V}_x(0,1^+,0) = pK_0$. By solving the HJB equation for this problem, one finds K_0 as unique positive solution to

$$\Big(\beta + \lambda - \frac{pb_L^2}{2(1-p)\sigma_L^2}\Big)K_0 - (1-p)p^{-\frac{1}{1-p}} K_0^{-\frac{p}{1-p}} = \lambda K_V, \tag{4.33}$$

and the corresponding optimal wealth process is

$$d\tilde{X}_t^* = -c_t^* dt + \pi_t^* \frac{dL_t}{L_t}, \tag{4.34}$$

where

$$c_t^* = p^{-\frac{1}{1-p}} K_0^{-\frac{1}{1-p}} \ \tilde{X}_t^*, \qquad \pi_t^* = \frac{b_L}{\sigma_L^2(1-p)} \ \tilde{X}_t^*. \tag{4.35}$$

Considering an agent with initial wealth $(1,\delta)$, who chooses the same investment/consumption strategy, we get

$$\widehat{V}(0,1,\delta) \ \geq \ \mathbb{E}\Big[\int_0^\infty e^{-(\beta+\lambda)t}\Big(\frac{(c_t^*)^p}{p} + \lambda G_\gamma[V](t,\tilde{X}_t^*,\tilde{Y}_t^{0,\delta})\Big)\,dt\Big].$$

Therefore

$$\begin{aligned}
\frac{\widehat{V}(0,1,\delta) - \widehat{V}(0,1,0)}{\delta} &\geq \frac{\lambda}{\delta}\,\mathbb{E}\Big[\int_0^\infty e^{-(\beta+\lambda)t}\big(G[V](t,\tilde{X}_t^*,\tilde{Y}_t^{0,\delta}) - G[V](t,\tilde{X}_t^*,0)\big)dt\Big] \\
&= \lambda K_V \int_0^\infty e^{-(\beta+\lambda)t}\,\mathbb{E}\Big[\frac{(\tilde{X}_t^* + \tilde{Y}_t^{0,\delta}J_t)^p - (\tilde{X}_t^*)^p}{\delta}\Big]\,dt.
\end{aligned} \tag{4.36}$$

Let $\delta \to 0$ in (4.36). Applying Fatou's Lemma and observing that $\tilde{Y}^{0,\delta}J_t = \delta I_t$,

$$\begin{aligned}
\widehat{V}_y(0,1,0^+) &\geq pK_V\lambda\int_0^\infty e^{-(\beta+\lambda)t}\mathbb{E}\Big[(\tilde{X}_t^*)^{p-1}I_t\Big]\,dt \\
&= pK_V\lambda\int_0^\infty \exp\Big(-\big(\lambda\frac{K_V}{K_0} - (b_I - \frac{\rho b_L\sigma_I}{\sigma_L})\big)t\Big)\,dt \\
&> pK_0 \ = \ \widehat{V}_x(0,1^+,0),
\end{aligned}$$

where the middle equality uses (2.1), (2.2), (4.33), (4.34) and (4.35), and the strict inequality uses $\frac{b_I}{\sigma_I} > \frac{\rho b_L}{\sigma_L}$. □

When the two assets are uncorrelated, the result above says, in particular, that there is investment in the illiquid asset even if the Sharpe ratio of the liquid asset is higher than that of the illiquid asset. Let us now deal with the optimal consumption and investment in the liquid asset. Set, for any $x \geq 0$, $y > 0$,[12]

$$C^*(s,x,y) \;:=\; y\,\widehat{C}^*\Big(s,\frac{x}{y}\Big), \qquad \Pi^*(s,x,y) \;:=\; y\,\widehat{\Pi}^*\Big(s,\frac{x}{y}\Big) + \frac{\rho\sigma_I}{\sigma_L}\frac{x}{y},$$

and set $\tilde{X}^{*,t,x} := Z^{*,t,x/y}\tilde{Y}^{t,y}$. From the above results, we get the following theorem.[13]

Theorem 4.13. *Let $\frac{b_I}{\sigma_I} > \frac{\rho b_L}{\sigma_L}$. The unique optimal control (α^*, c^*, π^*) for (2.10) is*

$$\begin{cases} \alpha_k^* = \dfrac{R_{\tau_k}}{1+z^*}, & k \in \mathbb{N}, \\ c_s^* = C^*(s-\tau_k, \tilde{X}_s^{*,\tau_k,R_{\tau_k}-\alpha_k^*}, \tilde{Y}_s^{\tau_k,\alpha_k^*}), & s \in [\tau_k,\tau_{k+1}), \quad k \in \mathbb{N}, \\ \pi_s^* = \Pi^*(s-\tau_k, \tilde{X}_s^{*,\tau_k,R_{\tau_k}-\alpha_k^*}, \tilde{Y}_s^{\tau_k,\alpha_k^*}), & s \in [\tau_k,\tau_{k+1}), \quad k \in \mathbb{N}. \end{cases}$$

Proof. The expression for α_k^* follows from Proposition 4.11. The proof that (c^*, π^*) is the unique optimal control in each random interval (τ_k, τ_{k+1}) follows by doing again, but in the opposite direction, the transformations leading to the equivalence between the original problem (2.10) and the transformed one (4.12) (via the first transformation (4.4)). Precisely, using Proposition 4.4, Theorem 4.9, Proposition 3.2, and the Markov nature of our problem, one shows, by induction, that $V(r) = \mathbb{E}\left[\int_0^{\tau_k} e^{-\beta s} U(c_s^*) ds\right] + \mathbb{E}\left[e^{-\beta\tau_k} V\left(R_{\tau_k}^*\right)\right]$, for all $k \in \mathbb{N}$. Then, (4.1) implies that the second term on the right hand side goes to 0 for $k \to \infty$, hence (α^*, c^*, π^*) is optimal. □

4.6 Numerical approximations

This subsection presents an iterative scheme to approximate K_V and Φ. The procedure is illustrated more extensively, in the case $\gamma = 0$, in [9, 13]. Because of the nonlocal term $\mathcal{H}_0[\varphi]$ in (4.17), we cannot approximate the value function Φ directly as a viscosity solution of a PDE, but need to define an iterative scheme. Fixing $T > 0$ and starting with $K_V^{0,T} := 0$, define, inductively on $n \in \mathbb{N}$, the sequence $(K_V^{n,T}, \Phi^{n,T})$ as follows.

- Given $n \in \mathbb{N}$ and $K_V^{n,T}$, let $\Phi^{n,T}$ on $\mathbb{R}_+^2$ be the unique constrained viscosity solution on $[0,T] \times \mathbb{R}_+$ to

$$-\Phi_t^{n,T} + K_\lambda \Phi^{n,T} - \lambda K_V^{n,T} f_\gamma(t,z) - \sup_{\hat{c}\geq 0, \hat{\pi}\in\mathbb{R}} H_{cv}^0(z, \Phi^{n,T}, \Phi_z^{n,T}, \Phi_{zz}^{n,T}; \hat{c}, \hat{\pi}) = 0, \quad (4.1)$$

[12] For the case $y = 0$, see the discussion after (4.5).

