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Essays in structural economics

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Chapter 1

A structural model for corporate liquidity

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Abstract

I analyse the patterns of the cross-sectional distribution of liquidity across Italian limited liability companies. I document that the level of liquidity is stable over business cycles and that firms react to fluctuations in economic conditions by adjusting their net trade debt position. To understand the impact of cash flow uncertainty, financial efficiency, and financing costs, I propose a fixed-cost model of liquidity management. I estimate the model by matching the moments of the cross-sectional distribution of liquidity. I use the estimated model to highlight the impact of inflows' uncertainty and refinancing costs on the optimal management policy. Lastly, I (am planning to) do counterfactual exercises to understand the impact of different policies on liquidity management choices.

[1]Luiss Guido Carli. I would like to thank my supervisor Francesco Lippi, Fabio Braggion, Federico Carlini, Maria Pia Lombardo, Paolo Santucci de Magistris, and Luana Zaccaria.

1.1 Introduction

Liquidity management is one of the core activities at the firm level. Entrepreneurs and managers need to plan the allocation of resources, manage the cash flow, and balance liquidity inflows/outflows. Some firms might be willing to keep liquidity buffers, while others might rely on their supply chain to survive liquidity crunches. Despite the importance of this topic, there is a lack of studies disentangling liquidity dynamics at the firm and the aggregate level, and the majority of those studies focus only on cash holdings, which are only a part of the firms' liquidity. In this paper I focus on the distribution of liquidity across firms, highlighting the importance of trade credit/debt, and propose a structural model to analyze liquidity management decisions that produce results coherent with the empirical findings.

In their seminal paper, Bates et alt. [18], documented a constant increase in US firms' cash holdings from 1980 to 2006. This finding is interesting and puzzling because one expects cash-holding decisions to be affected by business cycle fluctuations and other macroeconomic events. For example, it is not obvious why cash holdings are unaffected by changes in aggregate riskiness or changes in money supply. When the money supply is low and interest rates are high, increasing the incentive to reduce cash holdings. Additionally, when the uncertainty level is high, firms might find it optimal to increase their cash holdings to create a buffer to absorb negative shocks. To answer these findings, several empirical studies have been proposed. The main findings are that precautionary motives, agency costs and financial constraints influence cash holdings decisions at the firm level.

These studies highlight particular mechanisms influencing liquidity or find some correlation between firms' characteristics and cash holdings choices, but focus only on cash holdings, which is only part of the liquidity management problem at the firm level. The liquidity level of a firm depends not only on its cash holdings but also on credit lines available and trade credit/debt position. For example, a firm that has a credit line granted, which is not used, has a liquidity buffer that is not captured using cash holdings as the only measure of liquidity. On the other side, a firm could report a good level of cash holdings while having a high level of short-term trade debt. Looking only at cash holdings, the firm seems to have liquidity, while in reality this liquidity is temporally "borrowed" from suppliers, and vice versa for the suppliers. Measuring liquidity at the corporate level is not easy exactly because several factors need to be considered and because liquidity can be stored in multiple forms.

While credit lines have been widely studied as a contingent source of liquidity, only recently the importance of trade credit/debt position has been analysed, mostly because data are rarely available. A recent study on the importance of credit lines for liquidity management is by Nikolov et al.[60], where they estimate a dynamic model of liquidity management and highlight the importance of collateral assets to rely effectively on credit lines as liquidity buffers. Credit lines have the desirable characteristic of being contingent, meaning that firms use them when they need extra liquidity, but they are costly and not easy to access for most firms.

On the other side, every firm has commercial relationships within its supply chain, and often B2B transactions are not settled in cash but postponed for some days, raising trade credit/debt. This is a common practice for most firms and is a key component of corporate liquidity management. For example, a firm that is suffering a liquidity deficit can ask its supplier to delay payments, ask its customers to anticipate payments, or directly settle in cash the trades. Amberg et al. [11] documented this pattern using an exogenous liquidity downfall in Sweden and showed that the trade position is indeed relevant for liquidity management. In another working paper [12], they show evidence that the usage of trade credit is common across firms' characteristics.

On the theoretical side, despite the increasing interest in the topic, there is a scarcity of structural models built to disentangle liquidity decisions or that produce a cross-sectional distribution of liq-

uidity coherent with the data. The first key contribution to this literature is Miller M. H., and & Orr D. [57], which propose an inventory model for cash management decisions. Recently, there has been a growing literature that models firms' investment-liquidity-risk management decisions, like the model proposed by Bolton et al. [29]. However, to the best of my knowledge, those works either do not focus on firms' liquidity, do not consider the trade credit position, or fail to match empirical evidence. With this paper I contribute to this literature, generalizing and estimating the model proposed by Miller & Orr by including stochastic cash flows and asymmetric costs.
In the first part of the paper, I show that the distribution of liquidity across firms is stable over the business cycle if the net trade position is included. The net trade position of each firm includes the standard "Cash and cash equivalents" plus the sum of net trade credit[1]. This measure is closely related to the net working capital. I then perform a descriptive statistic analysis and focus on differences at the industry level. In the second part of the paper, I estimate a fixed-cost model to simulate the decision process of a representative firm managing its liquidity. This model is a generalization of the framework proposed by Miller and Or where the firm minimizes the costs of managing liquidity, which is given by the opportunity cost of holding liquid assets and the fixed cost to be paid to adjust its liquidity position. The model is estimated using Italian S.P.A. balance sheet data. The sample spans from 2013 to 2022, including the COVID-19 period. The solution to the firm's problem is to determine two critical levels of liquidity such that the firm does not change its assets allocation while liquidity is within these levels. If liquidity ends up outside this region, the firm pays the fixed cost and adjusts its liquidity to the optimal level. The lower level represents the lowest level of liquidity such that the firm prefers to pay the cost and externally raise its liquidity. The upper level represents the maximum level of assets that the firm finds optimal to keep liquid, hence above that level, it prefers paying the fixed cost needed to find the optimal investment opportunity and to invest the excess liquidity, enjoying the investment revenues. Finally, the level of liquidity that the firm desires to hold is the one that minimizes the total expected cost.
The solution is associated with an invariant distribution of liquidity. The empirical strategy of this paper is to match the empirical cross-sectional distributions with the structural distributions to estimate the costs associated with liquidity management. In section 1.3.1, I show that the cross-sectional distribution is stable, therefore it can be compared with the invariant one associated with the solution of the problem. In the last section of the paper I (am planning to) use the estimated model to run a sensitive analysis and some counterfactual exercises. I will estimate the increase in the cost of liquidity management due to the COVID-19 pandemic.

1.2 Literature review

This paper is related to two strands of the literature. The first one is the empirical literature on liquidity management, the second one is the structural corporate finance literature. In the liquidity management literature, one of the most important papers is Bates et al.[18], who was the first to shed light on the patterns of cash holdings at the firm level. They find that US firms have been increasing their cash holdings since the eighties. They point out that the average firm more than doubled its cash/asset ratio. They claim that this was due to the increase in cash flow riskiness, lower inventories and account receivables and the advent of R&D. Acherya et alt. [1] analysed the correlation between cash accumulation and banking lending spreads and concluded that this liquidity accumulation is coherent with precautionary motives for riskier firms. Part of the literature focuses

[1] In Italy, Italian limited liabilities company, S.P.A., are requested to report in their balance sheet short-term trade credit and debt.

on the different cash management policies between public and private companies. For the US, Gao et al. [44] studied the relationship between agency costs and cash accumulation and found that agency costs are generally associated with higher liquidity. Moreover, higher agency frictions are associated with higher target levels of liquidity and different management choices. Poti et al. [68] found similar results for European firms. They analysed the interconnection between cash management policies and stakeholders' risk attitudes and found evidence that supports the correlation between cash management and tolerance to risk exposure. Bigelli & Sánchez-Vidal [24] analysed the Italian economy and found similar results.
On the theoretical side, liquidity management was first studied in theoretical frameworks by Baumol [60] and Tobin [11], who used inventory control theories to model money demand at the household level. Following this literature, Miller & Orr [22] proposed an inventory model for firm cash holdings with fully stochastic cash flow. A generalized version of the model was proposed by Frenkel & Jovanovic [42]. Recently, this framework has been used to study cash management and cash decisions at the household level by Alvarez & Lippi [7] and firms' investment decisions by Baley & Blanco [14].
Another closely related strand of the literature focuses on asset allocation, investments and funding. For example, Riddick & Whited [69] linked the level of cash holdings to the investment opportunities of the firm and found that firms accumulate cash holdings to save to invest when the right investment opportunity arrives. Between income uncertainty and constrained access to external funds, they find that the former plays a major role in determining the firms' saving policy. Bolton, Chen, and Wang [55] extend the analysis by estimating a larger model that includes investment decisions, financing opportunity and cash management a lá Miller&Orr. In their model, the representative firm decides how much to invest, how much to borrow and the optimal payout policy to maximize shareholders' return. Each period the firm receives revenues from production net of the financing decisions. The relationship between corporate liquidity, investment and financing decisions is that the former influences financing and payouts. However, their model implies an invariant distribution of cash holdings that is not coherent with the real cross-sectional distribution of cash holdings.

1.3 Empirical evidences

In this section I describe in detail the dataset I used, I motivate my measure of liquidity, and I present the main statistics and empirical findings.
To perform the empirical analysis I use balance sheet data of the Italian limited liabilities company.[2] Both public and private companies are included.[3] The sample span is 2013-2022, characterized by expansionary monetary policy, the COVID-19 pandemic, and the post pandemic monetary tightening. In this period there existed about 40.000 S.P.A., however, I excluded firm-year observations if one of the balance sheet entries I need is missing or empty. I exclude government firms and corporations whose key activity is a primary sector activity, financial or commodity. Additionally, I exclude inactive firms that report zero sales over a year and firms that filed for bankruptcy. After the cleaning, I am left with about 14.000 observations per year. Appendix A includes table 1.6 and figure 1.10, which contain information about the geographical dispersion of the sample.

[2]These companies are called "società per azioni" (S.P.A.) or "società in accomandità per azioni" (S.A.P.A.) and correspond to limited corporations whose equity is divided into stocks.
[3]The data have been downloaded from "AIDA" and "Refinitiv", Bureau van Dijk platforms.

1.3.1 Measuring liquidity

The first step is to build a measure of corporate liquidity. In household literature, the main measure of liquidity is the amount of cash in hand (or in the bank account). This measure is also used for corporate studies, but it is standardized by total assets to correct for firm size. In particular, this measure is the ratio between "*Cash and cash equivalents*" over "*Total assets*", two balance sheet entries that are publicly available. Calling this measure ch_t, the cross-sectional distributions in 2013 and 2022 are shown in the following Figure:

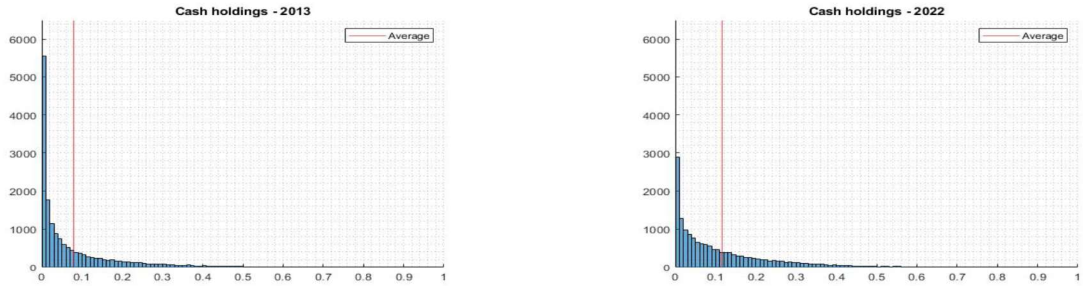

Figure 1.1: The figure contains the cross-sectional distribution of cash holdings in 2013 and 2022. The X-axis represents the cash holdings ratio, while the Y-axis represents the number of firms in each bin. Each bin represents a percentage point.

From Figure 1.1 it is possible to see how firms used to keep a small fraction of their assets as cash, as the mass is concentrated towards 0. In this sample period, there is a monotonic trend in increasing liquidity, a pattern already documented in the literature. This can be seen both by the shifting of the mass to the right and by the increase of the average cash holding, which happens in every year of the sample. In 2013, half of Italian corporations had less than 2% of their asset held as cash or cash equivalents. In 2022 the number of firms with liquidity below 2% halves, while the average firm now keeps more than 10% assets as cash or cash equivalents. This increase is constant across the sample period. The following table and figure contain details about the distribution of cash holdings across the sample:

Figure 1.2: Average cash holdings with confidence intervals and trend.

Year	Mean	Stand. Dev.	Median
2013	8.17	12.97	2.76
2014	8.61	13.14	3.13
2015	9.16	13.25	3.72
2016	9.82	13.54	4.44
2017	10.12	13.69	4.76
2018	10.07	13.74	4.68
2019	10.54	14.29	5.01
2020	13.04	14.92	8.06
2021	13.05	14.93	7.95
2022	11.61	14.17	6.68

Table 1.1: Statistics for the cash/asset ratio. All variables are in %.

From Table 1.2 it is clear that during the COVID-19 year, there was a sharp increase in cash holdings. One of the reasons is that the Italian government[4] adopted a series of policies to facilitate banks' lending and access to credit to small and medium firms. These policies aimed to guarantee that firms had enough resources, especially liquidity, to survive the pandemic.[5] After the pandemic, liquidity returned to the pre-pandemic trend level. The trend was calculated using OLS and observations from 2013 to 2019.

The main question of this paper is to understand why firms have only a small fraction of their asset as liquid, while theoretical works suggest that firms should keep a liquidity buffer to absorb shocks.[6] Before the pandemic, although monetary policy was loose, most firms had only 3-5% of their assets held as cash or cash equivalents. The answer is that cash holdings are an important component of the liquidity of a firm, but they are not the only ones to be considered, meaning that it is not a good proxy for liquidity.

Part of the literature defines liquidity as the sum of cash holdings and credit lines, as these financial contracts provide liquidity reserves that can be used when needed. The paper by Nikolov et al. [60] explores the relationship between cash holdings and credit line usage. They find that credit lines are used when cash is needed and external financing is costly, but have the downside that firms need collateral assets to have access to them.

In this paper I take a different perspective and study liquidity coming from the operational activities of the firms, focusing on liquidity inflows/outflows from the working capital. In particular, I use a measure of liquidity that corrects cash holdings for the net exposure on the supply chain, which I measure as the difference between short term trade credit and debit. The idea that firms use their trade position to manage their liquidity has been proposed and documented by several scholars, including Biais et al.[22], Wilner[86], Cuñat[35], Amberg et al.[11][12]. Nilsen [62] analyses the link between monetary policy, bank lending and trade credit. On the other side, trade credit financing is risky when the underlying trade activity slows down or is harmed, like during the pandemic. Bureau et al.[29] provides some insights on the impact of COVID-19 on this mechanism.

Following this literature, I propose a measure of liquidity that includes both cash holdings and the net trade position. The net trade position is the difference between commercial credit due by borrowers minus commercial debt due to suppliers, all due within the year.[7] The idea behind this is that firms manage their liquidity by adjusting their trade position. In particular, they can borrow liquidity from their supplier by postponing payments or by asking to receive inputs on credit and, similarly, firms can lend liquidity to their customer by giving them credit. Usually, these credits are short-term and granted to firms of the same production network or supply chains, based on commercial agreements and long standing relationships. Hence, trade credit/debt is a key and active part of liquidity management. The cost of this liquidity is a premium on the price of the goods/services.

[4] Together with EU.

[5] These policies are DL n. 23 dell'8 aprile 2020 (cd. "Decreto Liquidità"), Legge n. 178/2020 "("Legge di Bilancio 2021"), DL 73 del 25 maggio 2021 (cd. Decreto "Sostegni bis"), and Legge n. 234 del 30 dicembre 2021 (Legge di Bilancio) among others.

[6] For example, in [28], the optimal policy is to keep about 20% of assets liquid, with no firms having a low level of liquidity.

[7] In the Italian balance sheet firms report their commercial credit towards customers due within a year and debt towards supplies due within a year. The name of these entries are "Crediti verso clienti entro l'anno" and "Debiti verso fornitori entro l'anno".

To measure liquidity[8] I compute the net trade position and add it to the level of cash holdings:

$$m_t = \frac{Cash\ and\ C.\ Equiv._t + \overbrace{Trade\ credit_t - Trade\ debt_t}^{Net\ trade\ position_t}}{Tot.\ assets_t} \tag{1.1}$$

The following table and figure describe the differences between the standard measure of liquidity (ch_t) and the measure of liquidity including the trade position (m_t):

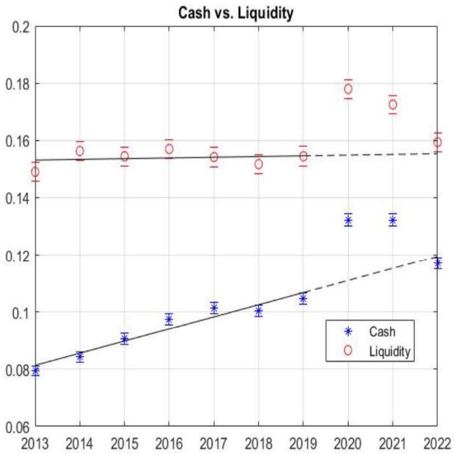

Figure 1.3: Average cash holdings with 95% confidence intervals.

