Scientia Iranica E (2015) 22(6), 2621–2633

SCIENTIA IRANICA

Sharif University of Technology
Scientia Iranica
Transactions E: Industrial Engineering
www.scientiairanica.com

Closed-form equations for optimal lot sizing in deterministic EOQ models with exchangeable imperfect quality items

M. Farhangi[a], S.T.A. Niaki[b,*] and B. Maleki Vishkaei[c]

a. *Department of Industrial Engineering, Qazvin Branch, Islamic Azad University, Qazvin, Iran.*
b. *Department of Industrial Engineering, Sharif University of Technology, Tehran, P.O. Box 11155-9414, Iran.*
c. *Young Researchers and Elite Club, Qazvin Branch, Islamic Azad University, Qazvin, Iran.*

Received 20 November 2013; received in revised form 29 March 2014; accepted 29 September 2014

KEYWORDS
Inventory;
Economic order quantity;
100% screening;
Imperfect items;
Exchangeable items;
Shortage.

Abstract. In this paper, the optimal lot size for batches with exchangeable imperfect items is derived where the delay time for the exchange process depends on the quantity of imperfect items. This delay in exchange may or may not lead into shortage. The initial received lot is 100% screened. After the screening process, an order to exchange defective products takes place. The imperfect items are held in buyer's warehouse until the arrival of the exchange lot from the supplier for which, after another 100% screening process, imperfect items are sold at a lower price in a single batch. Two possible situations in which 1) there will not be any shortage, and 2) there will be a shortage that is fulfilled before the end of the replenishment cycle, are investigated. Proper mathematical models are developed and closed-form formulae are derived. Numerical examples are provided not only to demonstrate application of the proposed model, but also to analyze and compare the results obtained employing the proposed model and the ones gained using the classical economic order quantity model.

1. Introduction

Determining the optimal order quantity of a product is a well-known problem investigated by many researchers in production planning and inventory control, where different assumptions are made to adapt lot sizing formulae in different real-world conditions. Imperfect, nonconforming, or defective items are inevitable to exist in most of the received batches because of unreliable supply processes. Rosenblat and Lee [1], and Salameh and Jaber [2] were among the first who used the assumption of existing imperfect quality items in an Economic Production Quantity (EPQ) model. Then, Cardenas-Barron [3] corrected their model. Moreover, Goyal and Cardenas-Barron [4] used a practical approach for determining the optimal EPQ with imperfect items. Yoo et al. [5] proposed a profit-maximizing EPQ model that incorporated both imperfect production and a two-way imperfect inspection.

Wee et al. [6] considered the existence of imperfect items in the arrived batches of a lot-sizing problem. In their model, items were screened at a constant rate and imperfect items were sold at a lower price. Another assumption of Wee et al.'s model was the occurrence of shortages. Chang and Ho [7] showed that the quantities of optimal order and optimal shortage in Wee et al.'s model could be obtained simultaneously.

*. *Corresponding author. Tel.: +98 21 66165740; Fax: +98 21 66022702*
E-mail addresses: Milad.Farhangi@gmail.com (M. Farhangi); Niaki@Sharif.edu (S.T.A. Niaki); behzad.maleki.v@gmail.com (B. Maleki Vishkaei)

Hsu and Hsu [8] corrected a flaw in Wee et al.'s model in which unscreened lots, that may contain imperfect items, fulfilled the shortage. Tai [9] extended Hsu and Hsu's [8] model considering two warehouses and multi-screening processes. Moreover, Taleizadeh et al. [10] proposed a non-deterministic EPQ model for a limited capacity production system in which the defective-rate followed either a uniform or a normal probability distribution.

Kevin Hsu and Yu [11] presented an Economic Order Quantity (EOQ) model with imperfect items in which the supplier offered a one-time only discount offer. Maddah and Jaber [12] rectified a mistake in Salameh and Jaber's research [2] and presented an EOQ model for imperfect quality items without shortage. Further, Jaber et al. [13] extended Salameh and Jaber's [2] model by assuming reduction in the fraction of imperfect items based on a learning curve. Khan et al. [14] extended Jaber et al.'s [13] model in two conditions of lost sales and backorders. A further extension is due to Wahab and Jaber [15], in which Salameh and Jaber [2], Maddah and Jaber [12], and Jaber et al.'s [13] models were extended to include two different holding costs being charged for defective and perfect items. Konstantras et al. [16] studied the effect of shortage in an EOQ model with learning in inspection. Khan et al. [17] investigated another EOQ model for imperfect quality items in which there were errors in inspections. In their work, in order not to confront shortage, sale-returns that were added to actual demands were equal to perfect screened out items at maximum. Hsu and Hsu [18] presented a modified model of Khan et al. [17] where inshortage was allowed and backordered.

In all the research performed so far on the topic of imperfect quality items, there are four scenarios for imperfect quality items after being screened out of receiving lots. They are:

1. Imperfect items are sold at a lower price;
2. Imperfect items are returned to the supplier;
3. They can be reworked; and
4. They can be purged from the inventory system.

Here, in this research, it is assumed that imperfect items are exchanged with a new lot from the supplier at no charge with the quantity equal to imperfect quality items found just for one time in each cycle. Besides, since ordering the exchange requires knowledge of the exact number of imperfect quality items, the time the supplier needs to provide them depend on the exchanged quantity.

This research is another extension to Maddah and Jabber's model [12], in which not only shortage is allowed, but also imperfect items are not sold exactly after the screening process. Instead, they are sent back to the supplier due to his/her commitment for one-time exchange of imperfect items. Then, when all the arriving exchanged items are screened, the batch of imperfect items (that are screened out in the second inspection process) is sold at a lower price exactly after the screening process terminates. The exchange process takes some time depending on the quantity of the imperfect items found. Besides, based on the number of defective items found and their required production time, the supplier may or may not be able to replenish the buyer on time, i.e. shortage may or may not happen. This results in investigating the problem in two cases of with shortage and without shortage. The aim is to determine the optimal order quantity in two cases of with shortage and without shortage.

In Section 2, the problem is defined and the parameters are introduced. In Section 3, two models are developed for the two possible cases of with and without shortage. Numerical examples are provided in Section 4 to demonstrate applicability of the derivations and to compare the solutions with the ones of the traditional EOQ model.

2. Problem definition

Consider a buyer who orders a quantity of Q items to a supplier. The batch, delivered by the supplier, contains a portion p of imperfect quality items, a random variable that follows a certain probability distribution. Based on an exchange agreement between the buyer and the supplier, the supplier assures an exchange of imperfect items once in each cycle. As a result, the delivered batches are 100% inspected by the buyer and the imperfect items are screened out. Consequently, an order to exchange imperfect items is placed by the buyer. The supplier begins to make new products right after finding out the quantity of imperfect items and replenishes the buyer with an inevitable delay to produce required items. This delay in exchange may cause a shortage in buyer's inventory. When producing all required exchangeable items is finished, the exchanging lot will be delivered to the buyer. The supplier exchanges the imperfect items with the newly produced ones at the exchange time (right after finishing production of the new items). In this way, the supplier's transportation vehicle is sent to the buyer just one time in each cycle; it delivers new items and takes the imperfect items back to supplier's facility to reduce transportation cost. This means the buyer should maintain the imperfect items in his/her warehouse until the exchange time. The exchanged batch contains the same portion of imperfect quality items where it is 100% inspected by the buyer, once again. However, this time, the imperfect items that are screened out are not returned to the supplier and are sold at a lower price.