[13] We make the assumption $\frac{b_I}{\sigma_I} > \frac{\rho b_L}{\sigma_L}$ to make the problem meaningful in view of Proposition 4.12 (see again the discussion after (4.5)). In this case $z^* < \infty$.

with boundary and terminal conditions

$$\Phi^{n,T}(t,0) \quad = \quad K_V^{n,T} \int_t^T e^{-K_\lambda(s-t)} \lambda f_\gamma(s,0) ds, \qquad t \in [0,T], \tag{4.2}$$

$$\Phi^{n,T}(T,z) \quad = \quad 0, \qquad z \geq 0. \tag{4.3}$$

- Given $n \in \mathbb{N}$ and $\Phi^{n,T}$, let $K_V^{n+1,T} := \mathcal{H}_0[\Phi^{n,T}]$.

Then one proves, following [13, Ch. 3, Sec. 6], the following estimate ensuring the convergence of our scheme when $T \to \infty$ and $n \to \infty$.

Proposition 4.14. *Let $\delta := \frac{\lambda}{\lambda+\beta-k_p}$. For all (t,z) in $[0,T] \times \mathbb{R}_+$,*

$$\left|(\Phi - \Phi^{n,T})(t,z)\right| \quad \leq \quad C_0 e^{k_{J,p} t}(1+z)^p \left(\delta^n + \frac{e^{-(\lambda+\beta-k_p)T}}{1-\delta} + e^{-(\lambda+\beta-k_p)(T-t)}\right),$$

where $C_0 = V_{Mert}^{(p)}(1)$ (see Remark 2.7).

Proposition 4.14 provides a rate of convergence sensitive to the value of λ:

- the larger is λ, the slower is the convergence in n;
- the smaller is λ, the slower is the convergence in T.

5 Discussion

This section provides and discusses some numerical experiments performed, in the case of power utility, by means of the iterative approximation procedure described in Subsection 4.6. The discussion is limited to key features, in order to show how our methodology can be applied. We choose the parameters

$$\beta \ = \ 0.2, \qquad p \ = \ 0.5, \qquad b_L \ = \ 0.15, \qquad \sigma_L \ = \ 1, \qquad b_I \ = \ 0.2, \qquad \sigma_I \ = \ 1.$$

With these values, Assumption 2.6 is satisfied for every $\rho \in (-1,1)$. It is reasonable to let the illiquid asset have a higher Sharpe ratio than the liquid one. This is economically intuitive and ensures that, for every value of the correlation ρ, it is always optimal to invest in the illiquid asset (Proposition 4.12(2)). We solved the PDE (4.1) using an explicit finite-difference scheme, after the change of variable $\mathbb{R}_+ \to [0,1)$, $z \mapsto \tilde{z} = \frac{z}{z+1}$, inducing a corresponding transformation $\Phi \mapsto \tilde{\Phi}$, to work with the bounded domain $[0,1)$. We fixed beforehand T between 1 and 10, depending on λ according to what we said at the end of Subsection 4.6: for the extreme cases $\lambda = 1$ and $\lambda = 50$, we took, respectively, $T = 10$ and $T = 1$, as with these choices the terms involving T and $T-t$ in Proposition 4.14 become reasonably negligible (resp., of order 10^{-5} and 10^{-22}). We used a uniform grid on $[0,T] \times [0,1]$ with time step length $5 \cdot 10^{-4}$ and space step length 0.02. The numbers $f_\gamma(t,\tilde{z})$ were computed beforehand at each point of the grid, using an L^2-optimal quantization grid for the gaussian law with $N = 5000$ points. Finally, almost all the numerical tests below were performed for $t = 0$, so Proposition 4.14 was used with $t = 0$ to estimate the error. The only numerical tests performed with $t > 0$ are the ones regarding the optimal consumption and investment in the liquid asset in Subsection 5.2, where $\lambda = 5$ and $t = 1$; in this case, $T = 5$ in order to control the error.

5.1 Value function and cost of illiquidity

To study the cost of illiquidity, we consider quantities related to the value function V. The first measure of illiquidity cost is the difference between the value functions corresponding to different values of λ.[14] As $V(r) = r^p V(1)$, we study $V(1)$. Table 1 reports it for different values of λ and for $\gamma = 0$ or $\gamma = 1$.

λ	1	5	10	50	Constr./Unconstr. Merton
$\gamma = 0$	1.66755	1.70493	1.71257	1.71945	1.72133
$\gamma = 1$	1.67179	1.71121	1.71656	1.72036	1.72133

Table 1: $V(1)$ for various γ, λ, and fixed $\rho = 0$.

Another measure of illiquidity cost is given by - see [23] - the extra amount of initial wealth $e(r)$ needed to reach the same level of expected utility as an investor without trading restrictions and with the same initial capital r. Hence, it corresponds to the solution to the equation $V(r + e(r)) = V_M(r)$, where V_M is the value function of the corresponding unconstrained Merton problem. As $e(r)$ is proportional to r, we study $e(1)$. Table 2 reports $e(1)$ for different values of λ and for $\gamma = 0$ or $\gamma = 1$.

λ	1	5	10	50
$\gamma = 0$	0.066	0.0193	0.0103	0.00218
$\gamma = 1$	0.060	0.0119	0.0056	0.00112

Table 2: $e(1)$ for various γ, λ, and fixed $\rho = 0$.

Concerning the impact of the observation parameter γ, we observe that:

1. The impact of γ on the absolute cost of liquidity, measured as $V(1)$ or as $e(1)$, is not high, but shows a peak for intermediate values of λ. This is expected: when λ is low, the illiquid asset is very rarely traded, so it is less useful to have information on it; when λ is high, the discrete information on I at trading dates is frequently updated, so the continuous part of the information on I is less relevant.
2. The relative impact of γ on $e(1)$, i.e. the quantity $\frac{e^{\gamma=0}(1) - e^{\gamma=1}(1)}{e^{\gamma=0}(1)}$, for $\lambda \geq 5$ is of the order of 50%.

5.2 Optimal policies

Again, without loss of generality, we assume that $r = 1$. Figure 1 represents the optimal allocation policy in the illiquid asset, expressed as a proportion of wealth - i.e. the quantity

[14] We stress that one cannot always expect convergence to Merton's unconstrained solution when $\lambda \to \infty$. The presence of illiquidity - even in the case of high λ (high trading frequency) - induces a constraint on the investment strategies in I, in order to satisfy the state constraint. This fact can produce a gap between the fully liquid case and the limit of the illiquid one. In general, the limit case for $\lambda \to \infty$ corresponds to the *constrained* fully liquid Merton problem, i.e. the Merton problem with the constraint that the investment in I does not admit borrowing or short selling. When the optimal solution of the unconstrained Merton problem satisfies this constraint, the constrained and unconstrained Merton problems are equivalent : hence, in the latter case, we have convergence to the (unconstrained) Merton solution for $\lambda \to \infty$.