Year	Mean		Stand. dev.		Median	
	ch_t	m_t	ch_t	m_t	ch_t	m_t
2013	7.95	14.91	11.75	21.73	2.93	13.01
2014	8.44	15.63	11.98	21.90	3.35	13.55
2015	9.06	15.45	12.28	21.67	3.99	13.55
2016	9.75	15.70	12.63	21.92	4.78	14.04
2017	10.12	15.42	12.92	22.13	5.13	13.75
2018	10.05	15.18	12.96	22.16	4.96	13.28
2019	10.46	15.45	13.38	22.07	5.60	13.57
2020	13.23	17.80	14.28	21.25	8.63	15.79
2021	13.02	17.26	14.29	20.80	8.46	14.85
2022	11.71	15.94	13.45	20.91	7.10	13.63

Table 1.2: All variables are in %.

Accounting for the short-term net trade position, the liquidity level is higher by about 5%. From Figure 1.3 it is possible to see that firms' liquidity is relatively stable over the business cycle.[9] Analyzing the patterns of ch_t and m_t, it is clear that during the business cycle firms trade off cash holdings with the trade position. Interest rates and the cost of credit decreased during the 2013-2019 period and it is possible to conclude that firms, on average, lowered their reliance on credit from suppliers increasing cash, which was easier to get from banks and markets. It is possible to see that in 2020 there was a parallel shift in both measures, but the net trade credit decreased in 2021 while cash holdings did not change. It is also worth noting that m_t allows us to observe firms that have negative liquidity, as the trade debt can be higher than cash and trade credit. With the other measure, ch_t, this is not possible because firms either have 0 or positive cash and cash equivalents. This is reflected in the fact that there is bunching at 0, as it is possible to see from Figure 1.1. Firms can have temporarily negative liquidity, for example when they need to make payments to suppliers that are worth more than their cash holdings. It is in those cases that trade credit/debt becomes an important tool for liquidity management and/or when credit lines are opened. The following figure contains the distribution of m_t:

[8] I would like to stress that this measure is an imperfect proxy for firms' liquidity but captures several dynamics missing in ch_t. With this measure, the cross-sectional distribution of cash holdings is comparable with the results in [28].

[9] The Kolmogorov Smirnov test can be found in the appendix.

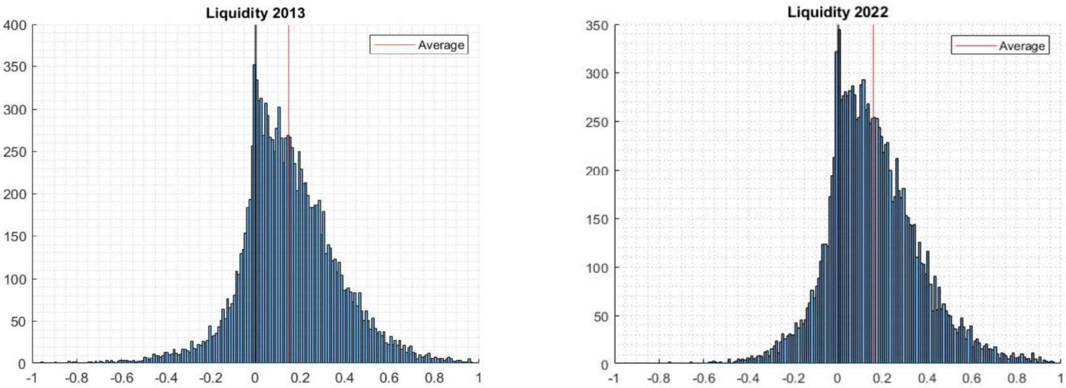

Figure 1.4: The figure contains the cross-sectional distribution of liquidity in 2013 and 2022. The X-axis represents the ratio m_t, while the Y-axis represents the number of firms in each bin. Each bin represents a percentage point.

Differently from cash holdings, which are exponentially distributed with the mass concentrated towards 0, m_t is tent-shaped, with the majority of the mass concentrated between 0 and the average level. Firms keep about 13-15% of their assets as liquid. In the next section, I analyze the pattern in liquidity, focusing on the relationship between cash and the trade position.

1.3.2 Liquidity decomposition (in progress)

In this section, I analyze the relationship between cash holdings, liquidity, and the patterns in their dynamics. From table 1.2 it is possible to see that from 2013 to 2019 firms were decreasing their net trade position and increasing their cash holdings. During the COVID-19 pandemic, liquidity increased, and this increase was driven by an increase in cash holdings. At the same time, net trade credit decreased. This was the effect of the government's interventions, that, despite the slowdown of economic activities, sustained corporations' liquidity. The dynamic is clear in the following figure:

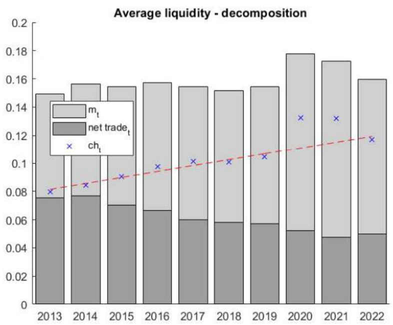

Figure 1.5: This figure contains the average level of liquidity m_t, of cash holdings ch_t and trade position *net* $trade_t$ as a fraction of total asset. The red line is the trend level computed excluding the COVID-19 years

The next question is to understand if the reduction in net trade credit is caused by a reduction in trade credit or an increase in trade debt:

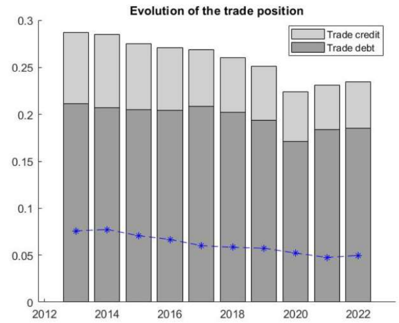

Figure 1.6: This figure contains the average level of trade credit and trade debt per year. The blue line is the net trade position.

From the figure 1.6 above, it is possible to see that the level of trade debt was stable between 2013 and 2019, with a sudden decrease during the pandemic. On the other side, the level of trade credit is decreasing over the same period, with a sudden decrease during the pandemic. In the appendix, Section 1.6.3, it is possible to see that these changes reflect shifts in the distribution of trade credit and debt across firms, hence it is possible to conclude that the change in the composition of liquidity during the 2013-2019 period was mostly due to a decline of trade credit. This decline, together with the expansionary monetary policy, led to a change in the composition of corporate liquidity in favour of cash holdings. Since the "demand" for trade debt seems to be relatively stable, in the future it would be worth disentangling the decrease in trade credit granted by S.P.A. due to lower demand (substitution effect, as documented in this paper) to the decrease due to other factors, for example integration to the global supply chain (trade credit granted by foreign corporations)[10].
In a preliminary attempt to corroborate the idea that firms use trade debit as a source of liquidity, I perform a panel analysis regression. In particular, I regress the level of cash holdings ch_t on trade credit and trade debt, controlling with fixed effects for sector, size and year. The results are listed in the table below:

[10] This channel can be relevant because access to alternative liquidity provider can alter bargaining power, and firms' resiliency. See for example Giannetti et al. [45]

	(A)	(B)	(C)	(D)	(E)
Constant	0.12	0.1234	0.1217	0.1347	0.01327
	(0.000)	*(0.000)*	*(0.000)*	*(0.000)*	*(0.000)*
Trade credit	0.03	0.019	0.0172	-0.0246	-0.0305
	(0.420)	*(0.555)*	*(0.492)*	*(0.247)*	*(0.150)*
Trade debit	0.10	0.098	0.11	0.0986	0.090
	(0.001)	*(0.001)*	*(0.000)*	*(0.000)*	*(0.000)*
Sector FE	No	No	Yes	Yes	Yes
Size FE	No	No	No	Yes	Yes
Year FE	No	Yes	Yes	Yes	Yes
2020 - 2021	Excluded	Excluded	Excluded	Excluded	Included
R^2 *(Within)*	10.7%	0.96%	1.00%	0.55%	0.45%

Table 1.3: Dependent variable is cash holdings *ch*.

The preliminary results suggest that firms with higher levels of cash holdings are also the firms that have more trade debt.

1.3.3 Industry level analysis (in progress)

In this section, I show differences across industries. Each industry is classified following the NACE2 definition. If a corporation operates in more than one industry, I only consider the main industry:

Sector	# firms
Agricultural	0
Mining/Quarrying	0
Manufacture	10,094
Natural res.	0
Constructions	2,226
Trade	5,041
Transports	1,493
Hospitality	569
IT & Communication	1,703
Finance	0
Prof. services	3,626
Other. Serv.	621
P.A.	0
Education	72
Health et co.	354
Art et co.	285
Others	130

Table 1.4: Number of firms per industry.

The majority of the corporations are in the manufacturing industry, then in trade and services. Different sectors have different business characteristics that influence operations management and liquidity management. For example, there is heterogeneity in the average liquidity holdings:

Sector	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022
Agricultural	-	-	-	-	-	-	-	-	-	-
Mining/Quarrying	-	-	-	-	-	-	-	-	-	-
Manufacture	14,9%	15,6%	15,8%	15,8%	15,2%	14,9%	15,0%	17,4%	15,5%	14,1%
Natural res.	-	-	-	-	-	-	-	-	-	-
Constructions	12,1%	13,2%	13,4%	14,3%	14,2%	13,8%	14,1%	16,1%	14,8%	13,5%
Trade	14,0%	14,5%	13,9%	14,1%	13,7%	13,6%	14,2%	17,5%	18,8%	16,5%
Transports	12,8%	13,8%	14,0%	14,0%	13,9%	14,6%	14,8%	15,9%	17,0%	17,9%
Hospitality	3,8%	4,6%	4,5%	5,9%	6,8%	5,9%	6,4%	6,5%	7,1%	8,5%
IT & Comm.	22,6%	23,2%	21,5%	21,9%	22,6%	22,6%	21,6%	23,4%	23,7%	22,6%
Finance	-	-	-	-	-	-	-	-	-	-
Prof. services	17,1%	20,5%	18,4%	18,9%	18,9%	18,8%	19,2%	21,1%	21,7%	21,1%
Other. Serv.	30,4%	30,0%	31,4%	32,9%	34,0%	32,6%	32,7%	34,7%	34,6%	34,5%
P.A.	-	-	-	-	-	-	-	-	-	-
Education	22,8%	21,0%	24,3%	28,7%	28,8%	25,9%	20,0%	16,5%	28,9%	21,9%
Health et co.	15,6%	13,5%	14,4%	14,8%	15,2%	14,3%	14,0%	13,8%	14,1%	12,5%
Art et co.	6,6%	7,5%	6,5%	7,6%	8,4%	9,1%	10,0%	11,6%	13,1%	14,4%
Others	14,3%	14,9%	14,2%	14,1%	14,3%	14,7%	13,6%	14,7%	14,1%	10,5%
Average	14,8%	15,6%	15,4%	15,7%	15,3%	15,1%	15,3%	17,6%	17,1%	15,7%

Table 1.5: Average liquidity by industry. Missing values for sectors not included in the analysis.

Lastly, the following figure contains the distribution of liquidity by sector in a given year. These distributions reflect the differences in the business model. For example, trade companies usually pay their supplier after they have realized their sales, financing themselves using trade credit, as it is possible to see from the 3rd panel of the figure 1.7.

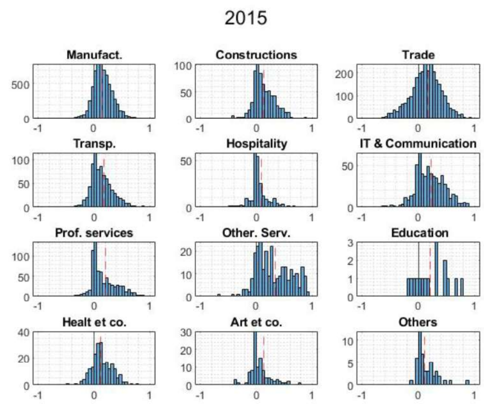

Figure 1.7: Distribution of liquidity across industries.

In this version of the paper I estimate the model on the overall distribution of m_t, but I am planning to estimate the model sector by sector and analyse the differences in terms of cost of liquidity management by industry.

1.4 A model for liquidity management

In this section of the paper, I propose a model for firms' liquidity management. The model belongs to the inventory models class [19] [80], and is a generalization of the one proposed by [57]. The solution of the model is the optimal policy for cash management that is associated with an invariant distribution. I estimate the model by matching the implied moments with the observed counterparts.

1.4.1 The problem of the firm

A representative firm minimizes the total expected cost of liquidity management. The total cost depends on two costs, the opportunity cost of keeping assets liquid and the fixed costs to be paid to adjust the liquidity level. The opportunity cost $R(m_t)$ is a flow cost and captures the forgone returns of investing liquid assets in longer-term activities. The firm can decide at any moment to raise liquidity, for example by borrowing or opening a credit line, but this is a costly operation, which is modelled as a fixed cost c_1 operation. Similarly, the firm can invest its liquidity in longer-term activities but needs to pay the fixed cost c_2, which captures the cost of finding the right investment opportunity. Sometimes, financing or investment opportunities arrive at the right moment, therefore with some arrival rate λ per unit of time, the adjustments will be free. I assume that the level of liquidity changes with the cash flow[11], which is modelled as a Brownian motion dW_t so that:

$$dm_t = \mu dt + \sigma dW_t \tag{1.2}$$

Given the cost structure and the assumption that liquidity evolves following (1.2), the problem of the firm minimizing the total expected cost of liquidity management is:

$$V(m_{t_0}) = \min_{\{\tau_j, m_t\}} \left\{ \mathbb{E}_{t_0} \left[\sum_{\tau_j} e^{-\rho \tau_j} c(\Delta m_{\tau_j}) \chi(\tau_j) + \int_{t_0}^{+\infty} e^{-\rho t} R(m_t) m_t dt \right] \right\} \tag{1.3}$$

where:

1. τ_j is a stopping time, i.e. the time when the firm decides to raise or invest its liquidity, and Δm_{τ_j} is the change in m;

2. χ_{τ_j} is a dummy variable that is 0 if the adjustment is free (with arrival rate λ);

3. $R(m)$ is the opportunity cost. When $m > 0$, R_2 is the opportunity cost of liquidity, while when $m < 0$, R_1 is the cost of borrowing through trade:

$$R(m) = \begin{cases} R_1 \ \textit{if } m < 0 \\ R_2 \ \textit{if } m \geq 0 \end{cases}$$

4. $c(\Delta m)$ is the fixed cost to be paid to adjust liquidity:

$$c = \begin{cases} c_1 \ \textit{if } \Delta m > 0 \\ c_2 \ \textit{if } \Delta m < 0 \end{cases}$$

[11] In this version of the model I focus on liquidity independently if it arrives (exits) as cash or trade credit (debt). I am currently writing a different specification of the model where I allow both cash and trade position, see the Section 1.4.5.

5. ρ is the discount rate.

From equation (1.3) I derive the associated Hamilton Jacobi Bellman equation:

$$(\rho + \lambda)V(m_t) = \lambda V(m^*) + R(m_t)m_t + \mu V'(m_t) + \frac{\sigma^2}{2}V''(m_t) \tag{1.4}$$

I solve (1.4) by guess and verify. To determine the constants I use standard boundary conditions (optimality of critical points, smooth pasting, and value matching). The system of boundary conditions is not linear, therefore I use a numerical solver. The solution to the problem consists of two critical levels of liquidity $\{m_1, m_2\}$ such that, if liquidity ends up below m_1 or above m_2, the fixed cost $c(\Delta m)$ is paid and the liquidity level is reset to the optimal one m^*. m^* is the level of liquidity that minimizes the problem in (1.3). Details about the solution procedure can be found in the Appendix, section 1.6.4.
What is the intuition behind the solution? For any amount of liquidity, the firm needs to decide whether to adjust or wait for the next realization of the cash flow, weighing the fixed cost to be paid with the expected opportunity cost. Suppose that the liquidity level is high, then the opportunity cost is high. If the firm thinks that the next period additional resources will flow in, then it might find it optimal to pay the fixed cost, looking for the right investment opportunity and decreasing the liquidity level to the optimal one. Doing so pays the fixed cost, but for the next periods, the opportunity cost will be lower. m_2 is exactly the level such that, in expectation, investing the excess liquidity $m_2 - m^*$ is equally costly as bearing the expected opportunity cost associated with $m_t = m_2$.[12] A similar logic applies to m_1. However, if a good investment or financing opportunity arrives for free, then the firm optimally adjusts without paying any fixed cost.