In the next subsections, the parameters and the decision variable required to model the problem are defined.

2.1. Decision variable

The only decision variable of the mathematical formulation is:

Q Order size of the product.

2.2. Parameters

D The demand rate;

x The screening rate, $x \geq D$;

c The purchasing cost per unit;

K The ordering cost per order;

p The defective percentage in Q (it follows a uniform distribution in $[0, \beta]$, where $0 \leq \beta < 1$);

$f(p)$ The probability distribution function of p;

s The selling price per unit;

V Salvage value per defective item, $V < c$;

y The supplier's production rate;

d The screening cost per unit;

b The backordering cost per unit per unit of time;

h The holding cost per unit per unit of time;

T The cycle time.

3. Mathematical model

Since the supplier begins producing the required exchanged items when the quantity of imperfect quality items is determined, the behavior of buyer's inventory depends on the rates of demand, screening, and exchange. During the exchange process, if the quantity of perfect items that are screened out in the first inspection reaches zero, then a shortage occurs in the buyer's inventory, otherwise not. Thus, the problem is investigated in two possibilities of either having a shortage or not. Note that to avoid shortage during the screening process, similar to Salameh and Jaber [2], it is assumed that the demand rate is less than the production rate of the perfect items.

3.1. Model without shortage (Model 1)

For the first possible situation, if the average rate of demand until the end of the exchange process is less than the average rate of received perfect items, then shortage will not occur. In other words, the time required for both the inspection and the exchange processes together, t_1, will be less than the time at which the inventory of perfect items reaches zero. This implies that:

$$D < \frac{(1-p)Q}{Q/x + pQ/y}.$$

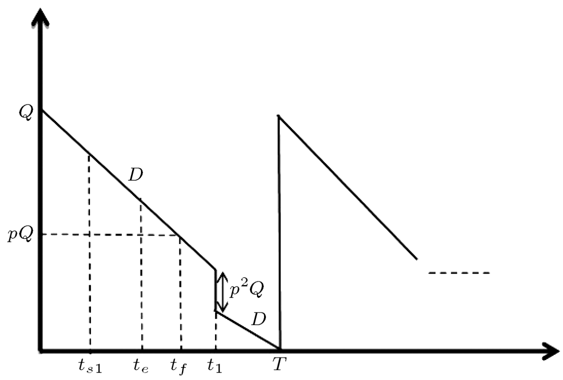

Figure 1. The behavior of the buyer's inventory without shortage.

For this case, the inventory behavior is shown in Figure 1, where:

t_{s1} The time in the cycle at which the first screening process is finished and the exchange order takes place by the buyer;

t_e The time in the cycle at which the imperfect items are exchanged with the new ones;

t_f The time in the cycle at which the initial perfect items are sold out;

t_1 The time in the cycle at which the second screening process is finished and the remaining imperfect items (which are screened out in the second screening process) are sold in a single batch;

T The cycle time.

Note that in this case, the following explicit assumptions are made.

3.1.1. Assumptions

The assumptions involved in the model without shortage are:

1. The demand rate is known and constant;
2. The replenishment is instantaneous;
3. The defective items are sold after finishing the second screening process;
4. The screening process takes place with the demand, simultaneously. However, it is assumed that the screening rate is greater than the demand rate, $x > D$;

5. To avoid shortage within screening time, t, $p \leq 1 - D/x$;
6. Shortage does not occur during the exchange period, i.e. $D < \frac{(1-p)xy}{y+px}$.

At the beginning of each cycle, the inventory begins with the order quantity, Q, and the delivered lot is screened at the rate of x to be sold to the customers at the rate of D. As $x > D$, there will not be any shortage during the screening process and the demand will be fulfilled. Right after termination of the first screening process, an order to exchange imperfect items with newly produced items takes place by the buyer at time t_{s1}. The supplier starts producing the exchange items, sends them to the buyer, and receives the imperfect items at time t_e. At this time, there are still perfect items that have been screened out during the first screening process. At the end of the second screening process (t_1), p^2Q items are found imperfect and are sold at a lower price in a single batch. The remaining perfect items are sold until the end of the cycle T.

Note that if the buyer does not confront shortage, there should be still perfect items left in the inventory at the time of exchange. Therefore, the time required for screening process plus the time needed for exchange process should be smaller than the time required for consuming all the perfect items screened out of the first screening process. Since $(1-p)Q$ is the quantity of perfect items found in the initial screening process, the required time for its consumption is $\frac{(1-p)Q}{D}$. Hence, $t_1 < \frac{(1-p)Q}{D}$ and $\frac{Q}{x} + \frac{pQ}{y} < \frac{(1-p)Q}{D}$, which results in $D < \frac{(1-p)xy}{px+y}$.

3.1.2. Costs

Three types of costs, procurement, screening, and holding, are involved in the first model. These costs are explained as follows:

Procurement cost, TPR. The procurement cost is the summation of ordering and purchasing the ordered lot.

$$TPR = K + Qc. \tag{1}$$

Screening cost, TSC. Since the screening process occurs twice in a cycle, its cost is obtained as:

$$TSC = dQ(1+p). \tag{2}$$

Holding cost, TH. The holding cost in the first model consists of the holding costs of two types of inventories, imperfect and perfect items. It is derived in Appendix A as:

$$TH = h\left\{\frac{\left((1-p^2)Q\right)^2}{2D} + p^2Q\left(\frac{Q}{x} + \frac{pQ}{y} + \frac{pQ}{x}\right)\right\}$$

$$= h\left\{\frac{\left((1-p^2)Q\right)^2}{2D} + \frac{p^2Q^2}{x} + \frac{p^3Q^2}{y} + \frac{p^3Q^2}{x}\right\}. \tag{3}$$

As a result, the Total Cost (TC) in the model without shortage becomes:

$$TC = K + cQ + dQ(1+p) + h\left\{\frac{\left((1-p^2)Q\right)^2}{2D} + \frac{p^2Q^2}{x} + \frac{p^3Q^2}{y} + \frac{P^3Q^2}{x}\right\}. \tag{4}$$