$\hat{z} := \frac{1}{1+z^*}$ - and as function of ρ, for fixed $\gamma = 0$. The different lines correspond to different values of λ. When λ is low, $\hat{z}$ is close to 0; when λ is high, $\hat{z}$ is close to the corresponding value in the constrained Merton problem. For increasing values of λ, the graphs lie between these two extreme cases, increasing with it.

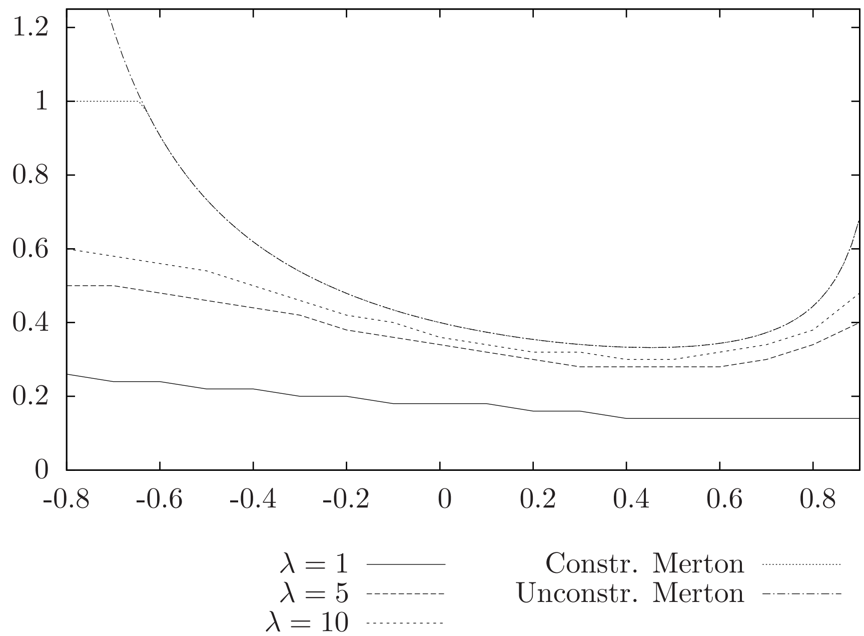

Figure 1: Optimal investment proportion $\hat{z}$ in the illiquid asset, as function of ρ, for $\gamma = 0$.

The impact of the observation parameter on the optimal allocation proportion in the illiquid asset is shown in Table 3 for different values of λ, considering the extreme cases $\gamma = 0$ and $\gamma = 1$, for fixed $\rho = 0$. The agent invests more in the illiquid asset if she/he can observe it continuously. The impact of γ is negligible in the extreme cases $\lambda = 1$ and $\lambda = 50$; of the order of 6%, when $\lambda = 3, 5, 10$.

	$\lambda = 1$	$\lambda = 3$	$\lambda = 5$	$\lambda = 10$	$\lambda = 50$	Constr./Unconstr. Merton
$\gamma = 0$	0.18	0.3	0.34	0.36	0.4	0.4
$\gamma = 1$	0.18	0.32	0.36	0.38	0.4	0.4

Table 3: Optimal investment (proportion over the wealth) in the illiquid asset, for various γ, λ and fixed $\rho = 0$.

Let us analyze the feedback maps C^*, Π^*. We need to be careful comparing these maps for different values of γ, when $t > 0$. Indeed, they are defined on y, which refers to the stochastic process $\tilde{Y}$. But $\tilde{Y}$ depends on γ, so the feedback maps C^*, Π^* do not read the same input for different values of γ. To overcome this problem, one would need to perform Monte-Carlo simulations to study the distributions of the optimal strategies.[15] However,

[15] As our auxiliary control problem is not autonomous, we cannot look at the stationary distribution as in [1].

such an approach would be numerically intensive; for simplicity, we consider the dependence of the feedback maps on the extra observation $B^{(1)}$, for different values of γ, which still enables us to illustrate the effect of partial observation. As we are mainly interested in the impact of γ on the strategies, we fix the other parameters, taking $\lambda = 5$ and $\rho = 0$. We consider an agent who, at time $t = 1$, owns a liquid wealth $\tilde{X}_1 = 0.5$, having invested $\alpha_0 = \tilde{Y}_0 = 0.5$ in illiquid wealth at time $t = 0$ (assuming that τ_1 has not occurred yet). We plot the optimal consumption and investment in the liquid asset as functions of the additional information $B_1^{(1)}$, which determines, together with γ, the value of $\tilde{Y}_1$.[16] To be more explicit, we compute $C^*(1, \tilde{X}_1, \tilde{Y}_1)$, $\Pi^*(1, \tilde{X}_1, \tilde{Y}_1)$ as functions of $B_1^{(1)}$ and γ. As, from (3.9), $\tilde{Y}_1 = \tilde{Y}_0 \ \exp\left(b_Y - \frac{(1-\rho^2)\gamma^2}{2} + \sqrt{1-\rho^2}\ \gamma B_1^{(1)}\right)$, we get the function to plot by substitution.

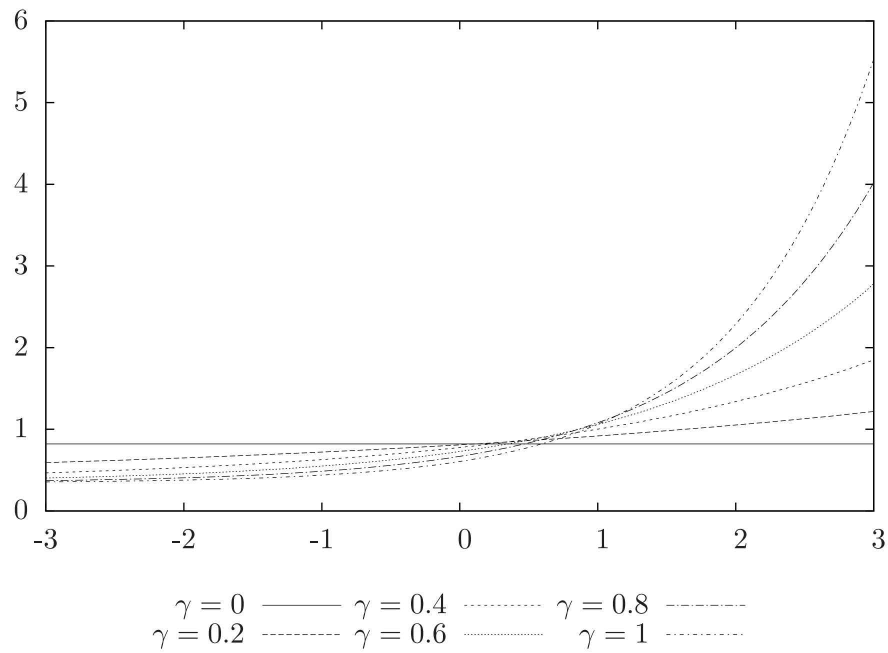

Figure 2: Optimal consumption rate, as function of $B_1^{(1)}$, for various γ (setting $\lambda = 5$, $\rho = 0$, $\tilde{X}_1 = .5$, $\tilde{Y}_0 = .5$).