1.4.2 Distribution of liquidity

The model is associated with an invariant distribution of liquidity $\phi(m)$. In particular, given that $dm_t \sim BM(\mu, \sigma)$ and the optimal policy $\{m_1, m^*, m_2\}$, the invariant distribution associated with the solution solves the following Kolmogorov forward equation:

$$\begin{cases} \lambda f(x) = -\mu f'(x) + \frac{\sigma^2}{2} f''(x) & \forall x \in [m_1, m^*) \\ \lambda g(x) = -\mu g'(x) + \frac{\sigma^2}{2} g''(x) & \forall x \in (m^*, m_2] \end{cases} \tag{1.5}$$

The density is piece-wise defined because the Kolmogorov forward equation does not hold at m^*, which is a reinjection point. Solving equation (1.5), the invariant distribution is:

$$\phi(m) = \begin{cases} f(m) = Ee^{\gamma m} + Fe^{\delta m} & \forall m \in [\underline{m}, m^*) \\ g(m) = Ge^{\gamma m} + He^{\delta m} & \forall m \in (m^*, \bar{m}] \\ 0 & \forall m \notin (m^*, \bar{m}) \end{cases} \tag{1.6}$$

Where $\{E, F, G, H, \gamma, \delta\}$ are constants that depend on the parameters of the model. Details about the derivation of the density can be found in section 1.6.5 of the Appendix. The interpretation of the invariant distribution is that, for any initial t_0, as $T \to +\infty$, the density that the firm will have

[12] This is one of the boundary conditions, the value matching:

$$V(m_2) = V(m^*) + c(m_2)$$

liquidity level $m_T = k$ is $\phi(k)$. An alternative intuition is that if N firms populate the economy, and they follow the same management policy, then, as time goes on and $N \to +\infty$, $\phi(m)$ is the density of the liquidity distribution across firms. In the next section, I describe the empirical strategy to estimate the model.

1.4.3 Estimation

To estimate the model I follow a generalized method of moments (GMM) procedure. Year by year, I find the set of parameters that minimize the GMM criterion. The parameters to be estimated are $\{\rho, R_1, R_2, c_1, c_2, \lambda, \mu, \sigma\}$. I fix ρ to be 1% annual. As a proxy for R_2, the opportunity cost when the firm has liquid resources, I use the differential between the median return on investment (ROI) and the Italian short term interest rate[13]:

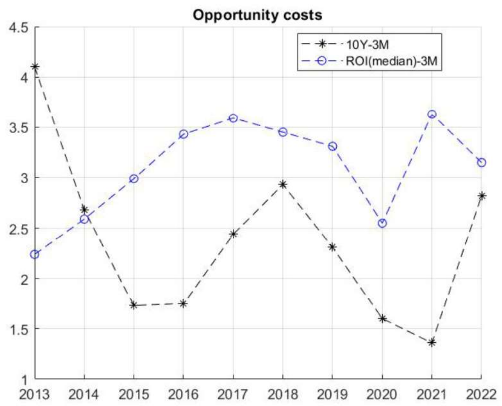

Figure 1.8: The following figure contains the opportunity cost measure (blue line) and the spread between the short-term and the long-term risk free bonds (black line). Yearly values in percentage.

It is worth noticing that in 2013 and 2014 the spreads between long-term and short-term government bond returns were higher than the median ROI. This is the case because Italy was in the aftermath of the EU sovereign debt crisis and, in 2013, there were some political turmoils, election included. ROI follows a cyclical behaviour, falling in 2020 and bouncing back in 2021. The estimation of R_1 is more complicated because this parameter captures the cost of having trade debt. From the literature, it is known that there are explicit costs, like the premium on the price of inputs, and implicit costs, for example in terms of bargaining power. I estimate R_1 by assuming that it is a fraction of the opportunity cost $R_2 \to R_1 = \iota R_2$. The parameters to be calibrated are:

$$\theta = \{c_1, c_2, \lambda, \mu, \sigma, \iota\}$$

Since the solution of the model is numerical, I define a grid $\Theta \subset R^+$ and perform the estimation over this grid of values. For each $\theta_i \in \Theta$, I compute the distance between the structural moments and their population equivalent. I use the distance to compute the score and search the global minimum of the criterion over the grid. The minimum is unique and the associated θ^* is the estimated set of parameters.

[13] I use the annualized 3 months interest rate reported by the OECD.

To construct the GMM criterion I use the non centered moments. The structural moment of order n is[14]:

$$\mathbb{E}\left[m^n;\theta\right] = \left[E\frac{e^{\gamma x}}{\gamma}\sum_{j=0}^{n}\left((-1)^j\frac{(m)^{n-j}}{\gamma^j}\frac{j!}{(n-j)!}\right)\right]_{m_1}^{m^*} + \left[F\frac{e^{\delta x}}{\delta}\sum_{j=0}^{n}\left((-1)^j\frac{(m)^{n-j}}{\delta^j}\frac{j!}{(n-j)!}\right)\right]_{m_1}^{m^*} + \\ + \left[G\frac{e^{\gamma x}}{\gamma}\sum_{j=0}^{n}\left((-1)^j\frac{(m)^{n-j}}{\gamma^j}\frac{j!}{(n-j)!}\right)\right]_{m*}^{m_2} + \left[H\frac{e^{\delta x}}{\delta}\sum_{j=0}^{n}\left((-1)^j\frac{(m)^{n-j}}{\delta^j}\frac{j!}{(n-j)!}\right)\right]_{m^*}^{m_2} \tag{1.7}$$

The population counterpart is:

$$M^n = \sum_j \left(\frac{m_j}{j}\right)^n \tag{1.8}$$

The GMM criterion is constructed using the percentage deviation of $\mathbb{E}\left[m^n|\theta\right]$ from M^n, giving equal weight to each moment:

$$J(\theta) = \left(\frac{\mathbb{E}\left[m^n;\theta\right] - M^n}{M^n}\right)' I \left(\frac{\mathbb{E}\left[m^n;\theta\right] - M^n}{M^n}\right) \tag{1.9}$$

θ^* is then defined as:

$$\theta^* \equiv \arg\min_{\theta\in\Theta} J(\theta) \tag{1.10}$$

I repeat this routine for each year. The set of estimated parameters is:

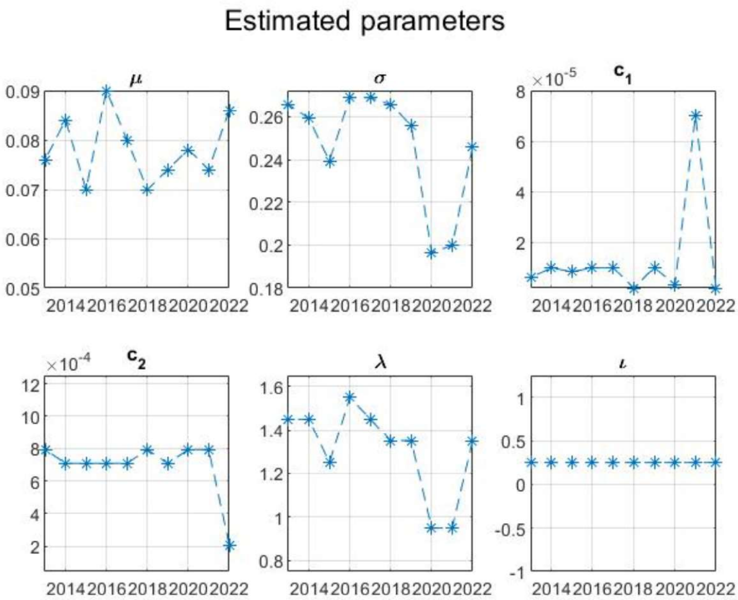

Figure 1.9: Caption

The following table contains details about the estimation performance:

[14]Without loss of generality, I omit θ in the definition.

Year	2013		2014		2015		2016		2017	
n	M^n	$\mathbb{E}[m^n;\theta^*]$	M^n	$\mathbb{E}[m^n;\theta^*]$	M^n	$\mathbb{E}[m^n;\theta^*]$	M^n	$\mathbb{E}[m^n;\theta^*]$	M^n	$\mathbb{E}[m^n;\theta^*]$
1	0,149	0,146	0,156	0,152	0,154	0,151	0,156	0,152	0,154	0,149
2	0,067	0,065	0,070	0,066	0,069	0,065	0,071	0,066	0,071	0,068
3	0,025	0,025	0,027	0,026	0,026	0,026	0,027	0,026	0,027	0,026
4	0,016	0,016	0,017	0,016	0,017	0,016	0,017	0,016	0,018	0,017
5	0,008	0,009	0,009	0,009	0,009	0,009	0,009	0,010	0,009	0,010
6	0,007	0,007	0,007	0,007	0,007	0,007	0,007	0,007	0,007	0,007
7	0,004	0,004	0,004	0,005	0,004	0,005	0,005	0,005	0,005	0,005

Year	2018		2019		2020		2021		2022	
n	M^n	$\mathbb{E}[m^n;\theta^*]$	M^n	$\mathbb{E}[m^n;\theta^*]$	M^n	$\mathbb{E}[m^n;\theta^*]$	M^n	$\mathbb{E}[m^n;\theta^*]$	M^n	$\mathbb{E}[m^n;\theta^*]$
1	0,151	0,145	0,153	0,149	0,176	0,175	0,170	0,170	0,157	0,158
2	0,070	0,067	0,070	0,067	0,075	0,070	0,071	0,069	0,067	0,067
3	0,026	0,026	0,027	0,026	0,031	0,031	0,030	0,029	0,028	0,027
4	0,018	0,017	0,018	0,017	0,019	0,018	0,018	0,017	0,017	0,017
5	0,009	0,010	0,009	0,010	0,011	0,011	0,010	0,010	0,010	0,010
6	0,007	0,007	0,007	0,007	0,008	0,008	0,007	0,007	0,007	0,007
7	0,005	0,005	0,005	0,005	0,005	0,006	0,005	0,005	0,005	0,005

The year-by-year estimation suggests that the costs of adjusting liquidity were stable before the pandemic. Following the pandemic the number of free adjustments decreased, meaning that the overall cost of liquidity management increased. At the same time, the cost of raising liquidity for a firm with high trade debt spiked in 2021.

Before the counterfactuals, it is important to highlight the role of each parameter on $\{m_1, m_2\}$.[15] Ceteris paribus, if the uncertainty of the liquidity inflow/outflow (σ) is higher, the effect on the optimal policy is to increase the inaction region. The logic is that firms expect to hit the critical thresholds more often and, therefore prefer to bear higher opportunity costs to reduce the number of times $c(m)$ is paid. A similar logic holds for the fixed costs. As it is more costly to adjust liquidity, the inaction region increases. Lastly, when the arrival rate of free adjustment opportunities λ increases, the inaction region increases because firms expect to receive more free adjustment opportunities so keep waiting longer before paying $c(m)$. It is possible to find the estimated cross-sectional distribution in the Appendix, Section 1.6.6.

1.4.4 Counterfactuals (in progress)

1) Shock to flows uncertainty (σ)
2) Changes in the cost structure & technology ($c(m)$, λ)
3) Transfer of money from steady state ($m^* \to m^* + \tau$)

1.4.5 Generalized version of the model (in progress)

In this section, I propose a generalized version of the model that accounts for cash holdings and trade credit/debt separately. In the baseline model, the firm chooses its liquidity ratio and decides when to adjust. In this section, I assume that in each period the firm receives a liquidity inflow/outflow. It needs to control its liquidity level by jointly managing cash holdings and the trade position. The inflows/outflows change both variables, and the firm faces different costs. Calling T_t the net trade position, the liquidity level is given by $m_t = ch_t + T_t$. As in the baseline model,

$$dm_t = \mu dt + \sigma dW_t$$

[15] Since these variables are not in close form, I give a qualitative description.

but this time dm_t realizes either as cash flow or as a trade credit/debt:

$$dch_t = \begin{cases} 0 & \textit{With prob. } \eta \\ \mu dt + \sigma dW_t & \textit{With prob. } 1-\eta \end{cases} \tag{1.11}$$

$$dT_t = \begin{cases} 0 & \textit{With prob. } 1-\eta \\ \mu dt + \sigma dW_t & \textit{With prob. } \eta \end{cases} \tag{1.12}$$

So that, in expectation:

$$\mathbb{E}[dch_t] = (1-\eta)\mu dt$$
$$\mathbb{E}[dT_t] = \eta\mu dt$$

The problem of the firm is to decide when to adjust either its trade position, its cash holdings, or both, and the composition of its liquidity upon adjustment:

$$\min_{\{ch_t, T_t, \tau_T, \tau_c, \tau_m\}} \mathbb{E}_t \left[\sum_{\tau_T} e^{\rho\tau_T} c_1 + \sum_{\tau_c} e^{\rho\tau_c} c_2 + \sum_{\tau_m} e^{\rho\tau_m} c_3 + \int_t^{+\infty} e^{\rho t}(R_1 ch_t + R(T_t)T_t)dt \right] \tag{1.13}$$

Where:

1. $\{\tau_T, \tau_c, \tau_m\}$ are the time when the firm decides to adjust the trade position/cash holdings/both respectively;
2. $\{c_1, c_2, c_3\}$ are the fixed costs to be paid to adjust the trade position/cash holdings/both respectively;
3. R_2 is the opportunity cost of cash holdings;
4. $R(T)$ is the opportunity cost of the trade position:

$$R(T) = \begin{cases} R_1 & T < 0 \\ R_2 & T \geq 0 \end{cases}$$

Differently from the baseline model, cash holdings are bounded at 0, which is a reflecting barrier for ch_t. Following the same logic used to derive (1.4) it is possible to compute the HJB associated with (1.13):

$$V(ch_t, T_t) = R_2 dt ch_t + R_i(T_t)dtT_t + \frac{1}{1+rdt}\mathbb{E}_t\left[V(ch_{t+d_t}, T_{t+dt})\right] \tag{1.14}$$

From the discrete time approximation:

$$\begin{aligned} \mathbb{E}_t\left[V(ch_{t+d_t}, T_{t+dt})\right] &= V(ch_t, T_t) + \mathbb{E}_t\Big[V'_{ch}(ch_t, T_t)dch_t + V'_T(ch_t, T_t)dT_t + \frac{1}{2}V''_{ch}(ch_t, T_t)dch_t^2 \\ &+ \frac{1}{2}V''_T(ch_t, T_t)d_T^2 + \frac{1}{2}V''_{ch,T}(ch_t, T_t)dch * dT\Big] = \\ &= V(ch_t, T_t) + V'_{ch}(ch_t, T_t)(1-\eta)\mu dt + V'_T(ch_t, T_t)\eta\mu dt + \\ &+ \frac{1}{2}V''_{ch}(ch_t, T_t)(1-\eta)\sigma^2 dt + \frac{1}{2}V''_T(ch_t, T_t)\eta\sigma^2 dt \frac{1}{2}V''_{ch,T}(ch_t, T_t)(1-\eta)\eta\sigma^2 dt \end{aligned} \tag{1.15}$$

Taking limit $dt \to 0^+$ and simplifying terms:

$$\begin{aligned} \rho V(ch_t, T_t) &= R_2 dt ch_t + R_i(T_t)dtT_t + V'_{ch}(ch_t, T_t)(1-\eta)\mu + V'_T(ch_t, T_t)\eta\mu + \\ &+ \frac{1}{2}\left[V''_{ch}(ch_t, T_t)(1-\eta)\sigma^2 + V''_{T_t}(ch_t, T_t)\eta\sigma^2\right] \end{aligned} \tag{1.16}$$

where $V''_{ch,T}(ch_t, T_t)$ has been omitted because the cross derivative is 0 by the structure of the problem. At the moment I am solving this ODE. The plan for this section is to compute the optimal policy and the associated invariant distribution. I will have two distributions, one for ch_t and one for T_t. I will use an estimation strategy similar to the one used in the baseline model but with some differences. The most important is that I am trying to gain access to a dataset containing details about corporations' invoices. This dataset includes the terms of trades, settlement times, prices and quantities. With this dataset I would be able to better identify the Brownian motion parameters (this is true also for the baseline model), but also the cost of trade debt (as a premium on inputs' prices) and the settlement terms (η). Instead of targeting the moments of the overall distribution of liquidity, I will be able to match the distribution of cash holdings and the distribution of the trade position, and this will allow me to better quantify the impact of cash holding shocks or trade credit changes on the overall problem of liquidity management.

1.5 Conclusions

In this paper I study the liquidity composition of Italian corporations and find that firms keep their liquidity level constant, changing the composition of the assets. I document that firms have on average 15% of their assets liquid, and, during the 2013-2019 period, firms increased their cash holdings and reduced trade credit allowances. The level of trade debt is stable over the period, while trade credit is declining. When the pandemic shock hit the economy, the trade position lost relevance, while cash holdings substantially increased, as the government pursued policies to help firms survive the pandemic.