3.1.3. Revenue (TR)

The revenue in each cycle is earned by selling perfect and imperfect items as:

$$TR = s\left(1-p^2\right)Q + V\left(p^2\right)Q. \tag{5}$$

3.1.4. Profit (TP)

The profit in a cycle is obtained by subtracting the total revenue per cycle and the total cost per cycle:

$$TP = s(1-p^2)Q + V(p^2)Q - K - cQ - dQ(1+p) - h\left\{\frac{\left((1-p^2)Q\right)^2}{2D} + \frac{p^2Q^2}{x} + \frac{p^3Q^2}{y} + \frac{p^3Q^2}{x}\right\}. \tag{6}$$

Thus, the expected profit per cycle is:

$$E(TP) = s\left(1-E(p^2)\right)Q + V\left(E(p^2)\right)Q - K - cQ - dQ\left(1+E(p)\right) - h\left\{\frac{Q^2E\left((1-p^2)\right)^2}{2D} + \frac{E(p^2)Q^2}{x} + \frac{E(p^3)Q^2}{y} + \frac{E(p^3)Q^2}{x}\right\}. \tag{7}$$

Besides, as p^2Q is the quantity of imperfect items that are screened out in the second screening process and are sold in a single batch at a lower price right after termination of the second screening process, the cycle time that is the duration of consumption of all the perfect items in the cycle is $T = (1-p^2)Q/D$ with the expected cycle length of:

$$E(T) = \frac{\left(1-E(p^2)\right)Q}{D}. \tag{8}$$

As a result, based on the renewal theorem, the expected profit per unit time will be:

$$ETPU = \frac{E(TP)}{E(T)} = sD + VD\left(\frac{E(p^2)}{1-E(p^2)}\right) - \frac{KD}{1-E(p^2)} - \frac{cD}{(1-E(p^2))}$$

$$-\frac{dD\,(1+E(p))}{(1-E(p^2))}-hQ\left(\frac{E\left((1-p^2)^2\right)}{2\,(1-E(p^2))}\right)$$
$$-\left(\frac{hQD}{x}\right)\left(\frac{E(p^2)}{1-E(p^2)}\right)-\left(\frac{hQD}{y}\right)$$
$$\left(\frac{E(p^3)}{1-E(p^2)}\right)-\left(\frac{hQD}{x}\right)\left(\frac{E(p^3)}{1-E(p^2)}\right). \quad (9)$$

The first and the second derivatives of $ETPU$ with respect to Q are:

$$\frac{dETPU}{dQ}=\frac{KD}{(1-E(p^2))\,Q^2}-(h)\left(\frac{E\left((1-p^2)^2\right)}{2\,(1-E(p^2))}\right)$$
$$-\left(\frac{hD}{x}\right)\left(\frac{E(p^2)}{1-E(p^2)}\right)-\left(\frac{hD}{y}\right)$$
$$\left(\frac{E(p^3)}{1-E(p^2)}\right)-\left(\frac{hD}{x}\right)\left(\frac{E(p^3)}{1-E(p^2)}\right), \quad (10)$$

$$\frac{d^2ETPU}{dQ^2}=\frac{-2KD}{(1-E(p^2))\,Q^3}<0. \quad (11)$$

Hence, $ETPU$ is concave in Q wherein the optimal lot size, Q^*, is obtained by equating Eq. (9) to zero and solving for Q as:

$$Q^*=\sqrt{\frac{KD}{h\left\{\frac{E((1-p^2)^2)}{2}+D\left(\frac{1}{x}\left(E(p^2)+E(p^3)\right)+\frac{E(p^3)}{y}\right)\right\}}}. \quad (12)$$

Note also that the optimal lot size will be equal to the one of the classical EOQ model by setting $p=0$, i.e. $Q^*=\sqrt{2KD/h}=Q^*_{EOQ}$.

3.2. Model with shortage (Model 2

The second model suits to the problem in which the buyer faces shortage in his/her inventory. In this problem, the delay for the exchange process leads into the time at which there is not any perfect quality item left in the warehouse. In this case, the demand rate is more than the average rate of perfect items being screened out until the end of the exchange process. Moreover, the ongoing shortage is fulfilled before the end of the cycle such that the demand rate becomes less than the average rate of perfect items being screened out until the end of the second screening process. In other words, the demand until the end of the second screening process is less than the quantity of perfect items in each cycle to cover the shortage before the end of the replenishment cycle. Therefore:

$$Dt_4\le(1-p^2)Q. \quad (13)$$

Hence:

$$D\le\frac{(1-p^2)Q}{\frac{Q}{x}+\frac{pQ}{y}+\frac{pQ}{x}}, \quad (14)$$

where, t_4 is the time in a cycle at which the second screening process is ended and the imperfect items are sold (shown later in Eq. (18)). Besides, the following assumptions are made in this case.

3.2.1. Assumptions

The assumptions involved in the model with shortage are:

1. To avoid shortage within the screening time, $p\le 1-D/x$;
2. The demand rate is known and constant;
3. The replenishment is instantaneous;
4. Defective items are sold after finishing the second screening process;
5. The screening process and demand proceed simultaneously, but the screening rate is greater than demand rate, $x>D$;
6. Shortage that occurs during the exchange delay period, i.e. $D\ge(1-p)xy/(y+px)$, is backordered;
7. Shortage is fulfilled before the cycle terminates, i.e., $D\le(1-p^2)xy/\left((1+p)y+px\right)$.

The behavior of inventory system for the second problem is shown in Figure 2.

where:

- t_{s1} The time in the cycle at which the first screening process is finished and the exchange order takes place by the buyer;
- t_1 The time in the cycle at which the primary perfect items are sold out;
- t_2 The time in the cycle at which the imperfect items are exchanged with new ones;
- t_3 The time in the cycle at which the shortage is fulfilled and, as stated previously;

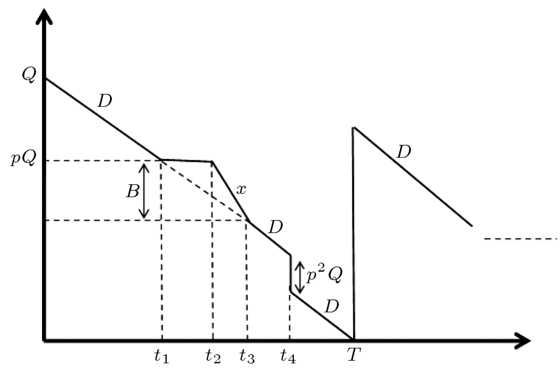

Figure 2. The behavior of the buyer's inventory with shortage.

t_4 The time in the cycle at which the second screening process is ended and the imperfect items are sold;

T The cycle time.

At the beginning of a cycle, the inventory begins with the order quantity Q, the delivered lot is screened by the rate of x to be sold to the customers at the consumption rate of D. As $x > D$ there will not be any shortage during the screening process and the demand is fulfilled. However, right after the termination of screening process, an order to exchange the imperfect items with new ones is taken place by the buyer at time t_{s1}. As the demand, during the exchange period, exceeds the number of available perfect items, the buyer confronts shortage at time t_1 before he/she receives the new items that are produced by the supplier. The supplier finishes producing pQ items ordered and sends the exchanging batch to the buyer while taking back the imperfect items at time t_2. Then, the buyer starts the second screening process at the rate of x. The perfect items that are screened out satisfy both the shortage and the demand during the screening process until t_3. The screening process continues until all the items are screened out at time t_4. Since the second batch contains the same p portion of imperfect items, we will have p^2Q imperfect items that are sold at a lower price in a single batch at time t_4. The remaining perfect items are sold by the end of the cycle T.