Figure 2 graphs the consumption rate: it is increasing in $B^{(1)}$, which illustrates the unsurprising fact that, when the agent knows that her/his illiquid investment is doing well, she/he should consume more. This effect is considerably stronger, when information about the illiquid asset increases (γ close to 1). The impact of the observation on the investment in the liquid asset can be studied in a similar way. We omit the resulting graph, as it has the same qualitative features as Figure 2 (when the agent knows her/his illiquid investment is doing well, she/he can take more risk with her/his investment in the other risky asset).

[16] This is true as $\rho = 0$, so that $\tilde{Y}$ does not depend on W.

5.3 Conclusions

From the analysis performed, we can extract the following behaviour with respect to the relevant parameters γ, λ.

1. The impact of γ on the cost of illiquidity and on the investment in the illiquid asset is negligible for low and high value of λ, and has a peak for intermediate values of λ.
2. The investment in the illiquid asset is increasing in λ and γ.
3. The consumption and the investment in the liquid asset are very sensitive with respect to γ for intermediate values of λ.

A Appendix

A.1 Technical results

Lemma A.1. *Given $r \geq 0$, for any $(c, \pi, \alpha) \in \mathcal{A}(r)$, there exists $(c^0, \pi^0) \in \mathcal{A}_0(r - \alpha_0)$ such that $(c, \pi)\mathbf{1}_{\{t \leq \tau_1\}} = (c^0, \pi^0)\mathbf{1}_{\{t \leq \tau_1\}}$ ($P \otimes ds$ – a.e.).*

Proof. Using the definition of $\mathbb{G}$, by a monotone class argument, for each $(\mathcal{G}_t)_{t\geq 0}$-predictable process (c, π), we can find a $(\mathcal{W}_t \vee \mathcal{B}_t^{(1)})_{t\geq 0}$-predictable process (c^0, π^0) satisfying $(c, \pi)\mathbf{1}_{\{t\leq\tau_1\}} = (c^0, \pi^0)\mathbf{1}_{\{t\leq\tau_1\}}$. The fact that $(c, \pi, \alpha) \in \mathcal{A}(r)$ implies $(c^0, \pi^0) \in \mathcal{A}_0(r - \alpha_0)$ is straightforward. □

Proposition A.2.

(i) G_γ *is well defined on the set of measurable functions with at most linear growth.*

(ii) G_γ *is linear and positive, in the sense that it maps positive functions to positive ones. As a consequence,* G_γ *is increasing, in the sense that* $\phi \leq \psi \Rightarrow G_\gamma[\phi] \leq G_\gamma[\psi]$.

(iii) G_γ *maps increasing functions to functions increasing with respect to both* x *and* y.

(iv) G_γ *maps concave functions to functions concave with respect to* (x, y).

(v) If $\psi(r) = r^p$, $p \in (0, 1)$, *then* ($k_{J,p}$ *is defined in* (3.17))

$$G_\gamma[\psi](t, \xi x, \xi y) = \xi^p G_\gamma[\psi](t, x, y), \quad \forall t \geq 0,\ \forall (x, y) \in \mathbb{R}_+^2,\ \forall \xi \geq 0. \tag{A.1}$$

$$0 \leq G_\gamma[\psi](t, x, y) \leq e^{k_{J,p}t}(x + y)^p, \quad \forall t \geq 0,\ \forall (x, y) \in \mathbb{R}_+^2. \tag{A.2}$$

(vi) Let $p \in (0, 1]$, *and let* ψ *be a p-Hölder continuous function on* $\mathbb{R}_+$. *Then, there exists some constant* $C \geq 0$ *such that, for all* $t \geq 0$, $x, x', y, y' > 0$, *and* $0 \leq h \leq 1$,

$$|G_\gamma[\psi](t, x, y) - G_\gamma[\psi](t, x', y)| \leq C|x - x'|^p, \tag{A.3}$$

$$|G_\gamma[\psi](t, x, y) - G_\gamma[\psi](t, x, y')| \leq Ce^{k_{J,p}t}|y - y'|^p, \tag{A.4}$$

$$|G_\gamma[\psi](t, x, y) - G_\gamma[\psi](t + h, x, y)| \leq C_1 e^{k_{J,p}t} y^p h^{p/2}, \tag{A.5}$$

Proof. It parallels [9], where the statement is proved in the case $\gamma = 0$. □

Lemma A.3. *Let $p \in (0,1)$ and let $k_{L,Y,p}$, $k_{J,p}$ be defined as in* (3.16)-(3.17)*. For every $(t,x,y) \in \mathbb{R}^3_+$ and $(c,\pi) \in \mathcal{A}_t(x)$, we have*

$$\mathbb{E}\left[(\tilde{X}_s^{t,x,c,\pi} + \tilde{Y}_s^{t,y})^p\right] \quad \leq \quad e^{k_{L,Y,p}(s-t)}(x+y)^p, \quad \forall s \geq t. \tag{A.6}$$

In particular, combining (A.6) *with Proposition A.2(v) and denoting $\varphi(r) := r^p$, we have*

$$\mathbb{E}\left[G_\gamma[\varphi](s, \tilde{X}_s^{t,x,c,\pi}, \tilde{Y}_s^{t,y})\right] \quad \leq \quad e^{k_{J,p}t} e^{(k_{L,Y,p}+k_{J,p})(s-t)}(x+y)^p. \tag{A.7}$$

Proof. It parallels [9], where the statement is proved in the case $\gamma = 0$. □

Lemma A.4. *Set*

$$f(u_L, u_I) \quad := \quad p(u_L b_L + u_I b_I) - \frac{p(1-p)}{2}(u_L^2\sigma_L^2 + u_I^2\sigma_I^2 + 2\rho u_L u_I \sigma_L \sigma_I).$$