To disentangle the drivers of liquidity management decisions, and to rationalize the cross-sectional distribution of liquidity, I propose a model of liquidity management based on [57]. I estimate the model matches the structural moments to the data and find. Preliminary results suggest that costs are stable over the business cycle, while the inflows/outflows process fluctuates.

1.6 Appendix

1.6.1 Geographical dispersion

The following table 1.6 and figure 1.10 highlight the geographical distribution of the observations.

Figure 1.10: Map of the distribution of firms across Italian regions.

Region	Firms
N.a.	5186
Abruzzo	165
Basilicata	51
Calabria	68
Campania	619
Emilia Romagna	1757
Friuli Venezia Giulia	361
Lazio	1049
Liguria	334
Lombardia	4993
Marche	362
Molise	29
Piemonte	1257
Puglia	244
Sardegna	167
Sicilia	275
Toscana	954
Trentino Alto Adige	323
Umbria	180
Valle d'Aosta	26
Veneto	1850

Table 1.6: Table containing the geographical distribution of surviving observations.

1.6.2 Kolmogorov Smirnov test

The following table contains the results of the Kolmogorov Smirnov test. For each possible couple of years, I test the hypothesis that the cross sectional distributions are drawn from two different distributions:

	2022	2021	2020	2019	2018	2017	2016	2015	2014	2013
2022	False	True	True	False	True	False	False	False	False	True
2021	True	False	True	True	True	True	True	True	True	True
2020	True	True	False	True	True	True	True	True	True	True
2019	False	True	True	False	False	False	False	False	False	False
2018	True	True	True	False	False	False	True	False	False	False
2017	False	True	True	False	False	False	False	False	False	True
2016	False	True	True	False	True	False	False	False	False	True
2015	False	True	True	False	False	False	False	False	False	True
2014	False	True	True	False	False	False	False	False	False	True
2013	True	True	True	False	False	True	True	True	True	False

It is possible to see that the test cannot reject the hypothesis that the distributions in 2014-2019 are different.

1.6.3 Trade credit/debit distribution

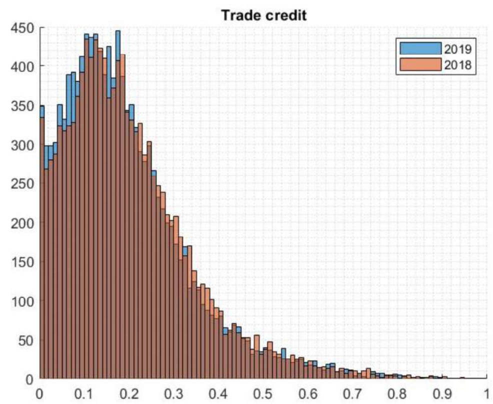

Figure 1.11: Trade credit distribution in 2018 and 2019. It is possible to see that there is a shift to the left in 2019, meaning that the allowance of trade credit was reduced.

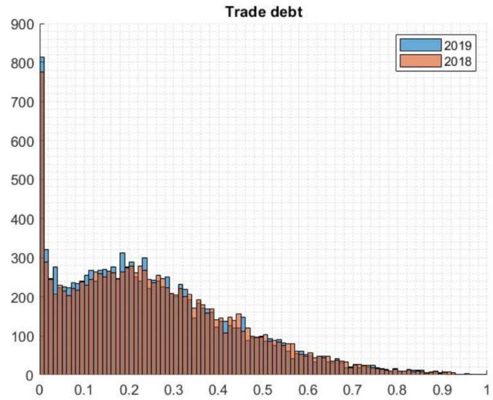

Figure 1.12: Trade debt distribution in 2018 and 2019. Differently from trade credit, there is no clear shift to the left.

1.6.4 Derivation of the HJB

In this section, I derive the HJB equation from (1.3) The optimal strategy for the firm policy is to adjust only when m_t ends up being too low or too high. Upon adjustment, the firm chooses to set $m_{\tau_j} = m^*$ set such that the expected future costs are minimized. In particular, let us assume that the crucial threshold below which the firm resets its liquidity is m_1, and similarly, m_2 is the crucial threshold value above which the firm prefers to reset. The optimal policy then can be summarized as waiting while $m_t \in [m_1, m_2]$ and paying $c(m)$ when $m_t \notin [m_1, m2]$ to reset to $m^* \in [m_1, m2]$. $\{m*, m1, m2\}$ are unknowns to be determined. To solve for the optima policy let us start with a discrete time formulation of the problem. Inside the inaction region (m_1, m_2), the firm does not control the state unless it can do it for free, with arrival rate λ, and wait for the next period only paying $R(m_t)m_t$. Since the flow cost, $R(m)$, depends on m_t, the value function is piece-wise defined. The following procedure is general for both branches, with the caveat that:

$$V(m_t) = Rm_t + E_t[V(m_{t+1})] \tag{1.17}$$

where m_{t+1} is either $m_t + \Delta m_t$ with probability (1-λ) or m^* with probability λ per unit of time, because the firms adjusts. This implies that:

$$V(m_t) = \Delta Rm_t + \frac{1}{1+\Delta r} E_t[\Delta\lambda (V(m^*) - V(m_t)) + (1-\Delta\lambda)V(m_t + \Delta m_t)] \tag{1.18}$$

Taking limit $\Delta \to 0 \Rightarrow \Delta m_t \to dm_t$, and using Itō's lemma to approximate $E_t[V(m_t + dm_t)]$:

$$V(m_t) = dtRm_t + \frac{1}{1+dtr}\left[dt\lambda (V(m^*) - V(m_t)) + (1-dt\lambda)\left[\mu V'(m_t)dt + \frac{1}{2}\sigma^2 dt V''(m_t)\right]\right] \tag{1.19}$$

Rearranging terms and dividing by dt:

$$(r+\lambda)V(m_t) = \lambda V(m^*) + Rm_t + \mu V'(m_t) + \frac{1}{2}\sigma^2 V''(m_t)$$

Which is the Hamilton Jacobi Bellman equation associated with the problem. The solution of the HJB equation will be the value function of the problem. Solving this problem also requires finding the optimal boundaries $\{m_1, m_2\}$ and the optimal resetting point m^*. To solve this differential equation, I use the guess and verify method. In particular, for each branch, I guess that the solution takes the following functional form:

$$V(m) = v + \frac{R}{r+p}m + A_1 e^{a_1 m} + A_2 e^{a_2 m} \tag{1.20}$$

this implies that the first derivative is:

$$V'(m) = \frac{R}{r+p} + A_1 a_1 e^{a_1 m} + A_2 a_2 e^{a_2 m} \tag{1.21}$$

and that the second derivative is:

$$V''(m) = A_1 a_1^2 e^{a_1 m} + A_2 a_2^2 e^{a_2 m} \tag{1.22}$$

substituting (1.20) (1.21) and (1.22) in (1.4) it is possible to find the following equation:

$$(r+\lambda)v+(r+\lambda)(A_1e^{a_1m}+A_2e^{a_2m}) = \lambda V(m^*)+\frac{R}{r+\lambda}+\mu(A_1a_1e^{a_1m}+A_2a_2e^{a_2m})+\frac{\sigma^2}{2}(A_1a_1^2e^{a_1m}+A_2a_2^2e^{a_2m}) \tag{1.23}$$

Notice that both hand sides have a linear part and an exponential part, thus, matching parts:

$$\begin{cases} (r+\lambda)v = \lambda V(m^*) + \frac{\mu R}{r+\lambda} \\ (r+\lambda)(A_1e^{a_1m} + A_2e^{a_2m}) = \mu(A_1a_1e^{a_1m} + A_2a_2e^{a_2m}) + \frac{\sigma^2}{2}(A_1a_1^2e^{a_1m} + A_2a_2^2e^{a_2m}) \end{cases} \tag{1.24}$$

Solving the system it is possible to derive the coefficients a_1 and a_2:

$$a_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 + 2\sigma^2(r+\lambda)}}{\sigma^2}$$

which have opposite signs because $\mu^2 \leq \mu^2 + 2\sigma^2(r+\lambda)$ and σ, r and λ are positive. The constant is equal to:

$$v = \frac{\lambda}{r+\lambda}V(m*) + \frac{\mu R}{(r+\lambda)^2}$$

To solve the HJB there are still 7 unknowns $\{m^*, m_1, m_2\}$ and 2 constants $\{A_1, A_2\}$ for each branches. For simplicity, let us rename A_1, A_2 for the second positive branch as $\{B_1, B_2\}$. To determine the unknowns, let us consider the boundary conditions implied in the problem:

1. Optimality at m^* since it is a minimum point;
2. Value matching at the lower boundary because when $m \leq m_1$ the agents resets to m^*;
3. Value matching at the lower boundary because when $m \geq m_2$ the agents resets to m^*;
4. Smooth pasting at the upper boundary m_1.
5. Smooth pasting at the lower boundary m_2.
6. Continuity at 0.
7. Differentiability at 0.

which are:

1. $$V'(m^*) = 0 \iff \frac{R}{r+\lambda} + A_1a_1e^{a_1m^*} + A_2a_2e^{a_2m^*} = 0$$

2. $$V(m_1) = V(m^*) + c_1 \iff \frac{R_1}{r+\lambda}m_1 + A_1e^{a_1m_2} + A_2e^{a_2m_2} = \frac{R_1}{r+\lambda}m^* + B_1e^{a_1m^*} + B_2e^{a_2m^*} + c_1$$

3. $$V(m_2) = V(m^*) + c_2 \iff \frac{R_2}{r+\lambda}m_2 + B_1e^{a_1m_2} + B_2e^{a_2m_2} = \frac{R_2}{r+\lambda}m^* + B_1e^{a_1m^*} + B_2e^{a_2m^*} + c_2$$

4. $$V'(m_1) = 0 \iff \frac{R_1}{r+\lambda} + A_1a_1e^{a_1m_2} + A_2a_2e^{a_2m_2} = 0$$

5.
$$V'(m_2) = 0 \iff \frac{R_2}{r+\lambda} + B_1 a_1 e^{a_1 m_2} + B_2 a_2 e^{a_2 m_2} = 0$$

6.
$$\lim_{m\uparrow 0} V(m) = \lim_{m\downarrow 0} V(m) \iff v_1 + A_1 + A_2 = v_2 + B_1 + B_2$$

7.
$$\lim_{m\uparrow 0} V'(m) = \lim_{m\downarrow 0} V'(m) \iff \frac{R_1}{r+\lambda} + A_1 a_1 + A_2 a_2 = \frac{R_2}{r+\lambda} + B_1 a_1 + B_2 a_2$$

This is a non-linear system of 7 equations with 7 unknowns that can be solved numerically. These boundary conditions are derived assuming that $m^* > 0$, however, in the code I generalize them such that $m^* < 0$ is possible. The estimated model will always be such that $m_t > 0$.

1.6.5 KFE derivation and solution

The second part of the model consists in estimating an implied cross sectional distribution. This is possible because of the hypothesis that cash balances evolve following a Brownian motion. This is convenient because, at the steady state, it implies a stationary distribution that can be compared with the empirical cross-sectional distribution. It is possible to derive this distribution by solving the Kolmogorov forward equation (KFE) associated to the firm's problem:

$$\begin{cases} 0 = -\mu f'(x) + \frac{\sigma^2}{2} f''(x) - \lambda f(x) & \forall x \in [m_1, m^*) \\ 0 = -\mu g'(x) + \frac{\sigma^2}{2} g'(x) - \lambda g(x) & \forall x \in (m^*, m_2]) \end{cases} \tag{1.25}$$

where $f(x)$ is the density if $m_t \in [m_1, m^*)$ and $g(x)$ is the density if $m_t \in (m^*, m_2]$. The main intuition behind these equations is that in steady state each instant the probability of m arriving in an infinitesimal interval is equal to the probability of leaving that interval. It is immediate to see that the solutions to the ODEs are exponential functions:

$$f(x) = Ee^{\gamma x} + Fe^{\delta x} \tag{1.26}$$

$$g(x) = Ge^{\gamma x} + He^{\delta x} \tag{1.27}$$

Using the same strategy used to determine $a_{1,2}$, it is possible to compute $\{\gamma, \delta\}$:

$$\{\gamma, \delta\} = \frac{\mu \pm \sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2}$$

Given $\{\gamma, \delta\}$, the constants to be determined are $\{E, F, G, H\}$. I use the following 4 boundary conditions associated with the problem:

1. No mass at the lower barrier;
2. No mass at the upper barrier;
3. Continuity at m^*;
4. Integrability to 1 since the solution is a density.

The boundary conditions translate into the following system:

$$\begin{cases} f(m_1) = 0 \iff Ee^{\gamma m_1} + Fe^{\delta m_1} = 0 \\ g(m_2) = 0 \iff Ge^{\gamma m_2} + He^{\delta m_2} = 0 \\ f(m^*) = g(m^*) \iff Ee^{\gamma m^*} + Fe^{\delta m^*} = Ge^{\gamma m^*} + He^{\delta m^*} \\ \int_{m_1}^{m^*} f(x)dx + \int_{m^*}^{m_2} g(x)dx = 1 \iff \left[\frac{E}{\gamma}e^{\gamma x} + \frac{F}{\delta}e^{\delta x}\right]_{m_1}^{m^*} + \left[\frac{G}{\gamma}e^{\gamma x} + \frac{H}{\delta}e^{\delta x}\right]_{m^*}^{m_2} = 1 \end{cases} \tag{1.28}$$

1.6.6 Estimated distributions

The fitted distributions are:

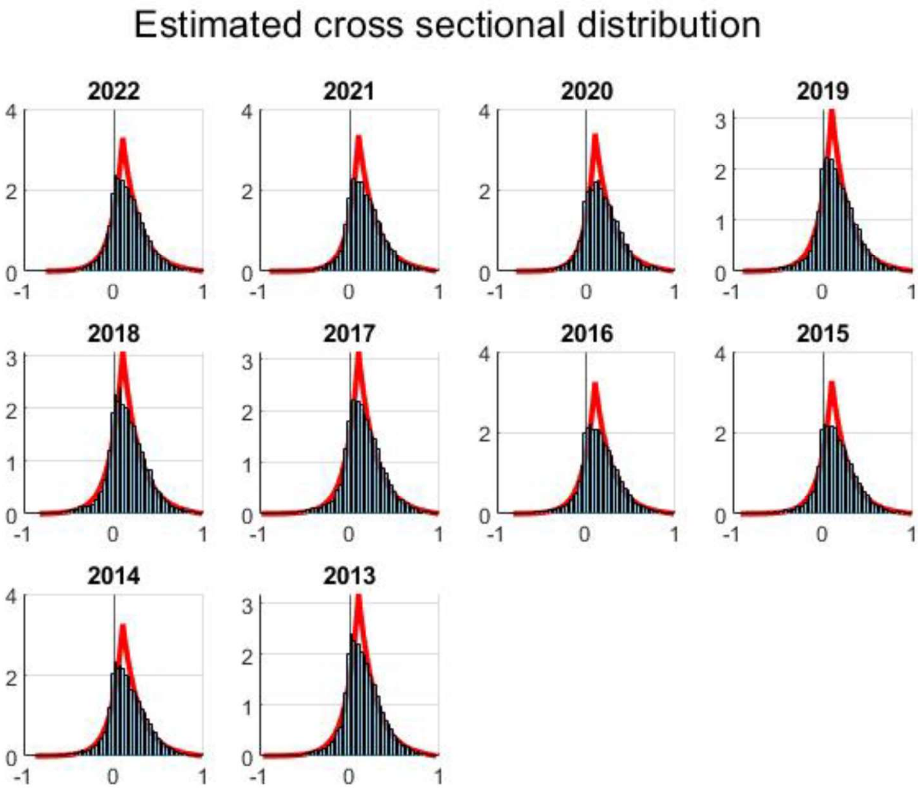

Figure 1.13: Fitted distributions. In red the model implied distribution, and in blue the histogram of the data, normalized to be a density.

Chapter 2

Algorithmic Pricing and Bank Lending

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2.1 Abstract

We study the impact of artificial intelligence in banking markets. We estimate a structural model of demand and supply for the credit lines market, characterized by information asymmetries and imperfect competition, since we found evidence of adverse selection and moral hazard in the literature. We then simulate the impact of the adoption of AI pricing algorithms at the bank level on market outcomes in a counterfactual exercise. In traditional concentrated markets, these algorithms learn to sustain collusive prices. However, charging high markups might not be optimal in selection markets. We find that AI generally increases prices also in these markets, but the relative increase strongly depends on the level of asymmetries. Additionally, we experiment with the robustness of the Q-learning algorithm as our application poses challenges to its learning process.

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[4]Luiss Guido Carli and EIEF. We would like to thank Emilio Calvano, Giacomo Calzolari, Cristina Gualdani, Luigi Pascali, and Nicola Pavanini.