Note that we have:

$$t_1 = \frac{(1-p)Q}{D}, \tag{15}$$

$$t_2 = \frac{Q}{x} + \frac{pQ}{y}. \tag{16}$$

Besides, as $D(t_3 - t_1) = (1-p)x(t_3 - t_2)$, then:

$$D\left(t_3 - \frac{(1-p)Q}{D}\right) = (1-p)x\left(t_3 - \frac{Q}{x} - \frac{pQ}{y}\right),$$

and:

$$t_3 = \frac{(1-p)pQ}{y\left(1-p-\frac{D}{x}\right)}, \tag{17}$$

$$t_4 = \frac{Q}{x} + \frac{pQ}{y} + \frac{pQ}{x}, \tag{18}$$

$$T = \frac{(1-p^2)Q}{D}. \tag{19}$$

3.2.2. Costs

Four types of costs, procurement, screening, holding, and shortage, are considered for the second problem. The procurement and screening costs are the same as the ones in the first model. They are:

$$TPR = K + cQ, \tag{20}$$

$$TSC = dQ(1+p). \tag{21}$$

Holding cost. Based on Figure 2, the holding cost in the second model is derived in Appendix B as:

$$\begin{aligned} TH =& h\Bigg\{\frac{(1-p)Q}{2}t_1 + pQt_2 + \bigg((1-p)pQ \\ & - \frac{1}{2}D(t_3 - t_1)\bigg)(t_3 - t_2) + \frac{1}{2}\bigg(p(1-p)Q \\ & - D(t_3-t_1)\bigg)(T - t_3) + p^2Q(t_4 - t_2)\Bigg\}. \end{aligned} \tag{22}$$

Substituting t_1, t_2, t_3 and t_4 by Eqs. (15), (16), (17), and (18), respectively, results in:

$$\begin{aligned} TH =& h\Bigg\{\frac{(1-p)^2Q^2}{2D} + \frac{p^2Q^2}{y} + \frac{pQ^2}{x} + \frac{p^3Q^2}{x} \\ & + \frac{1}{2}\Bigg[\frac{p^2(1-p)DQ^2}{y^2\left(1-p-\frac{D}{x}\right)} - \frac{2p(1-p)Q^2}{x} \\ & + \frac{p(1-p)DQ^2}{xy\left(1-p-\frac{D}{x}\right)} + \frac{p^2(1-p)^2Q^2}{y\left(1-p-\frac{D}{x}\right)} \\ & - \frac{2p^2(1-p)Q^2}{y} - \frac{(1-p)Q^2}{x} \\ & + \frac{(1-p^2)(1-p)Q^2}{D} - \frac{p(1-p)Q^2}{y} \\ & + \frac{p(1-p^2)(1-p)Q^2}{D} \\ & - \frac{p(1-p^2)(1-p)Q^2}{y\left(1-p-\frac{D}{x}\right)}\Bigg]\Bigg\} \\ =& h\Bigg\{\frac{(2-2p-p^2+p^4)Q^2}{2D} \\ & + \frac{(2p^3+p+2p^2-1)Q^2}{2x} + \frac{(2p^3-p+p^2)Q^2}{2y} \\ & + \frac{p^2(1-p)^2Q^2}{2y\left(1-p-\frac{D}{x}\right)} + \frac{p(1-p)DQ^2}{2xy\left(1-p-\frac{D}{x}\right)} \\ & + \frac{p^2(1-p)DQ^2}{2y^2\left(1-p-\frac{D}{x}\right)} \\ & - \frac{p(1-p^2)(1-p)Q^2}{2y\left(1-p-\frac{D}{x}\right)}\Bigg\}. \end{aligned} \tag{23}$$

Shortage cost. The shortage cost can be obtained by:

$$TS = \frac{b}{2}\left(D(t_2-t_1)(t_3-t_1)\right) = \frac{b}{2}\Bigg\{\frac{(1-p)pQ^2D}{xy\left(1-p-\frac{D}{x}\right)}$$

$$+\frac{p^2(1-p)DQ^2}{y^2\left(1-p-\frac{D}{x}\right)}-\frac{(1-p)^2pQ^2}{y\left(1-p-\frac{D}{x}\right)}-\frac{Q^2(1-p)}{x}-\frac{pQ^2(1-p)}{y}+\frac{Q^2(1-p)^2}{D}\bigg\}. \quad (24)$$

As a result, the total cost becomes:

$$TC=K+cQ+dQ(1+p)+h\bigg\{\frac{(2-2p-p^2+p^4)Q^2}{2D}+\frac{(2p^3+p+2p^2-1)Q^2}{2x}+\frac{(2p^3-p+p^2)Q^2}{2y}+\frac{p^2(1-p)^2Q^2}{2y\left(1-p-\frac{D}{x}\right)}+\frac{p(1-p)DQ^2}{2xy\left(1-p-\frac{D}{x}\right)}+\frac{p^2(1-p)DQ^2}{2y^2\left(1-p-\frac{D}{x}\right)}-\frac{p(1-p^2)(1-p)Q^2}{2y\left(1-p-\frac{D}{x}\right)}\bigg\}+\frac{b}{2}\bigg\{\frac{(1-p)pQ^2D}{xy\left(1-p-\frac{D}{x}\right)}+\frac{p^2(1-p)DQ^2}{y^2\left(1-p-\frac{D}{x}\right)}-\frac{(1-p)^2pQ^2}{y\left(1-p-\frac{D}{x}\right)}-\frac{Q^2(1-p)}{x}-\frac{pQ^2(1-p)}{y}+\frac{Q^2(1-p)^2}{D}\bigg\}. \quad (25)$$

3.2.3. Total profit

The total revenue per cycle is the same as the one in the first model and the total profit is obtained by subtracting total revenue and the total cost per cycle, i.e.:

$$TP=TR-TC=s(1-p^2)Q+V(p^2)Q-K-cQ-dQ(1+p)-h\bigg\{\frac{(2-2p-p^2+p^4)Q^2}{2D}+\frac{(2p^3+p+2p^2)Q^2}{2x}+\frac{(2p^3-p+p^2)Q^2}{2y}+\frac{p^2(1-p)^2Q^2}{2y\left(1-p-\frac{D}{x}\right)}+\frac{p(1-p)DQ^2}{2xy\left(1-p-\frac{D}{x}\right)}+\frac{p^2(1-p)DQ^2}{2y^2\left(1-p-\frac{D}{x}\right)}-\frac{p(1-p^2)(1-p)Q^2}{2y\left(1-p-\frac{D}{x}\right)}\bigg\}-\frac{b}{2}\bigg\{\frac{(1-p)pQ^2D}{xy\left(1-p-\frac{D}{x}\right)}+\frac{p^2(1-p)DQ^2}{y^2\left(1-p-\frac{D}{x}\right)}-\frac{(1-p)^2pQ-\frac{Q^2(1-p)^2}{x}}{y\left(1-p-\frac{D}{x}\right)}-\frac{pQ^2(1-p)}{y}+\frac{Q^2(1-p)^2}{D}\bigg\}. \quad (26)$$