Recalling (2.11)*, we have $k_p = \sup_{u_L \in \mathbb{R}, u_I \in [0,1]} f(u_L, u_I)$. For any b'_Y, b'_J such that $b'_Y + b'_J = b_I$, define*

$$f_{b'_Y}(u_L, u_Y) := \; p(u_L b_L + u_Y b'_Y) - \frac{p(1-p)}{2}(u_L^2\sigma_L^2 + u_Y^2\sigma_I^2(\rho^2 + \gamma^2(1-\rho^2)) + 2\rho u_L u_Y \sigma_L \sigma_I),$$

$$f_{b'_J}(u_J) \quad := \quad p b'_J u_J - \frac{p(1-p)}{2}\sigma_I^2(1-\rho^2)(1-\gamma^2)u_J^2,$$

and $k'_{L,Y,p} := \sup_{u_L \in \mathbb{R}, u_Y \in [0,1]} f_{b'_Y}(u_L, u_Y)$, $k'_{J,p} := f_{b'_J}(u_J)$. Then $k_p \leq k'_{L,Y,p} + k'_{J,p}$ and this inequality is an equality if we choose

$$b'_Y \quad = \quad \gamma^2 b_I + (1-\gamma^2)\frac{b_L \rho \sigma_I}{\sigma_L}. \tag{A.8}$$

Proof. As $f_{b'_Y}(u_L, u_I) + f_{b'_J}(u_I) = f(u_L, u_I)$, by definition of $k_p, k'_{L,Y,p}, k'_{J,p}$, we have

$$k_p \quad = \quad \sup_{u_L \in \mathbb{R}, u_I \in [0,1]} \left(f_{b'_Y}(u_L, u_I) + f_{b'_J}(u_I)\right) \quad \leq \quad k'_{L,Y,p} + k'_{J,p}.$$

The maximizers of $f, f_{b'_Y}, f_{b'_J}$ always exist, so the inequality above becomes an equality if and only if there exist a maximizer (u_L^*, u_Y^*) of $f_{b'_Y}$ and a maximizer u_J^* of $f_{b'_J}$ such that $u_Y^* = u_J^*$. If $\gamma \in (0,1)$, by strict convexity of $f'_{b'_Y}$ and $f_{b'_J}$, these maximizers are unique and can be computed explicitly with the first-order conditions, as

$$u_J^* \;=\; \mathrm{Proj}_{[0,1]}\left(\frac{b'_J}{(1-p)\sigma_I^2(1-\rho^2)(1-\gamma^2)}\right), \quad u_Y^* \;=\; \mathrm{Proj}_{[0,1]}\left(\frac{b'_Y - \frac{b_L\rho\sigma_I}{\sigma_L}}{(1-p)\sigma_I^2(1-\rho^2)\gamma^2}\right),$$

As $b'_Y + b'_J = b_I$, (A.8) can be rewritten as $\frac{b'_J}{(1-\gamma^2)} = \frac{1}{\gamma^2}\left(b'_Y - \frac{b_L\rho\sigma_I}{\sigma_L}\right)$, which implies $u_J^* = u_Y^*$. To conclude, it remains to notice that, under (A.8), for $\gamma = 0$ (respectively, $\gamma = 1$), the function $f_{b'_Y}$ does not depend on u_Y (respectively, the function $f_{b'_J}$ does not depend on u_J), so we can choose $u_Y^* = u_J^*$. □

Given $(\bar{t}, \bar{z}) \in \mathbb{R}^+ \times (0, +\infty)$ and $\varepsilon \in (0, \bar{z})$, we denote

$$D_\varepsilon(\bar{t}, \bar{z}) \quad := \quad [\bar{t}, \bar{t}+\varepsilon) \times (\bar{z}-\varepsilon, \bar{z}+\varepsilon) \subset \mathbb{R}^+ \times (0,+\infty). \tag{A.9}$$

Lemma A.5. *Let* $(\bar{t}, \bar{z}) \in \mathbb{R}^+ \times (0, +\infty)$ *and* $\varepsilon \in (0, \bar{z})$.

1. *There exists* $N_\varepsilon > 0$ *such that*

$$\limsup_{h \to 0^+} \left| \frac{\Phi(t+h, z) - \Phi(t, z)}{h} \right| \leq N_\varepsilon, \quad \forall\, (t, z) \in D_\varepsilon(\bar{t}, \bar{z}). \tag{A.10}$$

2. $\Phi(t, \cdot) \in C^1((\bar{z} - \varepsilon, \bar{z} + \varepsilon); \mathbb{R})$, *for every* $t \in [\bar{t}, \bar{t} + \varepsilon)$, *and there exist* $m_\varepsilon, M_\varepsilon > 0$ *such that*

$$m_\varepsilon \leq \Phi_z(t, z) \leq M_\varepsilon, \quad \forall (t, z) \in D_\varepsilon(\bar{t}, \bar{z}). \tag{A.11}$$

3. $\Phi(t, \cdot)$ *is twice differentiable a.e. in* $(\bar{z} - \varepsilon, \bar{z} + \varepsilon)$, *for every* $t \in [\bar{t}, \bar{t} + \varepsilon)$. *Moreover, denoting by* $\mathcal{O}_t^\varepsilon \subset (\bar{z} - \varepsilon, \bar{z} + \varepsilon)$ *the set where* $\Phi(t, \cdot)$ *is twice differentiable, there exists* $\delta_\varepsilon > 0$ *such that*

$$\Phi_{zz}(t, z) \leq -\delta_\varepsilon, \quad \forall t \in [\bar{t}, \bar{t} + \varepsilon),\ z \in \mathcal{O}_t^\varepsilon. \tag{A.12}$$

Proof. 1. Set

$$\mathcal{J}(t, z; \hat{c}, \hat{\pi}) := \widehat{\mathbb{E}} \left[\int_0^\infty e^{-K_\lambda s} \left(\frac{\hat{c}_s^p}{p} + \lambda f_\gamma(t + s, Z_s^{0,z,\hat{c},\hat{\pi}}) \right) ds \right]. \tag{A.13}$$

As the SDE for Z is autonomous, we have

$$\Phi(t, z) = \sup_{(\hat{c}, \hat{\pi}) \in \hat{\mathcal{A}}_0(z))} \mathcal{J}(t, z; \hat{c}, \hat{\pi}). \tag{A.14}$$

Recall that $f_\gamma(t, z) = K_V\ \mathbb{E}(z + J_t)^p$. Applying Dynkin's formula to $K_V(z + J_t)^p$, we see that $f_\gamma(\cdot, z)$ is differentiable and

$$\left| \frac{\partial}{\partial t} f_\gamma(t, z) \right| \leq C_{J,p} f_\gamma(t, z), \tag{A.15}$$

where $C_{J,p} = |b_J| p + \frac{1}{2} p(1 - p)\sigma_J^2$. So we can differentiate (A.13) with respect to t and, using (A.15), we get $\left| \frac{\partial}{\partial t} \mathcal{J}(t, x, y; c, \pi) \right| \leq C_{J,p}\ \Phi(t, z)$. The latter estimate is uniform in $(\hat{c}, \hat{\pi}) \in \hat{\mathcal{A}}_0(z)$, so from (A.14) and the fact that

$$|\Phi(t + h, z) - \Phi(t, z)| \leq \sup_{(\hat{c}, \hat{\pi}) \in \hat{\mathcal{A}}_0(z)} |\mathcal{J}(t + h, z; \hat{c}, \hat{\pi}) - \mathcal{J}(t, z; \hat{c}, \hat{\pi})|,$$

we get the claim with $N_\varepsilon = C_{J,p} \cdot \sup_{D_\varepsilon(\bar{t}, \bar{z})} \Phi(t, z)$.