2.2 Introduction

AI is considered a general purpose technology, as it is changing our world. Understanding its impact on the economy is a key challenge to address both the issues and the challenges that the adoption of this technology will have on every economic agent. The analysis of the economic impact of AI has already started, for example, in the famous paper by Calvano et al.[41]{CCDP}, which proves that under certain conditions AI can sustain collusive prices. In this project, we study the impact of AI adoption in financial markets. We focus on the banking market for credit lines, and we simulate AI pricing tools adoption within this market. We think that this is a relevant analysis because banking markets are selection markets, meaning that they are characterized by adverse selection and moral hazard. We study how these failures interact with the adoption of AI, which might generate another failure. The stability of the banking system is crucial for the economy, therefore understanding the impact of this technology on risk taking and credit rationing can help policymakers to regulate its adoption efficiently.

To simulate the adoption of AI pricing algorithms we first estimate a state-of-the-art model of demand and supply for credit lines, then we perform a counterfactual analysis where we assume that banks adopt AI to price their contracts. In particular, we follow the model proposed by Crawford et al.[34]{CPS.} that explicitly accounts for adverse selection and moral hazard in the equilibrium determination. After the estimation of the model, we run a counterfactual analysis where we introduce AI pricing tools that solve the pricing problem of the adopting bank. AI learns to play the pricing game by strategically interacting with other banks, which might have adopted AI as well. Within the structure of the model we can control what drives the level of asymmetries, and, in general, the fundamentals of the model, and relate the equilibrium with AI adoption to other equilibria. This exercise allows us to disentangle the interaction between AI adoption, moral hazard and adverse selection. We begin with a stylized exercise where the environment is closer to the one studied in [41], and find that when two banks adopt AI, they generally sustain over-competitive prices. We then run a comparative analysis with different degrees of adverse selection and find that AI's optimal policy is to decrease the relative cost of credit when asymmetries are higher, not rationing the marginal borrower. However, since our model includes different markets, there is heterogeneity in the results. Differently from the other papers in the literature, in our model banks can have losses when borrowers default. This poses a learning challenge to the AI algorithms because they approximate the optimal policy by gradual learning, and this learning might be harmed if defaults occur frequently. We experiment on the learning process by introducing state-dependent lotteries. We find that the algorithm needs sensibly more time to converge [...]

2.2.1 Literature review

Our work is related to two main strands of the literature, the literature on industrial organization of financial markets and the literature on AI and economics.

In the finance literature, the presence and the impact of asymmetries of information á la Akerlof[2], Rothschild and Stiglitz[49], and Stiglitz and Weiss[49] have been widely studied. This market failures influence access to credit and credit allocation, and can harm financial stability. Recently, the focus has shifted toward the interaction between asymmetries of information and other market failures. CPS analysed the interaction between adverse selection and market power and found that banks with higher market power may give up part of their mark-ups to mitigate the consequences of adverse selection. The idea that market structure is a crucial element in the determination of the market equilibrium dates back to Petersen and Rajan [66]. We contribute to this literature by analysing the interaction between market concentration, asymmetric information, and AI adoption. While AI represents a technological change and not a market failure, it has been pointed out by [41] that its adoption in concentrated markets can lead to tacit collusion, increasing mark-ups. In this paper, we include information asymmetries to understand whether the adoption of AI can worsen market allocation in selection markets. In traditional markets, AI increases markups, while in selection markets this might not be always optimal for the banks, as the analysis in CPS suggests.

2.3 Structural model

To disentangle the interaction between asymmetric information, market power and AI adoption we need a model where all these features are explicitly included. In particular, we need a demand and supply model that allows borrowers to have private information, banks to have market power, and us to introduce AI in the counterfactual analysis. For these reasons we follow the model and the methodology proposed by [34]. Their model combines the methodology to estimate demand in the spirit of Berry et al.[21] with the model proposed by Einav et al.[40]. Following CPS, we focus on first-time borrowers, and we only analyse the market for credit lines, which are not collateralized. Firms would like to borrow a certain quantity from the banks in their market, and the banks propose their price. Competition is assumed to be á la Nash-Bertrand. The borrower faces three choices. First, she chooses from which bank to borrow[1], then, conditional on borrowing from a given bank, how much to borrow and whether to default or not. The decisions will depend on a private component $\boldsymbol{\varepsilon}$ that is not observable by the bank and is the source of asymmetric information. Formally, the problem of the firm i is to choose the lender, how much to borrow, and whether to default maximizing the following utilities:

- Utility of borrowing from bank j (B):

$$U^B_{ijmt} = \bar{\alpha}^B_0 + X'^B_{jmt}\beta^B + \xi^B_{jmt} + \alpha^B P_{ijmt} + Y^B_{ijmt}\eta^B + \varepsilon^B_i + \nu_{ijtm} \tag{2.1}$$

- Utility of using the credit line (U):

$$U^U_{ijmt} = \bar{\alpha}^U_0 + X'^U_{jmt}\beta^U + \alpha^U P_{ijmt} + Y'^U_{ijmt}\eta^U + \varepsilon^U_i \tag{2.2}$$

- Utility from default (D):

$$U^D_{ijmt} = \bar{\alpha}^D_0 + X'^D_{jmt}\beta^D + \alpha^D P_{ijmt} + Y'^D_{ijmt}\eta^D + \varepsilon^D_i \tag{2.3}$$

where j is the bank, m stands for market, and t for year.[2] Each choice depends on observables at the firm level Y_{ijmt} and at the bank level X_{jmt}, the price P_{ijmt} and the unobserved components ε and ξ^B_{ijmt}. The latter represents the bank's characteristics known in the market but not observable in the data, while the former is the firm's private information. ν_{ijmt} is an unobservable shock to demand. Competition on the market is only on prices, while the granted amount is fixed and equal to the one we observe. Conditional on borrowing from a certain bank j, the firm decides the amount of the granted credit loan to use. The source of asymmetry in the loan use choice is ε^U_i. After deciding the lender and how much to borrow, the firm is left with the decision to repay or not the debt. The decision will be to default if the utility from defaulting is positive $U^D_{ijmt} > 0$.
Notice how the three frictions of interest are included in the model. First, banks have market power because they are differentiated and borrowers value their characteristics (X^B_{jmt}, ξ^B_{jmt}) in deciding

[1]The firm can also decide not to borrow as the outside option.

[2]Bank $j = 0$ represent the outside option. Firm i does not borrow if $U_{i0mt} > U_{ijmt}|j \neq 0$.

their lender. Second, asymmetries of information, in particular $\boldsymbol{\varepsilon} = \{\varepsilon_i^B, \varepsilon_i^U, \varepsilon_i^D\}$, lead to adverse selection. The underlying assumption is that each component is 0 mean, as they are unobservable, but can be correlated. Suppose the correlation between the unobserved component in demand (2.1) and in default (2.3) is positive. In that case, there is adverse selection because the firms that are more likely to borrow are also the ones that are more likely to default.[3] In particular, we assume that: $\boldsymbol{\varepsilon} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$:

$$\begin{pmatrix} \varepsilon_i^B \\ \varepsilon_i^U \\ \varepsilon_i^D \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_B^2 & \rho_{BU}\sigma_B\sigma_U & \rho_{BD}\sigma_B \\ \rho_{BU}\sigma_B\sigma_U & \sigma_U^2 & \rho_{UD}\sigma_U \\ \rho_{BD}\sigma_B & \rho_{UD}\sigma_U & 1 \end{pmatrix}\right)$$

where the adverse selection parameter is ρ_{BD}. $\boldsymbol{\Sigma}$ is jointly estimated with the demand and supply parameters. The last friction is moral hazard, captured by α^D, the sensitivity of the default choice to the price.
On the supply side, banks are differentiated and compete a la Nash-Bertand by choosing the price of the credit line. Their problem is to maximize the expected profits:

$$\Pi_{ijmt} = P_{ijmt}Q_{ijmt}\left(1 - F_{ijmt}(P_{ijmt})\right) - MC_{ijmt}Q_{ijmt} \tag{2.4}$$

where Q_{ijmt} is the expected quantity used, P_{ijmt} is the price charged, MC_{ijmt} is the marginal cost and F_{ijmt} is the probability of default. When choosing the price, each bank knows that the probability of default will be endogenous, but the sources of this endogeneity are two:

1. (Direct) The price charged by the bank influences the default decision (α^D);
2. (Indirect) The price charged by the bank influences the pool of borrowers (ρ_{BD}).

Both channels are taken into account by the banks, which will price the contract accordingly:

$$P_{ijmt} = \frac{MC_{ijmt}}{1 - F_{ijmt} + F'_{ijmt}M_{ijmt}} + \frac{(1 - F_{ijmt})M_{ijmt}}{1 - F_{ijmt} + F'_{ijmt}M_{ijmt}} \tag{2.5}$$

where F'_{ijmt} is the derivative of the default probability with respect to the price, and M_{ijmt} is the markup $-Q_{ijmt}/Q'_{ijmt}$. Equation (2.5) allows us to compute the implied marginal costs since we have P_{ijmt} and we can compute numerically F'_{ijmt} and Q'_{ijmt}.
In the next sections, we discuss in detail the data we use, the variables we include, and the estimation strategy.

2.3.1 Data

To estimate the model we use detailed microdata provided by the Bank of Italy. We use the credit registry data from 2010 to 2018. This dataset contains information about every contractual relationship between firms and banks, conditional on the total exposure being more than 75.000€. For the credit lines, the product we focus on, we observe the amount granted, the amount used by the firm, the stream of payments associated to the line, and delinquecies.[4] We have detailed balance sheet data for firms and banks. For banks, we have financial aggregates at the province level that we use as controls and to construct instruments for the estimation. At the province level, we have the number of deposits, the number of clients, the number of NPLs, information on the costs of deposits

[3] This is the definition of adverse selection following from [73]
[4] Following [49], we impute default to a credit line if there are delinquencies within 3 years.

and banks' demographics. We define a market as a province (m) year (t). A market is active if we observe at least one new borrower using a credit line. The banks active in each market are those who grant new credit lines.

We face a challenge in identifying which firms are non-borrowers in the market for the first credit line. Each year, most of the young firms in a market are non borrowers, but we need to distinguish between non-borrowers who are "active" in the market and non-borrowers who are not interested/capable of borrowing. To screen "active" non-borrowers from others, we impose that they must have more than 1 employee and be younger than 3 years[5]. Furthermore, we exclude from the sample "S.R.L.s.", a particular type of limited liability company that follows special accounting laws. However, after these cleanings, we are left with a non-borrower share ≥95% in many markets. We decided to do random sampling so that the non-borrowers share is at most 66%. We are working on strategies to improve the identification of the "active" non-borrowers. At the moment we are exploiting the fact that some firms are borrowing from banks but are below the 75.000€ threshold.[6] Calling these firms "non-registry" borrowers, our idea is to match the firms that report 0 borrowings with both "non-registry" and standard borrowers and exclude the ones that are relatively more similar to the "non-registry". The rationale is that these firms are not in the market of interest because their characteristics suggest that, if they borrow, they will be granted an amount below the credit registry threshold. Since we are still working to understand the effectiveness of this procedure, in the paper we show results that are obtained with random sampling.

A major caveat is that to estimate the model, we need to observe the prices of every possible bank-firm combination in each market and each year. However, this is not possible because some firm-bank relationships do not exist. Following the literature, we overcome this caveat by constructing prices for bank-firm relationships that we do not observe in the data. First, we use a propensity score matching procedure to match similar borrowers, we assign them with a firm fixed effect[7] and an amount granted, then we use the firm fixed effect to estimate a pricing equation:

$$P_{ijmt} = \gamma_0 + \gamma_1 \mathcal{D}_{ijmt} + \gamma_2 \mathcal{L}_{ijmt} + \lambda_{jmt} + \omega_i + \tau_{ijmt} \tag{2.6}$$

where $\mathcal{D}_{ijmt}$ is the distance between the firm and the bank, $\mathcal{L}_{ijmt}$ are dummies for the size of the credit line, λ_{jmt} is the market year fixed effect, and ω_i is the firm fixed effect. Using this pricing equation, we have all the bank-firm prices needed to estimate the model.

To estimate the model we include several variables at the firm and bank level that influence the borrowing process. At the firm level (Y_{ijmt}), we include total assets, profits, cash flow, trade debt, and the share of intangible assets. We also include the distance between banks and firms.[8] As banks' characteristics (X_{jmt}), we include the number of branches in the market, the share of branches in the market, and the number of years the bank has been active within that market. The idea is that each of these observables can directly influence the borrowing/lending decision. Finally, we include the firm, score[9], and bank/market/year fixed effects.

[5] An age comparable with the median borrower.

[6] We identify these firms because they are not in the credit registry but have positive bank debit in the balance sheet.

[7] We exploit an institutional feature of the Italian economy, where firms usually have multiple lending relationships from their first years.

[8] We use geographical coordinates to measure the distance between the municipality of the closest bank's branch and the municipality of the firm.

[9] Score is a discrete variable 1-10 that indicates the quality of the borrower using balance sheet information. The Bank of Italy provides it.

2.3.2 Estimation

We estimate the model using the maximum simulated likelihood, and we use instruments to identify $\{\alpha^B, \alpha^U, \alpha^D\}$. After including the pricing equation in equation (2.1), we can rewrite it as

$$U_{ijmt} = \underbrace{\delta_{jmt} + \alpha^B \tilde{P}_{jmt}}_{\hat{\delta}_{jmt}} + \underbrace{Y^B_{ijmt}\tilde{\eta}_{ijmt} + \varepsilon^B_i}_{V^B_{ijmt}} + \zeta_{ijmt} \tag{2.7}$$

where δ_{ijmt} is the standard mean utility in the spirit of [20], and $\alpha^B \tilde{P}_{jmt}$ comes after substituting the pricing equation within the model. $\tilde{\eta}_{ijmt}$ is a combination of η and α^B. Finally, ζ_{ijmt} is the sum of ν_{ijmt} and the pricing error, and we assume that it follows a Type I extreme value distribution. Since α^B enters in $\tilde{\eta}_{ijmt}$ in combination with η, it cannot be estimated directly in the maximum simulated likelihood procedure, therefore it is estimated separately in a two-stage IV regression exploiting the mean utility decomposition $\hat{\delta}_{jmt}$ and using proxies for banks' costs as instruments. The likelihood function depends on the probability of demand, the conditional probability of the amount used and the conditional probability of default. Since $\zeta \sim$ *Type* I, the demand probability takes the standard logit form:

$$Pr^D_{ijmt} = \int \frac{e^{\left(\hat{\delta}_{jmt} + V_{ijmt}\right)}}{1 + \sum_{k \neq \{j,0\}} e^{\left(\hat{\delta}_{kmt} + V_{ikmt}\right)}} f(\varepsilon^B_i) \partial \varepsilon^B_i \tag{2.8}$$

where the utility of the outside option is normalized to 0 and $f(\varepsilon^B_i)$ is the density of ε^B_i. To estimate (2.2) and (2.3) we only focus on the lending relationship that took place, meaning that we do not need to use the pricing equation and that we can directly estimate all the parameters. However, we have to address the omitted variable bias that would reflect on $\{\alpha^U, \alpha^D\}$. For this reason, we use market-year fixed effects and the Hausmann instrument. To estimate the loan use equation, conditional on demand $B = 1$ and utilization L, we use:

$$Pr^U_{ijmt|B=1} = \int \phi_{\varepsilon^U_i|\varepsilon^B_i} \left(\frac{L_{ijmt} - \alpha^U_0 - X^U_{jmt}\beta^U - \alpha^U P_{ijmt} - Y^U_{ijmt}\eta^U - \tilde{\mu}_{\varepsilon^U_i|\varepsilon^B_i}}{\tilde{\sigma}_{\varepsilon^U_i|\varepsilon^B_i}} \right) f(\varepsilon^B_i | B = 1) \partial \varepsilon^B_i \tag{2.9}$$

where $\phi_{\varepsilon^U_i|\varepsilon^B_i}$ is the associated probability density function:

$$\varepsilon^U_i|\varepsilon^B_i \sim N\Big(\underbrace{\frac{\sigma_L}{\sigma_D}\rho_{BU}\varepsilon^B_i}_{\tilde{\mu}_{\varepsilon^U_i|\varepsilon^B_i}}, \underbrace{\sigma^2_U\left(1 - \rho^2_{BU}\right)}_{\tilde{\sigma}_{\varepsilon^U_i|\varepsilon^B_i}}\Big)$$

A similar logic applies to the probability of default, but this time the density is conditioned on both demand and amount used:

$$Pr^D_{ijmt} = \int \Phi_{\epsilon^D_i|\varepsilon^B_i,\varepsilon^U_i} \left(\frac{\alpha^D_0 + X^D_{jmt}\beta^D + \alpha^D + Y^D_{ijmt}\eta^D - \tilde{\mu}_{\varepsilon^D_i|\varepsilon^B_i,\varepsilon^U_i}}{\tilde{\sigma}_{\varepsilon^D_i|\varepsilon^B_i,\varepsilon^U_i}} \right) f(\varepsilon^B_i | B = 1) \partial \varepsilon^D_i \tag{2.10}$$

where $\Phi_{\epsilon^D_i|\varepsilon^B_i,\varepsilon^U_i}$ is the associated cumulative distribution function:

$$\varepsilon^D_i|\varepsilon^B_i, \varepsilon^U_i \sim N\Big(\underbrace{A\varepsilon^B_i + C\varepsilon^U_i}_{\tilde{\mu}_{\varepsilon^D_i|\varepsilon^B_i,\varepsilon^U_i}}, \underbrace{\sigma^2_D - (A\rho_{BD} + C\rho_{UD})}_{\tilde{\sigma}_{\varepsilon^D_i|\varepsilon^B_i,\varepsilon^U_i}}\Big)$$

Calling d_{ijmt} the indicator for the actual credit line choice, and f_{ijmt} the indicator for default, the joint log-likelihood function l is:

$$l = \sum_i d_{ijmt} \left\{ log(Pr^B_{ijmt}) + log(Pr^U_{ijmt}) + f_{ijmt} log(Pr^D_{ijmt}) + (1 - f_{ijmt}) log(1 - Pr^F_{ijmt}) \right\} \quad (2.11)$$

We estimate the log-likelihood using 100 Halton random draws. After the simulation, we use IV to perform a two stage regression and identify α^B. First, using the contraction method proposed by [21], it is possible to find $\hat{\delta}_{jmt}$, which can be decomposed as follows:

$$\hat{\delta}_{jmt} = \bar{\alpha}^B_0 + \alpha^B \tilde{P}_{jmt} + X^B_{jmt} \beta^B + \xi^B_{jmt} \quad (2.12)$$

where ξ^B_{jmt} represents unobserved characteristics of the bank that borrowers value in their decision. We decompose this term using bank, market and time fixed effects:

$$\hat{\delta}_{jmt} = \bar{\alpha}^B_0 + \alpha^B \tilde{P}_{jmt} + X^B_{jmt} \beta^B + \xi^B_j + \xi^B_m + \xi^B_t + \Delta\xi^B_{jmt} \quad (2.13)$$

Finally, we estimate this equation using instrumental variables to address the remaining endogeneity issues. We use the number of deposits, the cost of deposits and the amount of deposits as cost shifters. Further information and details about the structural model and the estimation routine can be found in [34].