Then, using the renewal reward theorem, we have:

$$ETPU=\frac{E(TP)}{E(T)}=sD+VD\frac{E(p^2)}{1-E(p^2)}-\frac{KD}{(1-E(p^2))Q}-\frac{cD}{(1-E(p^2))}-\frac{dD(1+E(p))}{(1-E(p^2))}-h\bigg\{\frac{(2-2E(p)-E(p^2)+E(p^4))Q}{2(1-E(p^2))}+\frac{\left(2E(p^3)-E(p)+E(p^2)\right)QD}{2y(1-E(p^2))}+\frac{\left(2E(p^3)+E(p)+2E(p^2)-1\right)QD}{2x(1-E(p^2))}+\frac{D^2Q}{2y^2(1-E(p^2))}E\left(\frac{p^2(1-p)}{1-p-\frac{D}{x}}\right)+\frac{D^2Q}{2xy(1-E(p^2))}E\left(\frac{p(1-p)}{1-p-\frac{D}{x}}\right)+\frac{DQ}{2y(1-E(p^2))}E\left(\frac{p^2(1-p)^2}{1-p-\frac{D}{x}}\right)-\frac{DQ}{2y(1-E(p^2))}E\left(\frac{p(1-p^2)(1-p)}{1-p-\frac{D}{x}}\right)\bigg\}-\frac{b}{2}\bigg\{\frac{QD^2}{xy(1-E(p^2))}E\left(\frac{p(1-p)}{1-p-\frac{D}{x}}\right)+\frac{QD^2}{y^2(1-E(p^2))}E\left(\frac{p^2(1-p)}{1-p-\frac{D}{x}}\right)-\frac{QD}{y(1-E(p^2))}E\left(\frac{p(1-p)^2}{1-p-\frac{D}{x}}\right)-\frac{QD}{x}\frac{1-E(p)}{1-E(p^2)}-\frac{QD}{y}\frac{E(p)-E(p^2)}{1-E(p^2)}+Q\frac{1-2E(p)+E(p^2)}{1-E(p^2)}\bigg\}. \quad (27)$$

Defining:

$$A_1=E\left(\frac{p^2(1-p)^2}{1-p-\frac{D}{x}}\right), \quad (28)$$

$$A_2 = E\left(\frac{p(1-p)}{1-p-\frac{D}{x}}\right), \tag{29}$$

$$A_3 = E\left(\frac{p^2(1-p)}{1-p-\frac{D}{x}}\right), \tag{30}$$

$$A_4 = E\left(\frac{p(1-p)^2}{1-p-\frac{D}{x}}\right), \tag{31}$$

$$A_5 = E\left(\frac{p(1-p^2)(1-p)}{1-p-\frac{D}{x}}\right). \tag{32}$$

The first and the second derivatives of the expected profit per unit time, with respect to Q, are obtained as:

$$\begin{aligned}\frac{dETPU}{dQ} =& \frac{KD}{(1-E(p^2))\,Q^2} - \frac{h\left(2-2E(p)-E(p^2)+E(p^4)\right)}{2\,(1-E(p^2))} \\ &- \frac{hD\left(2E(p^3)-E(p)+E(p^2)\right)}{2y\,(1-E(p^2))} - \frac{hD\left(2E(p^3)+E(p)+2E(p^2)-1\right)}{2x\,(1-E(p^2))} \\ &- \frac{D^2hA_3}{2y^2\,(1-E(p^2))} - \frac{D^2hA_2}{2xy\,(1-E(p^2))} - \frac{DhA_1}{2y\,(1-E(p^2))} + \frac{hDA_5}{2y\,(1-E(p^2))} \\ &- \frac{bD^2A_2}{2xy\,(1-E(p^2))} - \frac{bD^2A_3}{2y^2\,(1-E(p^2))} + \frac{bDA_4}{2y\,(1-E(p^2))} + \frac{bD}{2x}\frac{1-E(p)}{1-E(p^2)} \\ &+ \frac{bD}{2y}\frac{E(p)-E(p^2)}{1-E(p^2)} - \frac{b}{2}\frac{1-2E(p)+E(p^2)}{1-E(p^2)},\end{aligned} \tag{33}$$

$$\frac{d^2ETPU}{dQ^2} = \frac{-2KD}{(1-E(p^2))\,Q^3}. \tag{34}$$

Since $\frac{d^2ETPU}{dQ^2} < 0$, the optimal lot size becomes:

$$Q^* = \sqrt{\frac{2KD}{h\left\{R_1+D\left(\frac{1}{y}\left(R_2+\frac{DA_3}{y}+\frac{DA_2}{x}+A_1-A_5\right)+\frac{R_3}{x}\right)\right\}+b\left(\frac{D}{y}\left(\frac{DA_2}{x}+\frac{DA_3}{y}-A_4-R_4\right)-\frac{D\,(1-E(p))}{x}+R_5\right)}} \tag{35}$$

where:

$$R_1 = 2-E(p^2)-2E(p)+E(p^4), \tag{36}$$

$$R_2 = 2E(p^3)+E(p^2)-E(p), \tag{37}$$

$$R_3 = 2E(p^3)+2E(p^2)+E(p)-1, \tag{38}$$

$$R_4 = E(p)-E(p^2), \tag{39}$$

$$R_5 = 1-2E(p)+E(p^2). \tag{40}$$

For a uniform distribution of p as an example, the terms in Eq. (35) are derived in Appendix C. Note that by setting $p = 0$, the optimal lot size becomes:

$$Q^* = \sqrt{\frac{2KD}{h\left(2-\frac{D}{x}\right)+b\left(1-\frac{D}{x}\right)}}. \tag{41}$$

However, since:

$$(1-p)xy/(y+px) \le D \le (1-p^2)xy \,/\left((1+p)y+px\right),$$

D will be equal to x. Hence, Q^* will be equal to the one obtained using the classical EOQ.

In the next section, numerical examples are solved in order to demonstrate application of the proposed methodology.

4. Numerical examples and sensitivity analyses

The proposed models are employed in this section to solve 36 generated numerical examples and to perform sensitivity analyses of Q^* on the parameters of each model. The results of solving these examples are shown in three groups of 12 examples, each having a different mean fraction of imperfect items. However, the parameters used in the three groups are the same in order for the comparison of the effect of the imperfect quality fraction on the optimal lot size to be justified.