2. Let $(t, z) \in D_\varepsilon(\bar{t}, \bar{z})$. $\Phi(t, \cdot)$ is strictly increasing for each t: this follows from the fact that it is concave, nondecreasing (it inherits these properties from $\widehat{V}$), and that $\lim_{z \to +\infty} \Phi(t, z) = +\infty$. Hence, strict monotonicity and concavity yield the existence of the left and right derivatives $\Phi_z^-(t, z)$, $\Phi_z^+(t, z)$, and the inequalities $\Phi_z^-(t, z) \geq \Phi_z^+(t, z) > 0$. Then, to show that $\Phi(t, \cdot)$ is differentiable at z, we need to prove that the first of the previous inequality is actually an equality. Assume, by contradiction, that $\Phi_z^-(t, z) > \Phi_z^+(t, z)$. Let $\delta > 0$ and consider the function, defined for $z_1 \in (\bar{z} - \varepsilon, \bar{z} + \varepsilon),\ t_1 \in [t, \bar{t} + \varepsilon)$,

$$\varphi^\delta(t_1, z_1) := \Phi(t, z) + \frac{\Phi_z^-(t, z) + \Phi_z^-(t, z)}{2}(z_1 - z) - \frac{1}{2\delta}(z_1 - z)^2 + (N_\varepsilon + \delta)(t_1 - t).$$

Due to item 1, the function $\Phi - \varphi^\delta$ has a local maximum at (t,z) in $(\bar z-\varepsilon, \bar z+\varepsilon)\times[t,\bar t+\varepsilon)$. Therefore, the subsolution viscosity property at (t,z) implies

$$- N_\varepsilon - \delta + K_\lambda \Phi(t,z) - K_2 z \frac{\Phi_z^-(t,z)+\Phi_z^-(t,z)}{2} - \lambda K_V f_\gamma(t,z) + \frac{K_\gamma^2}{2} z^2 \frac{1}{\delta} - \sup_{\hat c\geq 0,\, \hat\pi\in\mathbb{R}} H^1_{cv}\left(\frac{\Phi_z^-(t,z)+\Phi_z^+(t,z)}{2}, -\frac{1}{\delta}; \hat c, \hat\pi\right) \;\leq\; 0,$$

where $H^1_{cv}(r,q;\hat c,\hat\pi) = \frac{\hat c^p}{p} - \hat c r + K_1\hat\pi r + \frac{1}{2}\sigma_L^2\hat\pi_2^2 q$. Letting $\delta\to 0$, we get a contradiction as $\frac{K_\gamma^2}{2}z^2\frac{1}{\delta}\to+\infty$, whereas the other terms are bounded uniformly in δ. So $\Phi(t,\cdot)$ is differentiable at each $z\in(\bar z-\varepsilon,\bar z+\varepsilon)$ for every $t\in[\bar t,\bar t+\varepsilon)$. The fact that $\Phi(t,\cdot)\in C^1((\bar z-\varepsilon,\bar z+\varepsilon);\mathbb{R})$, for every $t\in[\bar t,\bar t+\varepsilon)$, follows from concavity. Finally, let us show (A.11). Let $\delta = \frac{\bar z-\varepsilon}{2}$. By concavity of $\Phi(t,\cdot)$, we have, for every $z\in(\bar z-\varepsilon,\bar z+\varepsilon)$ and $t\in[\bar t,\bar t+\varepsilon)$,

$$\frac{\Phi(t,\bar z-\varepsilon)-\Phi(t,\bar z-\varepsilon-\delta)}{\delta} \;\leq\; \Phi_z(t,z) \;\leq\; \frac{\Phi(t,\bar z+\varepsilon+\delta)-\Phi(t,\bar z+\varepsilon)}{\delta}, \qquad \text{(A.16)}$$

and, by concavity and strict monotonicity,

$$\frac{\Phi(t,\bar z-\varepsilon)-\Phi(t,\bar z-\varepsilon-\delta)}{\delta} \;<\; +\infty, \qquad \frac{\Phi(t,\bar z+\varepsilon+\delta)-\Phi(t,\bar z+\varepsilon)}{\delta} \;>\; 0.$$

Calling $M_\varepsilon := \sup_{t\in[\bar t,\bar t+\varepsilon)} \frac{\Phi(t,\bar z-\varepsilon)-\Phi(t,\bar z-\varepsilon-\delta)}{\delta}$, $m_\varepsilon \;:=\; \inf_{t\in[\bar t,\bar t+\varepsilon)} \frac{\Phi(t,\bar z+\varepsilon+\delta)-\Phi(t,\bar z+\varepsilon)}{\delta}$, by continuity of Φ, we have $0<m_\varepsilon\leq M_\varepsilon<\infty$, so the claim follows by (A.16).

3. Let $(t,z)\in D_\varepsilon(\bar t,\bar z)$. The fact that there exists a set $\mathcal{O}_t^\varepsilon$ with full Lebesgue measure such that $\Phi(t,\cdot)$ is differentiable at the points of $\mathcal{O}_t^\varepsilon$ follows from concavity of $\Phi(t,\cdot)$ and Alexandrov's Theorem. Assume $z\in\mathcal{O}_t^\varepsilon$. Letting $\delta>0$ and $\delta_1>0$, consider the function defined, for $z_1\in(\bar z-\varepsilon,\bar z+\varepsilon)$, $t_1\in[t,\bar t+\varepsilon)$, as

$$\varphi^\delta(t_1,z_1) \;:=\; \Phi(t,z)+\Phi_z(t,z)(z_1-z)+\frac{1}{2}(\Phi_{zz}(t,z)-\delta)(z_1-z)^2-(N_\varepsilon+\delta_1)(t_1-t).$$

Due to item 1, the function $\Phi-\varphi^\delta$ has a local minimum at (t,z) in $(\bar z-\varepsilon,\bar z+\varepsilon)\times[t,\bar t+\varepsilon)$, for each $\delta>0$. Therefore, the supersolution viscosity property at (t,z) and item 2 imply

$$N_\varepsilon+\delta_1+K_\lambda\Phi(t,z)-K_2 z m_\varepsilon-\lambda K_V f_\gamma(t,z)-\frac{K_\gamma^2}{2}z^2(\Phi_{zz}(t,z)-\delta) - \tilde U(M_\varepsilon)+\frac{1}{2}\frac{K_1^2}{\sigma_L^2}\frac{m_\varepsilon^2}{\Phi_{zz}(t,z)-\delta} \;\geq\; 0. \quad \text{(A.17)}$$