2.3.3 Results

Our results are in line with the literature. We report the coefficients of interest in the following table:

Model		Price	Tot. Ass.	Intang./T.A.	Profits	Cashflow	Trade debit	Distance	
Demand		**-11.0530**	0.5797	0.4600	0.0043	0.0110	-0.3471	-0.4784	
		(3.3101)	(0.0312)	(0.0362)	(0.0060)	(0.0144)	(0.0226)	(0.0228)	
Loan use		**0.1266**	0.0032	0.0043	-0.0005	0.0012	-0.0082	-0.0339	
		(0.0042)	(0.0009)	(0.0016)	(0.0002)	(0.0008)	(0.0016)	(0.0047)	
Default		**0.1514**	0.0432	-0.0042	-0.0033	-0.0071	-0.0340	-0.2590	
		(0.0283)	(0.0059)	(0.0095)	(0.0023)	(0.0053)	(0.0096)	(0.0255)	
Σ	Demand	Loan Use	Default	score FE	Sector FE	Amount FE	BMY FE	Bank FE	Market FE
Demand	0.9454			Yes	Yes	Yes	Yes	No	No
	(8.6256)								
Loan use	0.1138	0.2984		Yes	Yes	Yes	No	Yes	Yes
	(0.0414)	(0.0026)							
Default	**0.1318**	0.0695	1	Yes	Yes	Yes	No	Yes	Yes
	(0.0414)	(0.0046)	-						

Table 2.1: Main coefficient of interest from equations (2.1), (2.2), and (2.3). The bottom left of the table contains estimated correlations (Σ). Standard errors in brackets. Number of observations: 115226 for demand, 5765 for the used amount and default. Variables are scaled by their 95^{th} percentile.

We find evidence of adverse selection $\rho_{BD} = 13\%$ and moral hazard $\alpha^D = 0.1514$. In particular, firms with a positive propensity to borrow have, on average, a high propensity to default. It is worth noting that $\alpha^U > 0$, while, in theory, the amount used should be decreasing in the price of the line. We are currently working on understanding why this coefficient is positive and significant. In any case, the economic magnitude is small and in the counterfactual analysis, when we vary P_{ijmt}, is

about 2-3% in terms of loan use.
The overall fit of the model is summarized in the following table:

	Demand		Loan use		Default	
	Mean	Std.	Mean	Std.	Mean	Std.
Data	0.0977	0.2969	0.6509	0.3142	0.1275	0.3336
Model	0.0977	0.2204	0.6499	0.1287	0.1332	0.0900

Table 2.2: This table contains the average demand choice ($\mathrm{E}[d_{ijmt}]$)), the average usage rate, and the average default probability for the model and the one observed in the data.

2.4 AI adoption

In this section, we analyse the impact of artificial intelligence on bank lending. We use the estimated model to evaluate the effects of AI pricing tools on credit allocation. This allows us to control the different frictions and their interactions. We assume that banks adopt AI and train their AI simultaneously to find the optimal pricing strategy. This analysis was first proposed by CCDP, which analyse a standard market and do not estimate the underlying equilibrium model. In our study, we use an estimated structural model to quantify the effects of AI adoption since they can have policy relevant implications. Our model is different because the market for credit lines is characterized by asymmetric information that leads to adverse selection and moral hazard.[10] The pricing problem is then substantially different, as a price increase corresponds to a decrease in the quality of the marginal borrower. As the price of the contract increases, only firms with a high propensity to demand will stay in the market, while the others will choose the outside option. However, we have estimated that those firms have, on average, a higher propensity to default. The open question is to understand if AI will ration credit by increasing the prices or if it will decrease the prices and reduce the overall risk associated with the lending activity. To address this question we compare the estimated equilibrium with the equilibrium emerging if the pricing task had been delegated to a pricing algorithm which maximizes profits taking (2.1), (2.2), and (2.3) as given. In the next section, we describe the algorithm of choice, then we present the results.

2.4.1 Qlearning

Following the approach by [81] and [41], we model AI using a reinforcement learning algorithm based on [82], the Q-learning. This algorithm is appealing because it has a clear structure and simple intuition. The purpose of this algorithm is to estimate the Q-matrix, a matrix that maps each possible state of the word and each feasible action into a discounted payoff q:

$$Q(state, action) = \begin{bmatrix} q_{s=1,a=1} & q_{s=1,a=2} & \cdots & q_{s=1,a=A} \\ q_{s=2,a=1} & q_{s=2,a=2} & \cdots & q_{s=2,a=A} \\ \cdots & \cdots & \cdots & \cdots \\ q_{s=S,a=1} & q_{s=S,a=2} & \cdots & q_{s=S,a=A} \end{bmatrix} \tag{2.14}$$

where each entry q represents the discounted expected value of taking action a when the state is s. Estimating this matrix allows the algorithm to find the optimal policy associated with the problem, as the *Argmax* of this matrix is the value function associated with the problem. The estimation is gradual, by trial and error. The algorithm starts playing the game at random, collecting payoffs,

[10] This is a relevant issue for first time borrowers, as the information asymmetries are at the highest level.

and updating the correspondent entry of the Q-matrix. The updated value is a linear combination of the flow payoff (implied by the underlying model) and the old matrix entry.[11] During the learning phase these actions are taken either randomly or following the strategy implied by the current Q-matrix. To ensure that the algorithm updates the payoffs associated with each action in any possible contingency, the algorithms begin by playing random actions and collecting the associated payoff, but, over iterations, the frequency of random actions decreases in favour of the best actions. In particular, whether the algorithm takes a random action (exploration) or what it thinks is the best action (exploitation), depends on a time-dependent probability ε_t:

$$Action = \begin{cases} \text{"}\boldsymbol{Exploration}\text{"} \rightarrow random \text{ "}a\text{" } with\ prob. & \varepsilon_t \\ \text{"}\boldsymbol{Exploitation}\text{"} \rightarrow best \text{ "}a\text{" } with\ prob. & 1-\varepsilon_t \end{cases} \tag{2.15}$$

where ε_t evolves as $\varepsilon_t = e^{-\beta t}$, given the decay parameter β. Initially, taking random actions means exploring what happens in any contingencies, then, as t increases, the algorithm plays according to the policy it thinks is optimal, but, when it is forced to explore, it enters a different state of the world and updates the payoff accordingly. This deviation can trigger the algorithm to review the current "optimal" policy. Eventually, the algorithm will learn the optimal policy, which is invariant to the exploration. Following the literature, we define convergence as the contingency where the best response in any state of the world does not change for 200.000 iterations.[12]

We cap the training phase up to 60.000.000 iterations of the pricing game. Payoffs are assigned using the estimated model. In particular, at each interaction, we observe the prices $\boldsymbol{p}$ set by the banks in the market and assign the payoff accordingly to the market's equilibrium. In our application, the problem of the firm can be thought of as a repeated game where the firm maximizes its period profits π_t, which depends on the price set by the algorithm and by other agents. Calling $Q(s, p)$ the entry of the Q-matrix corresponding in charging price p when the state is s, the algorithm's learning equation is:

$$q_{t+1}(s,p) = \underbrace{(1-\alpha)q_t(s,p)}_{Past\ knowledge} + \underbrace{\alpha\left[\pi_t + \delta \max_p q_t(s', p)\right]}_{learning\ update} \tag{2.16}$$

where $\alpha \in [0, 1]$ is the learning rate.

The theoretical counterpart to $q(s, p)$ is:

$$Q(s,p) = \mathrm{E}\left[\pi | s,p\right] + \delta \mathrm{E}\left[\max_p Q(s', p') | s, p\right] \tag{2.17}$$

Notice that the $argmax_p\{Q(s, p)\}$ is the value function associated with the repeated profit maximization problem. From equation (2.17), it is clear that Q-learning is a way to estimate $Q(s, p)$ and, consequently, the optimal policy.

2.4.2 AI adoption and increase in adverse selection

We study the interaction between AI and adverse selection by analyzing two alternative scenarios. In the first scenario, we let the AI play the game according to the estimated equilibrium, in an alternative scenario we let the AI play the same game but with higher adverse selection (ρ_{BD}). We then compare the results between these two cases and see what is the AI reaction to an increase in

[11] The matrix can be randomly initialized.

[12] Theoretical results on convergence can be found in [83]. In this application, like in [41], there is no theoretical guarantee of convergence, therefore we rely on this definition of convergence.

adverse selection, all else equals. Since we are also interested in understanding the impact of AI in selection markets per sé, we begin our analysis by focusing on duopolies and we assume that both banks simultaneously adopt and train their AI, as in [41]. In our data, a duopoly is a province-year market where only two banks are active. For each borrower in the market and each market, we train two Q-learning algorithms to play the game conditional on the set of characteristics of the borrower. Thus, the state of the game is only the current prices charged by the banks. At each iteration, the algorithms choose the price to charge to the borrower. We assign payoffs following the endogenous expected demand, expected usage and expected default probabilities.[13] We have 32 markets that are duopolies. The number of firms in these markets is 239, and 100 are borrowers. In the baseline scenario, we replicate the results in the literature, but the algorithm does not always converge to a stable best response. In particular, among 100 borrowers, the algorithm exited the learning phase in 57 cases. Among these 57 cases of convergence, only 27 are to a fixed best response, meaning that the algorithms play always the same prices, which are each other best responses. [14] The following figure summarizes the results in the baseline scenario:

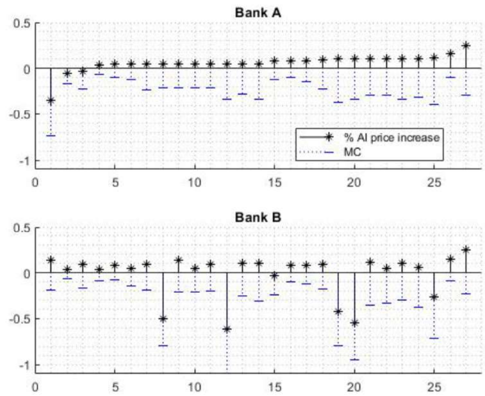

Figure 2.1: Average % price increase from the estimated equilibrium price. Each entry on the X-axis represents a borrower, while the Y-axis is the % deviation from the Nash-Bertrand equilibrium. Ordering is Bank A % price deviation. For example, for the first borrower, bank A decreased the price, while bank B increased it.

In most cases, the algorithm converged to an over-competitive equilibrium, but we occasionally observe a decrease in prices. We also look at the cost of credit resulting from the market equilibrium:

[13] We return to this point in the section "Future developments".

[14] We are investigating why the convergence rate is lower than in CCDP despite the learning being more persistent.

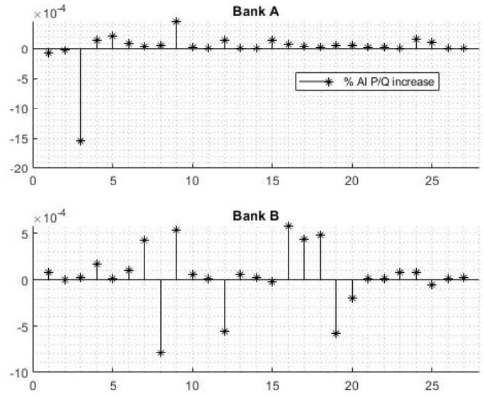

Figure 2.2: Average % cost of credit increase from the estimated equilibrium price. Each entry on the X-axis represents a borrower, while the Y-axis is the % deviation from the Nash-Bertrand equilibrium. Ordering is Bank A % price deviation as in Figure 2.1.

We repeat the same analysis in an alternative scenario where $\rho \uparrow$. The results are summarized in the following figures:

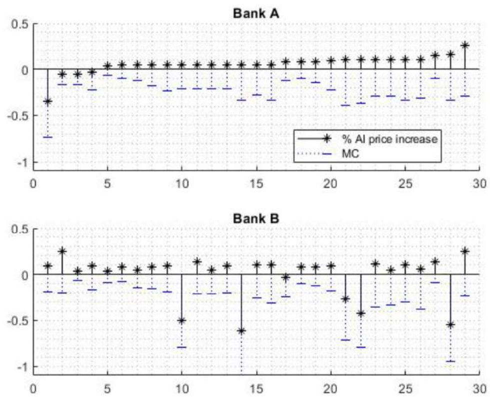

Figure 2.3: Average % price increase from the estimated equilibrium price. Each entry on the X-axis represents a borrower, while the Y-axis is the % deviation from the Nash-Bertrand equilibrium. Ordering is Bank A % price deviation. For example, for the first borrower, bank A decreased the price, while bank B increased it.

and

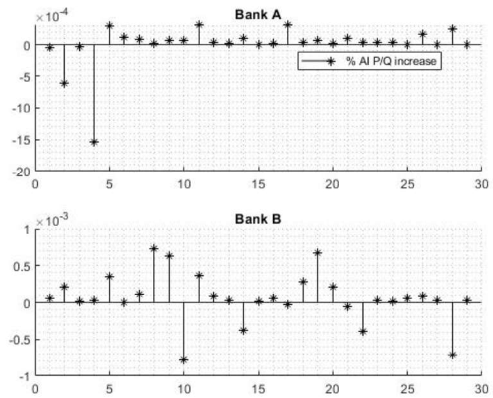

Figure 2.4: Average % cost of credit increase from the estimated equilibrium price. Each entry on the X-axis represents a borrower, while the Y-axis is the % deviation from the Nash-Bertrand equilibrium. Ordering is Bank A % price deviation as in Figure 2.3.

A preliminary analysis suggests that AI tend to decrease the cost of credit when adverse selection increases, however we see heterogeneous behaviour because the markets are heterogeneous.

2.4.3 Future developments

We are working on extending the analysis in two directions. First, we are currently working on the payoff update procedure. In the baseline case, we assign payoffs using expectations, but, given the nature of the game, another reasonable procedure would be to use lotteries. Since we condition on borrower's characteristics, the idea is to draft two state-dependent lotteries, one that assigns demand and one that determines if the firm defaults. We perform this analysis because we have reasons to believe that the algorithm does not understand the nature of the game, i.e. the selection problem. In particular, we suspect that the AI sees the problem only as a "steeper" demand curve, where an increase in price leads to a relatively higher decrease in expected profits. This is the case because, contrary to standard markets, as $P \uparrow \rightarrow q \downarrow, def \uparrow \rightarrow \mathrm{E}[\pi] \downarrow\downarrow$. We implemented lotteries to investigate if this conjecture is correct. With lotteries, the update of the payoff for the bank winning the demand lottery is either the full profit or the default loss, nudging the algorithms to understand that increasing prices means increasing the number of losses. Ideally, if the number of iterations goes to $+\infty$, the two alternative ways of performing the Q-entry updates should be equivalent, however, since the algorithm learns the optimal strategy in finite time, we might observe differences. We have experimented with the lotteries but the learning process becomes harder. We train the algorithm with the same learning parameters used in section 2.4.2, but, with lotteries, it did not converge. The reason is that with default, the algorithm faces a new learning challenge: π_t is much more volatile and can be either 0 or negative, meaning that every time a state is visited the update $\alpha\pi_t(s, p)$ can sensibly change the Q-entry $Q_{t+1}(s, p)$, changing the best response. This instability can harm the full process, making the algorithm unsuitable for this kind of application. This is a relevant issue that we would like to address, in particular, analyzing convergence with lotteries under different sets of meta parameters α, β, and understanding whether the results with lotteries are equivalent to the one in 2.4.2.