4.1. Numerical examples

The fixed parameters of all examples are:

$$c = \frac{\$300}{\text{unit}}, \qquad K = \frac{\$4000}{\text{unit}}, \qquad b = \frac{\$7}{\text{unit}},$$

$$d = \frac{\$1}{\text{unit}}, \qquad s = \frac{\$500}{\text{unit}}, \qquad h = \$4\frac{\text{unit}}{\text{time}},$$

$$V = \$200.$$

Other parameters of the models are determined such that both the optimal batch size of the first model, Q_1^*, and the one of the second model, Q_2^*, are obtained for most of the examples. Using these parameters, and based on different mean percentages of defective items, the solutions are obtained and are summarized in Table 1.

The results in Table 1 indicate that there is a small difference between Q_1^* and the one of the classical EOQ. However, the difference between Q_2^* and the one of the classical EOQ is more tangible.

As changes in y, x, D, and p may cause a shift between the two models or even may lead to a situation where the shortage cannot be fulfilled during the cycle time (as shown in Table 1), a sensitivity analysis is performed on the values of these parameters; the results are summarized in Table 2.

The results in Table 2 show that changes in x lead to a slight increment in both Q_1^* and Q_2^*, where Q_2^* shows more sensitivity than Q_1^*. Moreover, increments in the screening rate result in increases in EOQ. This is expected, since higher screening-rate causes lesser depleting time of imperfect items inventory, and hence, decreasing the holding cost. Nevertheless, Q_2^* is the least sensitive factor to x in comparison with the other parameters. Further, while the effect of change of x on $EPTU$ is the same as the one of EOQ, the change in y has a similar effect on EOQ as x does. Meanwhile,

Table 1. Solutions obtained based on various mean percentages of defective items.

$E(p)$	x	D	EOQ		$y = 900$	$y = 1400$	$y = 2950$	$y = 6800$
0.01	25000	19400	6228.96	Q^*	$Q_2^* = 144.87$	$Q_1^* = 6228.97$	$Q_1^* = 6229.06$	$Q_1^* = 6229.11$
				$EPTU(Q^*)$	2789501.92	3835225.52	3835225.88	3835226.07
				$EPTU(EOQ)$	-19152173.7	3835225.52	3835225.88	3835226.07
	30000	22300	6678.32	Q^*	$Q_2^* = 139.80$	$Q_1^* = 6678.33$	$Q_1^* = 6678.44$	$Q_1^* = 6678.50$
				$EPTU(Q^*)$	3161691.08	4410459.79	4410460.24	4410460.47
				$EPTU(EOQ)$	-26022556.9	4410459.79	4410460.24	4410460.47
	40000	21000	6480.74	Q^*	$Q_1^* = 6480.84$	$Q_1^* = 6480.95$	$Q_1^* = 6481.05$	$Q_1^* = 6481.10$
				$EPTU(Q^*)$	4152581.12	4152581.55	4152581.96	4152582.17
				$EPTU(EOQ)$	4152581.12	4152581.55	4152581.96	4152582.17
0.03	25000	19400	6228.96	Q^*	–	–	$Q_2^* = 446.39$	$Q_1^* = 6229.41$
				$EPTU(Q^*)$	–	–	3511886.73	3832719.19
				$EPTU(EOQ)$	–	–	1448874.84	3832719.19
	30000	22300	6678.32	Q^*	–	–	$Q_2^* = 434.77$	$Q_1^* = 6678.92$
				$EPTU(Q^*)$	–	–	4025790.87	4407581.20
				$EPTU(EOQ)$	–	–	1300895.10	4407581.20
	40000	21000	6480.74	Q^*	–	$Q_2^* = 280.45$	$Q_1^* = 6481.75$	$Q_1^* = 6483.16$
				$EPTU(Q^*)$	–	3577668.26	4149870.93	4149876.58
				$EPTU(EOQ)$	–	-2729786.43	4149870.92	4149876.57
0.06	25000	19400	6228.96	Q^*	–	–	–	$Q_2^* = 900.08$
				$EPTU(Q^*)$	–	–	–	3680948.40
				$EPTU(EOQ)$	–	–	–	3267208.58
	30000	22300	6678.32	Q^*	–	–	–	$Q_2^* = 896.98$
				$EPTU(Q^*)$	–	–	–	4229968.90
				$EPTU(EOQ)$	–	–	–	3700303.81
	40000	21000	6480.74	Q^*	–	–	$Q_2^* = 562.37$	$Q_1^* = 6485.27$
				$EPTU(Q^*)$	–	–	3870313.44	4141474.22
				$EPTU(EOQ)$	–	–	2459276.26	4141474.22

Table 2. Effects of parameter changes on the economic order quantity and the expected profit per unit time.

Parameters	% Changes in parameters	% Changes in			
		Q_2^*	$EPTU(Q_2^*)$	Q_1^*	$EPTU(Q_1^*)$
p	+5	-0.0231	-0.00125	+0.00017	-0.00028
	+3	-0.01384	-0.00074	+0.0001	-0.00016
	-3	+0.013767	+0.000713	-0.00011	+0.00016
	-5	–	–	-0.00018	+0.00026
x	+5	+0.001408	+0.0000133	+0.000028	+0.00000039
	+3	+0.000861	+0.00000815	+0.000018	+0.00000024
	-3	-0.00091	-0.0000086	-0.000022	-0.00000027
	-5	-0.00155	-0.000015	-0.000035	-0.00000045
y	+5	+0.025704	+0.000142	+0.000072	+0.00000094
	+3	+0.015681	+0.000088	+0.000044	+0.00000058
	-3	-0.01649	-0.000098	-0.00005	-0.00000062
	-5	-0.02794	-0.00017	-0.000082	-0.0000011
D	+5	-0.00405	+0.049959	+0.02458	+0.050326
	+3	-0.00225	+0.029978	+0.014821	+0.030195
	-3	+0.001658	-0.02998	-0.01505	-0.03019
	-5	+0.002403	-0.04998	-0.02521	-0.05032

Q_2^* shows more sensitivity on y than x, because not only the supplier's production rate affects the depletion time of imperfect items, but also it affects the time a shortage occurs.

Besides, Q_1^* has the most sensitivity to the changes in D among all parameters. Changes in D, however, have an inverse effect on Q_2^* where an increment in demand causes more shortage to happen during the exchange period. Thus, a reduction in an order quantity results in a decrease of imperfect item quantity and hence a decrease in the delay due to exchanging imperfect items. Note also that not only a demand change has a direct impact on $EPTU$ of both models, but also a change in the mean of the fraction of imperfect items has a direct effect on the first model's EOQ. In this case, the holding cost reduces due to a decrease in inventory, causing more quantities to be ordered. Finally, Q_2^* has the most sensitivity to the changes of p, where a change in p has an inverse effect on Q_2^*. This is because the higher p, the longer the exchange period and hence the higher the shortage cost is. Moreover, changes in p have a similar effect on $ETPU$ of both models. In general, while $ETPU$ of both models have a similar reaction towards the changes in the parameters, the degrees of the effects differ from each other.