Note that, given $a_0,b_0>0$ and $c_0\in\mathbb{R}$, there exists $\alpha_0>0$ such that

$$a_0\xi-\frac{b_0}{\xi} \;\leq\; c_0,\;\; \xi\;\leq\;0 \implies \xi\;\leq\;-\alpha_0. \qquad \text{(A.18)}$$

As $\Phi_{zz}\leq 0$, from (A.17) we see that (A.18) holds for $\xi=\Phi_{zz}(t,z)-\delta$. So we get the existence of $\delta_\varepsilon>0$, independent of $(t,z)\in D_\varepsilon(\bar t,\bar z)$ and of δ, such that $\Phi_{zz}(t,z)\leq\delta-\delta_\varepsilon$. By arbitrariness of δ we get the claim. □

Proposition A.6. Φ *is a viscosity solution in* $D_\varepsilon(\bar{t}, \bar{z})$ *of*

$$-\varphi_t + K_\lambda \varphi - \lambda f_\gamma(t,z) K_V - \sup_{\hat{c}\in[0,c_M],\, \hat{\pi}\in[-\pi_M,\pi_M]} H^0_{cv}(z, \varphi_z, \varphi_{zz}; \hat{c}, \hat{\pi}) \;=\; 0, \qquad \text{(A.19)}$$

where $c_M := (U')^{-1}(m_\varepsilon)$, $\pi_M := \frac{|K_1| M_\varepsilon}{\sigma_L^2 \delta_\varepsilon}$.

Proof. The fact that Φ is a supersolution of (A.19) in $D_\varepsilon(\bar{t}, \bar{z})$ follows from the fact that it is a supersolution of (4.16), as the supremum is taken over a smaller set in (A.19). Let us show that it is a subsolution in $D_\varepsilon(\bar{t}, \bar{z})$. Take $(t,z) \in D_\varepsilon(\bar{t}, \bar{z})$ and let $\varphi \in C^{1,2}(D_\varepsilon(\bar{t}, \bar{z}); \mathbb{R})$ be such that $\varphi(t,z) = \Phi(t,z)$ and $\varphi \geq \Phi$ in $D_\varepsilon(\bar{t}, \bar{z})$. As Φ is differentiable with respect to z, it must be $\varphi_z(t,z) = \Phi_z(t,z)$. If $\varphi_{zz} \leq -\delta_\varepsilon$, then

$$\sup_{\hat{c}\geq 0,\, \hat{\pi}\in\mathbb{R}} H^0_{cv}(z, \varphi_z, \varphi_{zz}; \hat{c}, \hat{\pi}) \;=\; \sup_{\hat{c}\in[0,c_M],\, \hat{\pi}\in[-\pi_M,\pi_M]} H^0_{cv}(z, \varphi_z, \varphi_{zz}; \hat{c}, \hat{\pi}), \qquad \text{(A.20)}$$

so we have the desired subsolution inequality. Otherwise, assume $\varphi_{zz}(t,z) > -\delta_\varepsilon$ and consider the function $\tilde{\varphi}$ defined, for $z_1 \in (\bar{z} - \varepsilon, \bar{z} + \varepsilon)$, $t_1 \in [t, \bar{t} + \varepsilon)$, as

$$\tilde{\varphi}(t_1, z_1) \;:=\; \varphi(t_1, z) + \Phi_z(t_1, z)(z_1 - z) - \frac{1}{2}\delta_\varepsilon (z_1 - z)^2. \qquad \text{(A.21)}$$

We have

$$\tilde{\varphi}(t_1, z) \;\geq\; \varphi(t_1, z) \;\geq\; \Phi(t_1, z), \qquad \forall\, t_1 \in [t, \bar{t} + \varepsilon). \qquad \text{(A.22)}$$

Fix $t_1 \in [t, \bar{t} + \varepsilon)$. Consider, for $z_1 \in (\bar{z} - \varepsilon, \bar{z} + \varepsilon)$, the Dini derivative of Φ_z at z_1, i.e. $D^+_z \Phi_z(t_1, z_1) \;:=\; \limsup_{h\to 0} \frac{\Phi_z(t_1, z_1 + h) - \Phi_z(t_1, z_1)}{h}$. As $\Phi(t_1, \cdot)$ is concave, we have

$$D^+_z \Phi_z(t_1, z_1) \;\leq\; 0, \qquad \forall\; z_1 \in (\bar{z} - \varepsilon, \bar{z} + \varepsilon). \qquad \text{(A.23)}$$

Moreover, by Lemma A.5(3), we have

$$D^+_z \Phi_z(t_1, z_1) \;\leq\; -\delta_\varepsilon, \qquad \forall\; z_1 \in \mathcal{O}^\varepsilon_{t_1}. \qquad \text{(A.24)}$$

From (A.23)-(A.24), from the fact that $\mathcal{O}^\varepsilon_{t_1}$ has full measure, and from Lemma 3.3 in [11], we get, by integrating twice (A.24),

$$\Phi(t_1, z_1) \;\leq\; \Phi(t_1, z) + \Phi_z(t_1, z)(z_1 - z) - \frac{1}{2}\delta_\varepsilon (z_1 - z)^2. \qquad \text{(A.25)}$$

Combining (A.25) with (A.21)-(A.22), we get $\tilde{\varphi}(t,z) = \Phi(t,z)$ and $\tilde{\varphi} \geq \Phi$ in $(\bar{z} - \varepsilon, \bar{z} + \varepsilon) \times [t, \bar{t} + \varepsilon)$. As Φ is a viscosity subsolution of (4.16), we have

$$-\tilde{\varphi}_t + K_\lambda \tilde{\varphi} - \lambda K_V f_\gamma(t,z) - \sup_{\hat{c}\geq 0,\, \hat{\pi}\in\mathbb{R}} H^0_{cv}(z, \tilde{\varphi}_z, \tilde{\varphi}_{zz}; \hat{c}, \hat{\pi}) \;\leq\; 0.$$

On the other hand,

$$\sup_{\hat{c}\geq 0,\, \hat{\pi}\in\mathbb{R}} H^0_{cv}(z, \tilde{\varphi}_z, \tilde{\varphi}_{zz}; \hat{c}, \hat{\pi}) \;=\; \sup_{\hat{c}\in[0,c_M],\, \hat{\pi}\in[-\pi_M,\pi_M]} H^0_{cv}(z, \tilde{\varphi}_z, \tilde{\varphi}_{zz}; \hat{c}, \hat{\pi}),$$

so also

$$-\tilde{\varphi}_t + K_\lambda\tilde{\varphi} - \lambda K_V f_\gamma(t,z) - \sup_{\hat{c}\in[0,c_M],\ \hat{\pi}\in[-\pi_M,\pi_M]} H^0_{cv}(z,\tilde{\varphi}_z,\tilde{\varphi}_{zz};\hat{c},\hat{\pi}) \ \le\ 0. \tag{A.26}$$