The second direction we are working on is in generalizing the AI adoption results. We want to extend the analysis in the case where the state space of the game includes the characteristics of the borrowers. In the structural model proposed by [34], the demand depends on prices as well as firms' characteristics, but we begin our analysis within borrowers (fixing characteristics) because we wanted to benchmark our results with the literature. Now we would like to depart from the baseline case and we are implementing an algorithm that can learn to play the full game, where the borrower changes over iteration t. The main limitation of this exercise is that standard Q-learning cannot handle the dimension of the problem, as the state space will include 10 variables plus the prices.[15] For this reason, we will implement a different AI: deep Q-learning. With this algorithm, we can train a general AI that can play the pricing game independently of the borrowers' characteristics, and have a general view of the impact of AI on the overall market allocation. In the analysis in section 2.4.2 we can compare differences in pricing strategies on a single borrower, but each borrower is priced by a different AI, while, with a generalized version of the AI we can analyse the impact on credit allocation, rationing, and risk-taking, both in the baseline scenario and in the scenario with higher adverse selection.

2.5 Conclusions

We analyse the impact of artificial intelligence in lending markets. The question is relevant because it is not clear how AI interacts with the typical friction present in financial markets. To answer this question we estimate a state-of-the-art model of demand and supply for credit lines with asymmetric information and later used these estimates as primitives in the AI adoption analysis. The estimates are in line with the results in [34], and the Q-learning implementation led to results similar to the findings [41], however with some key differences. First, the algorithm does not always converge, with the convergence rate being at about 60%. Second, the algorithm mostly plays the Nash-Bertrand prices or increases the markup, but, sometimes, the prevailing equilibrium price can be below Nash. In conclusion, we confirm that AI raises the prices, however, the increase is relatively lower because the algorithm anticipates the selection problem. Finally, we are working on the generalization of these results to study the effect of AI adoption on market outcomes.

[15] Standard Q-learning, also known as tabular Q-learning, suffer the curse of dimensionality: as the state space increases the entry of the Q-matrix increases geometrically, making it unfeasible to explore and learn the optimal policy.

Chapter 3

Inventory models in monetary economics

Carmelo Genovese[1]

Abstract

I review the use of inventory models in economic theory. These models were introduced in economics during the fifties, and through the years they have been adopted to model the economic behaviour of different kinds of agents. They have risen in popularity in recent years because the availability of detailed microdata allowed researchers to estimate the underlying parameters and understand the mechanisms that drive economic decisions. I review the theoretical contributions, focusing on the main contributions that generalized the early version of the models. For the early works, I focus on the works that contributed to generalising the framework, while, I focus on their recent success for empirical applications.

[1]Luiss Guido Carli.

3.1 Introduction

Inventory models, also known as fixed cost models, are theoretical models developed in operation research during the first half of the twentieth century. These models were created to analyse the problem of firms' inventory keeping, a fundamental source of costs in the fast growing US economy. Examples of early works are Davis (1925, [37]) and Meller (1925, [54]). Inventory modelling was a hot topic during these years, with scientific conferences on logistics and inventory keeping held annually all over the US.[1]

In its simplest formulation, an inventory problem is a problem where an agent needs to manage a stock, x, which is evolving over time. Managing this stock is costly: on one side, keeping the stock costs $A * x$ per period, on the other side, getting (or getting rid) of this stock costs B. For some reason, which will become clear within applications, the agent needs to keep the stock but would like to minimize the flow cost $A * x$, keeping in mind that she will need to pay B to adjust the stock. Hence, the problem of the agent, given the process driving the changes in x, is to balance the amount of stock, decide when to pay the fixed cost to adjust the stock, and, upon adjustments, decide the optimal level of stock x^*. In the case of a firm, x is the amount of goods in the warehouse, A is the cost of keeping the goods in the warehouse, and B is the cost of restocking the stock. The stock of goods decreases when sales are made. When deciding how many units of the good to keep, the firm needs to take into account that increasing the stock will increase the cost of managing the goods, while keeping low inventories implies that the fixed cost will be paid often to procure new goods.

Soon, scholars in economics started to read this literature and understood that a key aspect of inventory management was overlooked: the demand for goods. The models proposed by Arrow et al. (1951, [13]) and by Within (1952, [84]) are the first that jointly analyse the problem of inventory management with the demand for goods driving inventory's depletion. In particular, the contribution of Arrow et al. is to derive the optimal inventory policy in a setting with stochastic demand and both fixed and flow costs associated with inventory management. At the time, the most well-known result was to keep a constant sale/inventory ratio. Their contribution was to show that this policy might not be optimal and that fluctuations should be taken into account when making inventory decisions. Moreover, Within proposed a model where the demand for goods was stochastic, showing that demand uncertainty can lead to periods of depletion, and argued that the optimal management policy for inventory keeping should be based on expected demand. Within also wrote a survey on the evolution of inventory theories up to the fifties (1954 [85]).

These models are important contributions to the general theory of inventory keeping and changed the way researchers without an economic background conjectured the problem. In the following years, researchers have been generalizing the results adding structure to the baseline version of the problem. In particular, engineers and mathematicians have been developing generalized stochastic models for optimal inventory, logistics and transports, which are still relevant topics in operation research.[2]

The work by Within and Arrow et al. gave a second, and perhaps more important "contribution" to the economic literature: the introduction of inventory models in economic research. In the fifties, researchers realized that inventory theory could be extended to non-business applications: every stock/item that is managed as an inventory can be potentially analysed using this theory. The most famous application is for household cash management, which was first conjectured by Allais [4], then formalized independently by Baumol [20] and by Tobin[79]. Baumol himself stated in [20]:

"*T. M. Whitin informs me that the result in question goes back to the middle of the 1920s when it seems to have been arrived at independently by some half dozen writers. [...] Whitin analyzed them*

[1] For example, the RAND corporation used to organize a conference on logistics.

[2] It is possible to look at Perera and Sethi (2023 [64]) for a recent survey.

in his forthcoming [84] which, incidentally, first suggested the subject of this note to me."
In this paper I analyze the evolution of inventory models in economic research, highlighting the main contributions. The survey is organized as follows: in section 3.2 I describe the early application in monetary theory, including the criticisms made to this approach, then, in section 3.3 I summarize the contributions that generalized the baseline framework. Finally, I describe recent empirical applications in section 3.4.

3.2 Inventory models for money demand

The most well-known early application of inventory theories in economics is the famous Baumol-Tobin model for money demand. Both authors realized that the household's cash management problem can be thought of as an inventory problem: keeping the right amount of cash to pay for consumption given that holding and withdrawing cash is costly. The underlying assumption in both models is that households have a cash-on-hand constraint.
When deciding how much cash to hold, households need to balance the fixed cost associated with withdrawing cash[3], b, with the opportunity cost of holding cash, r. This opportunity cost can be thought of as the difference between the return from deposit/invest cash and the (possible) return on cash kept on hand. Notice how this problem is similar to an inventory keeping problem: the cost of raising cash is equivalent to the cost of restocking the inventory of goods, while the opportunity cost is equivalent to the cost of keeping the inventory. The inventory of money decreases as goods are purchased.
In [20] and [79], the household consumes T worth of goods per period, that are paid with cash on hand. This stock of cash decreases at every transaction, eventually running out. At that moment, the household must pay b and make a withdrawal. The problem of the household is then to choose the amount of cash C to withdraw and the frequency of withdrawals, leading to n withdrawals per period.[4] The optimal solution of the model is that when withdrawing, the household chooses:

$$C^* = \sqrt{\frac{2bT}{r}} \tag{3.1}$$

which is known as the "*square root rule*". To highlight the contribution of each paper in this survey, I will show how this optimal rule evolves. Given C^*, the optimal frequency of withdraws f per period is:

$$f^* = \frac{1}{n^*} = \sqrt{\frac{2b}{Tr}} \tag{3.2}$$

While the first to publish these results was Baumol, Tobin realized that this model is a micro foundation of households' money demand.[5] In particular, he realized that the inventory framework could be used to give a micro foundation of the money demand curve as a function not only of income[6] but also of the interest rate and transaction costs. Soon, in macroeconomic models, especially in Keynesian models, the use of interest-sensitive money demand schedules became the standard. For the rest of the paper, I will refer to their model as BT. [7]

[3] In the early works this cost is thought to be the cost of borrowing this cash or to disinvent part of the asset holdings, in modern days this can be thought of as the cost of going to the ATM to withdraw.
[4] For simplicity, it is assumed that T is uniformly consumed over the period.
[5] He stressed this point in the introduction of the paper [20]
[6] Tobin idea is that T, the need for transactions, reflect the level of income.
[7] The success of this theory is due to the articles by Baumol and Tobin, hence the name, but both authors agreed that the real father of this theory is Allais [48].

Following these works, Miller and Orr (1966, [58]), MO henceforth, had the idea to apply this framework to model cash holdings management at the firm level. While this exercise might seem straightforward, the essence of the problem is different. Firms' cash holdings evolve stochastically. They have cash expenditures to pay for costs and inputs and receive cash inflows from sales. Given this difference, they modelled cash holdings as a random Bernoulli variable. This is the opposite approach to BT, where cash was constantly decreasing in a deterministic way. Miller and Orr were well aware that both the assumptions were too extreme and tried to have a version of the model with both random and deterministic components, however, due to mathematical limitations, they focus on the case where the cash flow is a martingale.[8] The second contribution of their paper, which is a direct consequence of the martingale assumption, is that they define the optimal policy in terms of an inaction region. In contrast to BT, in the model of MO, firms' cash holdings can increase, and, consequently, the inventory of cash may become so costly to manage that it is more advantageous to pay the fixed cost b and deposit/invest the extra cash. An approach to solve this problem is to conjecture that the representative firm chooses an inaction region such that, while cash holdings are within this region, it does not pay b. To be specific, it needs to decide a threshold, M_2, such that if the level of cash holdings ends up being above this threshold, then it optimally pays b and invests the cash in excess, enjoying the higher return. On the other side, when cash holdings run out, the firm must withdraw, as in BT. Finally, the last decision of the firm is how much cash to hold when adjusting its cash holdings C^*_{MO}. Notice that uncertainty raises a precautionary motive for the agent: she needs to take into account that with some probability she will be paying b earlier than expected.

Inaction region solutions are also known as steady state solutions because, despite the dynamic nature of the agent's problem, the optimal policy is fixed.[9] In MO the inaction region is $[0, M_2]$, and the optimal amount held upon adjustment is

$$C^*_{MO} = \sqrt[3]{\frac{3b\mu^2 f}{4r}} \tag{3.3}$$

where μ is the size of the period change in cash holdings.[10] The following figure, taken from the paper, gives a graphic intuition behind the model:

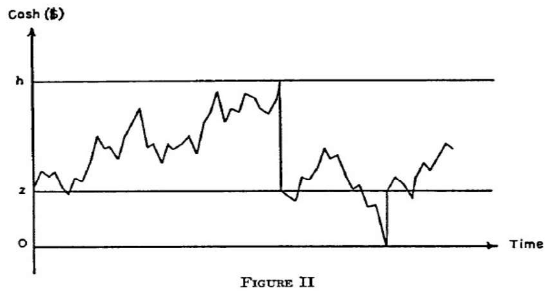

Figure 3.1: Optimal cash holdings policy from the paper by Miller and Orr $\{M_2 = h, C^*_{MO} = z\}$. Notice how the cash holding level evolves stochastically but is immediately reset at z when it hits one of the two boundaries $\{0,h\}$.

[8] Their attempt to have a model with variation around a deterministic trend can be found in the appendix of the paper.

[9] This kind of solution was proposed in mathematics by Karlin [52].

[10] In MO these change is driven by a Bernoulli process that determines whether cash holdings increase or decrease by m.

While the models discussed so far have had a lot of success, several authors distrusted them, highlighting that they were very stylized and that the assumptions behind the model were inconsistent with reality. For example, Sprenkle published an article criticizing this approach to modelling economic behaviours (1969 [78]) (1977 [77]). In this article, he argues against the credibility of modelling cash holdings decisions as inventory decisions, pointing out that several assumptions are too restrictive to have credible results. In particular, supported by the results in some papers trying to test the BT model, like Meltzer (1963 [55]), Sprenkle criticized both the assumption that the transaction flow is deterministic and that cash holdings cannot be negative within periods. These critiques are based on two facts. First, these models need detailed and high frequency data to be estimated correctly. Second, even if data were available, the process underlying inventory depletion is too simplistic to capture these dynamics. For example, the elasticity of C^* to r is:

$$\eta_{C^*,r} = -\frac{1}{2} \tag{3.4}$$

which is independent of other economic quantities. Of course, this result does not hold in reality. While these were valid points, advancements in mathematics as well as the availability of powerful storage and computation machines made it possible to generalize and estimate models much more sophisticated. Nowadays, it is possible to replicate fluctuations and have rich dynamics within these models, but whether a steady state policy based on an inventory model can realistically describe economic behaviours is still an open debate. In the next section, I will discuss the main contributions that addressed these critiques. The focus of these contributions will remain on cash holdings management and the implied money demand. Only later these models will be used for other economic studies.

3.3 From Baumol Tobin to the general stochastic model

While it was clear that the BT model was too stylized to produce results consistent with other economic theories and empirical evidence, its popularity did not fall. Soon, the basic model was enriched, partly-addressing the common critiques. These contributions can be divided into two categories. The first one includes contributions aimed at addressing the elasticity problem, while the other includes contributions aimed at generalising the dynamics of the problem. To this extent, the paper by Miller and Orr is an early attempt to generalize the framework by including stochastic stock variation. Despite their similarities, for several years, BT and MO were seen as complements: BT was the model for households' money demand, MO for firms' cash management.

3.3.1 Early contributions: focus on elasticities

One of the first authors that attempted to enrich the baseline versions of the BT model is Johnson (1969 [50]), who proposed a modified version that looked at the effect of interest earnings and fixed costs disbursement on the static budget constraint. In this way, managing costs are reflected in current consumption. The optimal cash holding in his model is:

$$C^*_{Johnson} = \frac{\sqrt{(2+r)bt}}{2r\left(1-\frac{k}{2}\right)} \tag{3.5}$$

where $\frac{k}{2}$ is a proportional cost of withdrawal. The elasticity of $C^*_{Johnson}$ with respect to r, denoted by $\eta_{\{C^*_{Johnson},r\}}$ is not constant as in BT:

$$\eta_{\{C^*_{Johnson},r\}} = -\frac{1}{2+r} \tag{3.6}$$

However, the author himself pointed out that $\eta_{\{C^*,r\}} \approx \frac{1}{2}$ for reasonable levels of r.
A similar motivation is behind the Karni (1973, [53]) version of the BT model. Karni generalized the model by rethinking the transaction level T as dependent on income Y. In particular, inspired by the new value of time economic theories that were taking over in microeconomics, he wanted to integrate an effort dimension in the fixed cost b. His idea was based on the fact that time is valuable, hence the fixed cost to withdraw should include this opportunity cost component:

$$b = \bar{b} + t_e W \tag{3.7}$$

where $\bar{b}$ is the fixed component of the adjustment cost, W is the hourly wage and t_e is the time lost to make a withdraw. Using the same logic, he decomposes the value of transaction T as fraction α of income

$$T = \alpha(\bar{Y} + t_w W) \tag{3.8}$$

where $\bar{Y}$ is fixed income, W is the hourly wage, t_w are hours worked. This attempt to generalize the model does not change the fact that both the demand and the effort enter the cost structure in a deterministic way, therefore, the implied optimal level of cash upon withdrawal still follows the square root role:

$$C^*_{Karni} = \sqrt{\frac{\alpha\left(\bar{Y} + t_w W\right)\left(\bar{b} + t_e W\right)}{2r}} \tag{3.9}$$

However, the elasticity now does depend on the relative share of fixed and total income:

$$\eta_{\{C^*_{Karni},r\}} = -\frac{1}{2}(1 - \frac{\bar{Y}}{Y}) \in (-\frac{1}{2}, 0) \tag{3.10}$$

From equation (3.9) it is possible to compute other relevant sensitivities, for example, the elasticity to hours worked t_w or fixed income $\bar{Y}$.
This kind of generalization allowed to extend the money demand theory, and to give microfoundation to the relationships between cash holdings and other relevant economic quantities, such as the mentioned income and wages. This added depth to the framework and its implications for money demand theories.[11]
In this period, the inventory concept was finally applied to a different economic problem. Sheshinski and Weiss (1977, [75]) adopted this framework to model the problem of the firm setting prices.[12] The problem was formulated with the following logic: A firm needs to decide the price P_t of its product. Once the price is set, changing it is costly b, therefore, the firm needs to take into account that the price level in the economy is expected to increase[13] and that $P_t < P^*_{t+1}$, where P^*_{t+1} is the price the firm would choose if adjusting P_t was cost-free. This difference generates a profit lost. Hence, the firms need to trade-off the lost in profits with the cost of adjusting the price, which is the typical inventory model trade-off. This is the first version of the famous *"menu cost"* model, a macroeconomic model were firms need to optimally choose P given that changing the prices in the menu is costly.[14] Notice that this early version of the menu cost model is in a deterministic setting, like in the BT.