5. Summary and conclusion

In this paper, an EOQ model for items with imperfect quality was investigated, in which the supplier agrees to exchange imperfect items with new ones. This exchange occurs with a delay that varies based on the quantity of imperfect items found. The delay in exchange leads to the following three possible scenarios:

I. The exchanged lot arrives before the time the initial perfect items stored in the warehouse are totally consumed so that there is no shortage;

II. The exchange occurs after depletion of perfect items so that the system faces shortage, which can be fulfilled before the end of the cycle; and

III. The shortage is larger than the difference between the quantities of perfect items and demand so that the system eventually ends with shortage.

In this paper, Scenarios I and II were first formulated, mathematically, and closed-form formulae were obtained for the optimal order quantity in both cases. Then, the proposed models were analyzed by solving 36 numerical examples, in which the effects of changes in models' parameters on the quantities of the order size were discussed. Results showed that if the classical EOQ were used instead of the model derived for scenario II, not only the ideal profit would not be achieved, but also it might result in a loss.

The proposed models in this research can be extended using some other assumptions such as different probability distributions of imperfect items, partial backordering, and the like.

References

1. Rosenblat, M.J. and Lee, H.L. "Economic production cycles with imperfect production process", *IIE Transactions*, **18**, pp. 48-55 (1986).
2. Salameh, M.K. and Jaber, M.Y. "Economic order quantity model for item with imperfect quality", *International Journal of Production Economics*, **64**, pp. 59-64 (2000).
3. Cárdenas-Barrón, L.E., Observation on: "Economic production quantity model for items with imperfect quality", *International Journal of Production Economics*, **67**, pp. 59-64 (2000).
4. Goyal, S.K. and Cárdenas-Barrón, L.E. "Note on economic production quantity model for items with imperfect quality - A practical approach", *International Journal of Production Economics*, **77**, pp. 85-87 (2002).
5. Yoo, S.H., Kim, D.S. and Park, M.S. "Economic production quantity model with imperfect-quality items, two-way imperfect inspection and sales return", *International Journal of Production Economics*, **121**, pp. 255-265 (2009).
6. Wee, H.M., Yu, J. and Chen, M.C. "Optimal inventory model for items with imperfect quality and shortage backordering", *Omega*, **35**, pp. 7-11 (2007).
7. Chang, H.C. and Ho, C.H. "Exact closed-form solutions for optimal inventory model for items with imperfect quality and shortage backordering", *Omega*, **38**, pp. 233-237 (2010).
8. Hsu, J.T. and Hsu, L.F. "An EOQ model with imperfect quality items, inspection errors, shortage backordering, and sales returns", *International Journal of Production Economics*, **143**, pp. 162-170 (2013).
9. Tai, A.H., *An EOQ Model for Imperfect Quality Items with Multiple Screening and Shortage Backordering*, Cornell University Library arXiv, pp. 1302-1323, USA (2013).
10. Taleizadeh, A.A., Najafi, A.A. and Niaki, S.T.A. "Economic production quantity model with scrapped items and limited production capacity", *Scientia Iranica, Transaction E: Industrial Engineering*, **17**, pp. 58-69 (2010).
11. Kevin Hsu, W.K. and Yu, H.F. "EOQ model for imperfective items under a one-time-only discount", *Omega*, **37**, pp. 1018-1026 (2009).
12. Maddah, B. and Jaber, M.Y. "Economic order quantity for items with imperfect quality: Revisited", *International Journal of Production Economics*, **112**, pp. 808-815 (2008).
13. Jaber, M.Y., Goyal, S.K. and Imran, M. "Economic production quantity model for items with imperfect quality subject to learning effects", *International Journal of Production Economics*, **115**, pp. 143-150 (2008).
14. Khan, M., Jaber, M.Y. and Wahab, M.I.M. "Economic order quantity model for items with imperfect quality with learning in inspection", *International Journal of Production Economics*, **124**, pp. 87-96 (2010).
15. Wahab, M.I.M. and Jaber, M.Y. "Economic order quantity model for items with imperfect quality, different holding costs, and learning effects: A note", *Computers & Industrial Engineering*, **58**, pp. 186-190 (2010).
16. Konstantaras, I., Skouri, K. and Jaber, M.Y. "Lot sizing for a recoverable product with inspection and sorting", *Computers & Industrial Engineering*, **58**, pp. 452-462 (2010).
17. Khan, M. Jaber, M.Y. and Bonney, M. "An economic order quantity (EOQ) for items with imperfect quality and inspection errors", *International Journal of Production Economics*, **133**, pp. 113-118 (2011).
18. Hsu, J.T. and Hsu, L.F. "A note on optimal inventory model for items with imperfect quality and shortage backordering", *International Journal of Industrial Engineering Computations*, **3**, pp. 939-948 (2012).

Appendix A

The holding cost of maintaining the perfect items in buyer's inventory during the cycle time in Model 1 is:

$$TH_1 = h \times \frac{(1-p^2)Q}{2} \times T = h \times \frac{(1-p^2)Q}{2} \times \frac{(1-p^2)Q}{D} = h \times \frac{\left((1-p^2)Q\right)^2}{2D}. \quad \text{(A.1)}$$

The holding cost of preserving imperfect items in buyer's inventory until the end of the second screening process in Model 1 is:

$$TH_2 = h \times p^2 Q \times t_1 = hp^2 Q\left(\frac{Q}{x} + \frac{pQ}{y} + \frac{pQ}{x}\right). \quad \text{(A.2)}$$

Hence, the total holding cost per cycle in Model 1 is:

$$TH = h\left\{\frac{\left((1-p^2)Q\right)^2}{2D} + p^2 Q\left(\frac{Q}{x} + \frac{pQ}{y} + \frac{pQ}{x}\right)\right\}$$

$$= h\left\{\frac{\left((1-p^2)Q\right)^2}{2D} + \frac{p^2Q^2}{x} + \frac{p^3Q^2}{y} + \frac{p^3Q^2}{x}\right\}. \quad \text{(A.3)}$$

Appendix B

In Model 2, the holding cost of perfect items, from the time they are screened out in the first screening process to the time they are fully consumed, is:

$$TH_1 = h \times \frac{(1-p)Q}{2} t_1. \quad \text{(B.1)}$$

In Model 2, the holding cost of pQ perfect items, from the time they are screened out in the first screening process to their exchange time, is:

$$TH_2 = h \times pQ \times t_2. \quad \text{(B.2)}$$

In Model 2, the holding cost of perfect items that are screened out in the second screening process is:

$$TH_3 = h \times \left((1-p)pQ - \frac{1}{2}D(t_3 - t_1)\right) \times (t_3 - t_2). \tag{B.3}$$

In Model 2, the holding cost of perfect items that are screened out of the second screening process, from the time the shortage has been fulfilled to the time they are fully consumed, is:

$$TH_4 = h \times \frac{1}{2}\left(p(1-p)Q - D(t_3 - t_1)\right) \times (T - t_3). \tag{B.4}$$