Noting that

$$\varphi(t,z) \ =\ \tilde{\varphi}(t,z), \quad \varphi_z(t,z) \ =\ \tilde{\varphi}_z(t,z), \quad \varphi_{zz}(t,z) \ >\ -\delta_\varepsilon \ =\ \tilde{\varphi}_{zz}(t,z), \tag{A.27}$$

and taking into account that H^0_{cv} is nondecreasing in the last argument, combining (A.26) and (A.27), we get the subsolution inequality for φ. □

Lemma A.7. *Let $a<b$ and let $F:[0,T)\times(a,b)\times\mathbb{R}^3\to\mathbb{R}$, $(t,x,r,p,q)\mapsto F(t,x,r,p,q)$, be continuous, continuously differentiable in (x,r,p,q), and proper degenerate elliptic (i.e. nondecreasing in r and nonincreasing in q). Let $u\in C^{1,2}([0,T)\times(a,b);\mathbb{R})$ be a classical solution in $[0,T)\times(a,b)$ to*

$$u_t + F(t,x,u,u_x,u_{xx}) \ =\ 0. \tag{A.28}$$

Then the space derivative $v:=u_x$ is a viscosity solution in $[0,T)\times(a,b)$ to

$$v_t + \nabla F(t,x,u(t,x),v,v_x)\cdot(1,v,v_x,v_{xx}) = 0, \tag{A.29}$$

where $\nabla F=(F_x,F_r,F_p,F_q)$.

Proof. For $x\in(a,b)$ and sufficiently small $h>0$, define $u^h(t,x):=u(t,x+h)$ and $v^h:=\frac{u^h-u}{h}$. Then, due to continuous differentiability of u, we have $v^h\to v$ locally uniformly in $[0,T)\times(a,b)$, when $h\to 0^+$. Furthermore, as u is a solution to (A.28) and using the differentiability of F, we see that

$$\begin{aligned} v^h_t \ &=\ \frac{1}{h}\Big(F(t,x,u^h,u^h_x,u^h_{xx}) - F(t,x,u,u_x,u_{xx})\Big) \\ &=\ \Big(\nabla F(t,x,u,v^h,v^h_x) + \mathcal{E}^h(t,x)\Big)\cdot(1,v^h,v^h_x,v^h_{xx}), \end{aligned}$$

where

$$\begin{aligned} &\mathcal{E}^h(t,x) := \\ &\int_0^1 \nabla F(t,x+hs,(1-s)u^h(t,x)+su(t,x),(1-s)u^h_x(t,x)+su_x(t,x),(1-s)u^h_{xx}(t,x)+su_{xx}(t,x))ds \\ &-F(t,x,u(t,x),v^h(t,x),v^h_x(t,x)). \end{aligned}$$

By continuity of F and of u, u_x, u_{xx}, and by the fact that v^h (respectively, v^h_x) goes to u_x (respectively, to u_{xx}), as $h\to 0^+$, we see that $\mathcal{E}^h\to 0^+$, as $h\to 0^+$, locally uniformly in $[0,T)\times(a,b)$. Applying the stability result for viscosity solutions (see, e.g., [27, Prop. 5.9, Ch .4]), we get that v is a viscosity solution to (A.29). □

A.2 A result by Kryolv on existence of classical solutions to fully non-linear parabolic equations

Theorem A.8. *Let Θ be an index set and let $Q := (0,T) \times \mathcal{O}$, with $\mathcal{O} \subset \mathbb{R}^N$ open. Let*

$$a = (a^{i,j})_{i,j=1,\dots,N} : \Theta \times (0,T) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^{N\times N},$$

$$b : \Theta \times (0,T) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R},$$

and call (θ, t, x, r, q) their formal arguments. Assume the following conditions.

1. *For every $\theta \in \Theta$, the functions a, b are continuously differentiable with respect to (t, x, r, q), and, for every $(\theta, t) \in \Theta \times (0,T)$, they are twice continuously differentiable with respect to (x, r, q).*

2. *The first derivatives of a, b with respect to t and the second derivatives of a, b with respect to (x, r, q) are bounded in every set of the form*

$$S_M := \{(\theta, t, x, r, q) \in \Theta \times (0,T) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \ : \ \theta \in \Theta, (t,x) \in Q, r + |q| \leq M\}.$$

3. *The function a satisfies a uniform ellipticity condition : for some constants $\Lambda \geq \varepsilon > 0$,*

$$\varepsilon|\xi|^2 \leq \sum_{i,j} a^{i,j}\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall(\theta, t, x, r, q) \in \Theta \times (0,T) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N, \ \forall \xi \in \mathbb{R}^N.$$

4. *There exists a continuous function h such that, for every $(\theta, t, x, r, q) \in \Theta \times (0,T) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N$,*

$$|D_q a^{i,j}|(1 + |q|) + |D_r a^{i,j}| + |D_x a^{i,j}|(1 + |q|)^{-1} \leq h(r), \quad \forall i, j = 1, \dots, N,$$

$$|D_q b|(1 + |q|) + |b| + |D_r b| + |D_x b|(1 + |q|)^{-1} \leq h(r)(1 + |q|^2).$$

5. *There exist constants $\delta_0 > 0$ and $M_0 > 0$ such that*

$$b(\theta, t, x, -M_0, 0) \geq \delta_0, \qquad b(\theta, t, x, M_0, 0) \leq -\delta_0, \quad \forall(\theta, t, x) \in \Theta \times Q.$$

Then, for each ϕ in $C(\bar{Q})$, there exists a unique $u \in C^{1,2}(Q) \cap C(\bar{Q})$ solution to

$$-\partial_t u - \sup_{\theta \in \Theta} \left\{ \mathrm{Tr}\left(a(\theta, t, x, u, Du)D^2 u\right) + b(\theta, t, x, u, Du) \right\} = 0 \quad \text{in } Q,$$

with Dirichlet boundary condition $u = \phi$ on $\mathcal{P}Q := \{T\} \times \mathcal{O} \cup (0,T) \times \partial\mathcal{O}$.

Proof. See [19, Th. 3, Sec. 6.4, p. 301]. The conditions are those in Example 8, Section 6.1, p. 279, of the same book. □

Acknowledgements. The authors thank the Editor, the Associate Editor and two anonymous Referees for their careful reviews and their useful suggestions. The authors are sincerely grateful to Silvia Faggian for her kind help for the editing work and Claudio Tebaldi for valuable comments. The authors also thank, for useful comments, the participants to the seminars where this paper has been presented, in Berne, Florence, Levico Terme, Pise, Bruxelles, Berlin, Manchester.

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