[11] This is the freshwater vs. saltwater economics period, where the most important schools in the US were arguing about fundamentals in macroeconomic theories.
[12] They took inspiration from Barro (1972 [16]), which, inspired by inventory models, included a fixed cost to adjust prices in the pricing problem.
[13] They assume a constant and known in advance inflation rate.
[14] Think about a restaurant that needs to print a new menu each time the price of one dish should be changed.

3.3.2 Stochastic and general equilibrium versions of the model

In the eighties, the focus shifted from structure to generalization. So far, one of the crucial limitations of using the BT/MO framework was the inability to have an underlying process which included both a trend, or a deterministic need for transactions (BT), and uncertainty in the sizes of the transactions (MO). Without a general process, the dynamics of the model was far from reality. For example, any household expects to have some expenses over time, but the exact amount fluctuates. The same is true for the firms, which have uncertain cash flows. The solution to this problem arrived from progress in mathematics.[15]

The first economists who had the intuition to use stochastic calculus to solve a general inventory model were Frenkel and Jovanovic (1980 [43]).[16] Their model is the core of today's applications. They formulated the inventory problem in continuous time and assumed that the underlying process evolves as a Brownian motion with drift. To solve the model, they followed the strategy suggested by MO by assuming that the household follows a steady state policy. Moreover, they assumed that households have a continuous outflow of cash dm, which is stochastic:

$$dm_t = -\mu dt + \sigma dW_t \tag{3.11}$$

where μ is the drift[17] and dW_t is a Brownian motion. This means that the household never knows how much cash will be needed at $t_n > t$, but he knows the distribution:

$$m_{t_n} \sim \mathrm{N}\left(\mu(t_n - t), \sigma^2(t_n - t)\right) \tag{3.12}$$

The essence of the problem is still the same as BT: decide how much cash to hold given the cash in advance constraint. The household pays an opportunity cost r per unit of cash in hand m and each withdrawal costs her b. If cash on hand runs out, she is forced to pay b and withdraw. Of course, upon withdrawing, the household chooses to hold m^*, the optimal level of cash on hand. The optimal policy is $\{m^*, m_2\}$, where m_2 is the level of cash on hand such that the households prefer to pay b and deposit excess liquidity $m_2 - m^*$, like in MO. Several aspects of this contribution deserve a discussion. First, the optimal level of cash holdings m^* is:

$$m^* \approx \sqrt{\frac{2b\sigma^2}{\sqrt{\mu - 2r\sigma^2} - \mu}} \tag{3.13}$$

which is sensitive to both the cost structure and the parameters of the underlying diffusion process. Second, the model can nest both the BT and the MO models. The model reduces to BT when $\sigma = 0$, while reduces to MO when $\mu = 0$. The implied elasticity of cash holdings to the opportunity cost is:

$$\eta_{m^*,r} = -\sqrt{\frac{r\sigma^2}{4\left(\mu^2 + 2r\sigma^2 - \mu\sqrt{(\mu^2 + 2r\sigma^2)}\right)}} \tag{3.14}$$

which is more convoluted but can still nest the $-1/2$ value by Baumol. From equation (3.13) is it possible to compute several other relevant elasticities. Finally, another benefit of using continuous time modelling is the possibility of computing the invariant distribution of the state variable (cash holdings) associated with the solution. The invariant distribution is the distribution of cash holdings assuming that emerges if the household sticks to the $\{0, m^*, m_2\}$ rule:

[15] For example, Cox and Miller published in 1966 a book on stochastic calculus [33].

[16] Stochastic calculus was already in use in economic papers, for example in Fisher (1975 [74]).

[17] Usually the drift is assumed to be negative, so it represents the expected expenses per period. $c = \mu$ can be interpreted as the value of consumption per period.

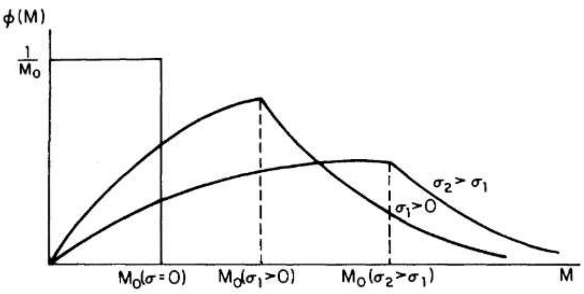

FIGURE II
The Steady-State Probability Density of Money Holdings for Alternative Values of σ

Figure 3.2: Invariant distribution for different values of the parameter σ. Image taken from [43].

This distribution, in theory, can be used to assess whether the implication of the theoretical model is consistent with the data. However, it depends on the Brownian motion parameters $\{\mu, \sigma\}$, which can be estimated only using transaction data at the household level, which were not available yet.[18]
The second kind of generalization made in the eighties was to include the inventory style cash management within macroeconomic models. First, Grossman and Weiss (1983 [47]) proposed a dynamic model where utility-maximizing households have a cash-on-hand constraint and can withdraw only at an exogenously given frequency. For simplicity, they do not include the fixed cost component of inventory theory, but they force households to wait between withdrawals, as standard inventory theory implies. This simplified their problem as the households' problem did not include the optimal choice of n. In particular, In their model, there are two types of agents that can withdraw cash from an interest-earning deposit any other period. These agents have non-financial wealth and can invest their money in a risk free bond that yields the same return as the deposit. The main result of their analysis is that changes in money supply can lead to real effects because some households cannot adjust their cash holdings right away. Despite this result coming from the exogenous assumption of synchronized adjustments, it was clear that monetary non-neutrality, typical in Keynesian models with sticky prices, could be obtained using fixed cost models. Rotemberg (1983 [72]) is another example. This logic that a fixed cost causes inaction and, consequently, real effects of nominal changes, like monetary policy changes, is behind the "menu cost" literature as well as other literatures.[19]
In the same years, Jovanovic published another important paper. In (1982 [51]), he proposed a model with endogenous opportunity cost: a general equilibrium version of the inventory model. In particular, in this paper, the opportunity cost r, which is a key component in inventory theory, is determined in equilibrium. Households need cash to pay for consumption, and the cash can be obtained by selling the asset. Of course, while converting asset into cash, the households incur a fixed cost, b. The average cash holdings $\bar{m}$ still follows the square root rule:

$$\bar{m} \approx \sqrt{\frac{bT}{2(\rho + \pi)}} \tag{3.15}$$

where ρ is discount rate and π is inflation.
This general equilibrium version of the inventory model was reformulated by Romer (1986 [70]). He

[18] A good introductory reference to deepen the knowledge in stochastic calculus and its application in economics is Dixit [39].
[19] For example see Cooper et al. (1993 [32]) or Bloom (2009 [26]).

built an overlapping generation (OLG) model where households can convert a risk free asset into cash by paying a fixed cost. Thus, in planning their consumption stream and their asset holdings, they need to consider the inventory trade-off. A novelty in this model is that the fixed cost is now expressed in utility loss rather than in real terms. The consequence of this assumption is that the frequency of adjustment is independent of wealth. The level of money holdings in this model is:

$$m = \left[\frac{e^{-\pi\tau} + \pi\tau - 1}{(\pi\tau)^2}\right]\left[\frac{(\pi\tau)E}{(1 - e^{-\pi\tau})T}\right]\tau \tag{3.16}$$

Where π is the inflation rate, τ is the time between withdrawals, E is the initial endowment of each agent and T is the lifespan of each individual.[20] Equation (3.16) is the analogue of the square root role, but in a general equilibrium OLG setting. The optimal level of cash holdings is given by the product of three quantities, the ratio of average cash holdings to initial withdrawn amount, the average amount used per unit of time, and the interval between adjustments τ. In particular, their counterpart in the BT square root rule are:

$$\left[\frac{e^{-\pi\tau} + \pi\tau - 1}{(\pi\tau)^2}\right] \to \frac{1}{2} \qquad \left[\frac{(\pi\tau)E}{(1 - e^{-\pi\tau})T}\right] \to Y \qquad \tau \to \sqrt{\frac{2b}{rY}}$$

This general equilibrium formulation of C^* implies that the cash holdings elasticity to the interest rate is now more convoluted, and, depending on parameters and assumption on the utility functional form, can even be positive.[21]

Other relevant papers written in this period are Chant (1976 [31]), which highlights the chaotic behaviour of early model away from steady state, Milbourne et al. (1983, [56]), that generalize and attempt to empirically validate the MO model, and Smith (1986, [76]), that shows how to construct aggregate money demand from individuals' inventory problem á la [56]/[43]. Akerlof and Milbourne (1980 [3]) analyse money demand when the inventory is subject to random jumps. Romer used his general equilibrium inventory model to study monetary policy transmission, generalizing the results in [47] and [72]. Blanchard returned to the origin of inventory theory and proposed a model to rationalize the empirical evidences coming from the automobile industry (1983, [25]). Finally, Caballero and Engel (1991 [30]) proposed a theory to study aggregate dynamics with agents following steady state policies.

3.4 Contemporaneous applications

In this section of the paper I will discuss some recent contribution to the BT/MO models and some modern application of inventory theory to model different economic problems. Before that, I would like to point out that the "*menu cost*" literature, started by [75], can be seen as an application of inventory theories to the price setting problem. However, this literature has been evolving separately since the eighties, therefore I will not cover it in the remaining of this survey.[22]

The idea that fixed costs lead to inaction and to the violation of the nominal/real dichotomy was discussed in the previous section 3.3. Inspired by these intuitions, Alvarez et al. (2009, [5]) proposed a general equilibrium inventory model to explain lumpiness in the reaction of prices and inflation to monetary shocks. This paper marks another important contribution in the general equilibrium

[20] To derive analytical results Romer assumed logarithmic utility and no discouting of the future.

[21] See section "*Money demand*" of the paper by Romer, pages 667-679, for a detail discussion.

[22] One of the most well known "*menu cost*" model is Golosov and Lucas (2007 [46]). In this article the reader will find a brief history of that literature and, in note 2, page 174, the main references needed to understand the evolution of the "*menu cost*" literature from the fixed cost framework.

inventory model literature because, it relaxed the OLG assumption of [70] and extends the findings in [71].
Another relevant contribution is Alvarez and Lippi (2009 [8]), that proposed a version of BT with the possibility of free adjustments. These free adjustments are the source of variation in the model and have a non trivial role in the analysis: the latter allowed the authors to derive non trivial equation for elasticities and other relevant statistics, like average cash balances at withdrawals. They model these free adjustment opportunities as the possibility to jump to the optimal inventory level m^* independently of the current level of cash m and the fixed cost b.[23] One immediate consequences of the possibility of free adjustments is that the precautionary motive to hold cash decreases because with some positive probability cash will be replenished. This also means that the inaction region increases and that the optimal level of cash on hand when withdrawing decreases. In this model aggregate money demand is:

$$M = \frac{m^*}{1 - e^{\frac{-\theta}{b} m^*}} - \frac{\mu}{\theta} \tag{3.17}$$

Another important point of this paper is that the authors had detailed micro data on households' cash holdings and cash management decisions, such as cash holdings at the moment of withdrawing or the size of the withdraws. This was the first empirical application where the critique by Sprenkle was no longer relevant. Alvarez, Lippi and co-authors are still working on extensions of this framework: [10], [6], and [9].
Other authors were inspired by these contributions. In particular, Bailey and Blanco (2021, [15]) realized that this framework was suited to model firms' investment decision. They found empirical evidence that firms' investment were lumpy, therefore coherent with a fixed cost model. Their contribution was to formalize the estimation strategy, mapping model implied moments and statistics to the empirical counterparts. This allowed them to do a strong impulse response analysis, quantifying the effects of an aggregate productivity shock on the capital to productivity ratio:

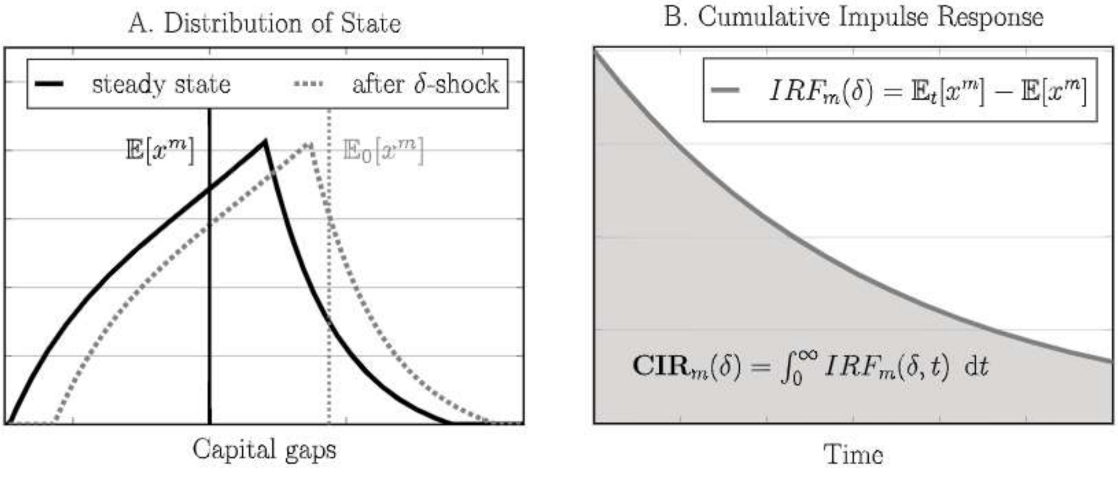

Figure 3.3: This figure, taken from their paper, shows the impact of an aggregate shock δ on the distribution of the state variable x(left panel) and the implied impulse response function of a given moment E$[x^m]$. See the paper for the empirical application on Chilean industrial plants.

[23] Practically, when solving his problem, the household knows that with an arrival rate θ he can jump from the current "value" of the problem $V(m_t)$ to the optimal one, $V^{(}m^*)$, where the "value" of the problem is the value function.

The last application of inventory model I would like to discuss is in finance, in the paper by Bolton et al. (2011, [27]). This is a well established paper in the financial literature that deals with the problem of a firm allocating its assets and deciding the optimal internal/external financing ratio. In particular, the firm needs to decide when to use external financing. The authors include a fixed financing cost that is proportional to capital[24] and assume that liquidity is managed as an inventory, and that can be used instead of external financing. Hence, liquidity has an intrinsic opportunity value. In terms of liquidity management, the implied invariant distribution of liquidity when the firm decides to use external financing is:

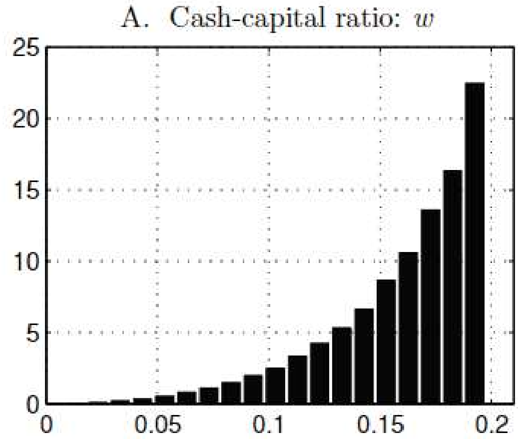

Figure 3.4: Figure taken from the paper. Invariant distribution of cash holdings when the firms is externally financing.

The paper by Bolton et al. touches deals with different interesting points and has other relevant results which are not shown because they do not belong to the inventory framework.

3.5 Concluding remarks

This is a short and non exhaustive sourvey of the use of inventory models in economics. After their introduction by [84] and [13], they become popular tools thanks to the work by [48] and [79], that used this framework to give microfoundation to money demand. The basic model was extended and generalized, and nowadays is used for different economic problems. The underlying concept is easy to understand, but can generate complex dynamics and interactions in sufficiently complicated models. If an economic problem can be reasonably modeled using inventory theories, the only drawback remaining is the availability of detail micro data to be able to conduct robust estimations. In their stochastic versions, these models can produce interesting results that can be tested, as it has been shown that it is possible to have a clear mapping between empirical and models' implied statistics/moments.

[24] Fixed in the sense that does not depend on the financing amount.

Chapter 4

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