Finally in Model 2, the holding cost of p2Q imperfect items that are screened out in the second screening process is:

$$TH_5 = h \times p^2 Q \times (t_4 - t_2). \tag{B.5}$$

Thus, the total holding cost per cycle in Model 2 is obtained as:

$$\begin{aligned} TH &= TH_1 + TH_2 + TH_3 + TH_4 + TH_5 \\ &= h\Big\{\frac{(1-p)Q}{2}t_1 + pQt_2 + \big((1-p)pQ \\ &- \frac{1}{2}D(t_3 - t_1)\big)(t_3 - t_2) + \frac{1}{2}\big(p(1-p)Q \\ &- D(t_3 - t_1)\big)(T - t_3) + p^2Q(t_4 - t_2)\Big\}. \end{aligned} \tag{B.6}$$

Appendix C

Assuming p to follow a uniform distribution in $(0, \beta)$, we have:

$$E(p) = \int_0^\beta pf(p)dp = \int_0^\beta \frac{p}{\beta}dp = \frac{\beta}{2}, \tag{C.1}$$

$$E(p^2) = \frac{\beta^2}{3}, \tag{C.2}$$

$$E(p^3) = \frac{\beta^3}{4}, \tag{C.3}$$

$$E(p^4) = \frac{\beta^4}{5}. \tag{C.4}$$

As a result:

$$\begin{aligned} A_1 &= E\left(\frac{p^2(1-p)^2}{1-p-\frac{D}{x}}\right) = \int_0^\beta \frac{p^2(1-p)^2}{1-p-\frac{D}{x}}f(p)dp \\ &= \int_{1-D/x-\beta}^{1-D/x} \frac{(1-z-D/x)^2(z+D/x)^2}{z}f(z)dz \\ &= \left(\frac{D}{x}\right)^3 - \left(\frac{D}{x}\right)^2 - \frac{\beta}{2}\left(\frac{D}{x}\right)^2 + \frac{\beta^2}{3}\left(\frac{D}{x}\right) \\ &+ \frac{\beta^2}{3} - \frac{\beta^3}{4} + \frac{1}{\beta}\left(\frac{D}{x}\right)^2 \ln\left(\frac{1-D/x}{1-D/x-\beta}\right) \\ &+ \frac{1}{\beta}\left(\frac{D}{x}\right)^4 \ln\left(\frac{1-D/x}{1-D/x-\beta}\right) - \frac{2}{\beta}\left(\frac{D}{x}\right)^3 \\ &\ln\left(\frac{1-D/x}{1-D/x-\beta}\right). \end{aligned} \tag{C.5}$$

Using the same method, A_2, A_3, A_4, and A_5 become:

$$\begin{aligned} A_2 =& E\left(\frac{p(1-p)}{1-p-\frac{D}{x}}\right) = \frac{\beta}{2} - \left(\frac{D}{x}\right) \\ &+ \frac{1}{\beta}\left(\frac{D}{x}\right)\ln\left(\frac{1-D/x}{1-D/x-\beta}\right) \\ &- \frac{1}{\beta}\left(\frac{D}{x}\right)^2\ln\left(\frac{1-D/x}{1-D/x-\beta}\right), \end{aligned} \tag{C.6}$$

$$\begin{aligned} A_3 =& E\left(\frac{p^2(1-p)}{1-p-\frac{D}{x}}\right) = \left(\frac{D}{x}\right)^2 - \left(\frac{D}{x}\right) \\ &- \frac{\beta}{2}\left(\frac{D}{x}\right) + \frac{\beta^2}{3} + \frac{1}{\beta}\left(\frac{D}{x}\right)\ln\left(\frac{1-D/x}{1-D/x-\beta}\right) \\ &- \frac{2}{\beta}\left(\frac{D}{x}\right)^2\ln\left(\frac{1-D/x}{1-D/x-\beta}\right) \\ &+ \frac{1}{\beta}\left(\frac{D}{x}\right)^3\ln\left(\frac{1-D/x}{1-D/x-\beta}\right), \end{aligned} \tag{C.7}$$

$$\begin{aligned} A_4 =& E\left(\frac{p(1-p)^2}{(1-p-\frac{D}{x})}\right) = \frac{\beta}{2} - 2\left(\frac{D}{x}\right)^2 \\ &+ \frac{\beta}{2}\left(\frac{D}{x}\right) + \left(\frac{D}{x}\right)^3 + \frac{\beta}{2}\left(\frac{D}{x}\right)^2 \\ &+ \frac{\beta^2}{3}\left(\frac{D}{x}\right) - \frac{\beta^3}{4} + \frac{2}{\beta}\left(\frac{D}{x}\right)^2 \\ &\ln\left(\frac{1-D/x}{1-D/x-\beta}\right) - \frac{3}{\beta}\left(\frac{D}{x}\right)^3 \\ &\ln\left(\frac{1-D/x}{1-D/x-\beta}\right) + \left(\frac{D}{x}\right)^4 \\ &\ln\left(\frac{1-D/x}{1-D/x-\beta}\right), \end{aligned} \tag{C.8}$$

$$A_5 = E\left(\frac{p(1-p^2)(1-p)}{1-p-\frac{D}{x}}\right) = \frac{\beta}{2} - \left(\frac{D}{x}\right)^2 + \frac{1}{2}\beta\left(\frac{D}{x}\right) - \beta^2 + \frac{1}{\beta}\left(\frac{D}{x}\right)^2 \ln\left(\frac{1-D/x}{1-D/x-\beta}\right) - \frac{1}{\beta}\left(\frac{D}{x}\right)^3 \ln\left(\frac{1-D/x}{1-D/x-\beta}\right). \quad \text{(C.9)}$$

Biographies

Milad Farhangi received his BS and MS degrees in 2010 and 2013, respectively, both in Industrial Engineering from Islamic Azad University, Qazvin branch. His research interests include inventory control and supply chains.

Seyed Taghi Akhavan Niaki received his BS degree in Industrial Engineering from Sharif University of Technology, Tehran, Iran, in 1979, and his MS and PhD degrees, both in Industrial Engineering, from West Virginia University, USA, in 1989 and 1992, respectively. He is, currently, Professor of Industrial Engineering at Sharif University of Technology, Tehran, Iran. His research interests include: simulation modeling and analysis, applied statistics, multivariate quality control, and operations research. Before joining Sharif University of Technology, he worked as a Systems Engineer and Quality Control Manager at the Iranian Electric Meters Company. He is also a member of $\alpha\pi\mu$.

Behzad Maleki Vishkaei received his BS and MS degrees in 2010 and 2013, respectively, both in Industrial Engineering from Islamic Azad University, Qazvin branch. He is currently a member of Young Researchers and Elite Club at Qazvin Islamic Azad University. His research interests include operations research, inventory control, and supply chains.