L(2, 1)-labeling of oriented planar graphs

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ABSTRACT

The $L(2, 1)$-labeling of a digraph D is a function l from the vertex set of D to the set of all nonnegative integers such that $|l(x) - l(y)| \geq 2$ if x and y are at distance 1, and $l(x) \neq l(y)$ if x and y are at distance 2, where the distance from vertex x to vertex y is the length of a shortest dipath from x to y. The minimum over all the $L(2, 1)$-labelings of D of the maximum used label is denoted $\vec{\lambda}(D)$. If $\mathcal{C}$ is a class of digraphs, the maximum $\vec{\lambda}(D)$, over all $D \in \mathcal{C}$ is denoted $\vec{\lambda}(\mathcal{C})$.

In this paper we study the $L(2, 1)$-labeling problem on oriented planar graphs providing some upper bounds on $\vec{\lambda}$. Then we focus on some specific subclasses of oriented planar graphs, improving the previous general bounds. Namely, for oriented prisms we compute the exact value of $\vec{\lambda}$, while for oriented Halin graphs and cacti we provide very close upper and lower bounds for $\vec{\lambda}$.

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1. Introduction

The *$L(h, k)$-labeling* of a graph G is a function l from the vertex set $V(G)$ to the set of nonnegative integers $\{0, 1, \ldots, \sigma_l\}$ such that:

- $|l(x) - l(y)| \geq h$ if x and y are adjacent in G, and
- $|l(x) - l(y)| \geq k$ if x and y are at distance 2 in G; the minimum value of σ_l over all $L(h, k)$-labelings is denoted by $\lambda_{h,k}$. This problem has been introduced by Griggs and Yeh [9,17] in the particular case in which $h = 2$ and $k = 1$ as a variation of the frequency assignment problem of wireless networks introduced by Hale [10]. Since its definition, a huge number of works have been produced on this topic (some surveys looking at the problem from different point of views are [1,3,18]) and particular interest has been raised by the special case of the $L(2, 1)$-labeling. A natural extension, recently introduced by Chang and Liaw [5], is the $L(2, 1)$-labeling on directed graphs. Here the definition is the same as in the undirected case in which the distance from vertex x to vertex y is defined as the length of the shortest dipath from x to y. In agreement with the undirected case, the $L(2, 1)$-labeling number $\vec{\lambda}(D)$ of D is the smallest number μ such that D has an $L(2, 1)$-labeling with $\max\{l(v) : v \in V(D)\} = \mu$. It is immediately clear that if G is the underlying graph of the oriented graph $\vec{G}$, then $\vec{\lambda}(\vec{G}) \leq \lambda(G)$. Finally, by extension, for a class $\mathcal{C}$ of digraphs, we denote by $\vec{\lambda}(\mathcal{C})$ the maximum $\vec{\lambda}(D)$ over all $D \in \mathcal{C}$.

In [5] it is proved that $\vec{\lambda}(T) \leq 4$ for any oriented tree T. The general $L(h, k)$-labeling problem is considered in [4] for oriented graphs whose longest dipath is of length at most 2 and some specific results are given in case of oriented trees and bipartite graphs with longest dipath of length 3. In [7] the same problem is considered for those orientations for which the underlying graphs are paths, cycles or trees. Orientations of regular grids are considered in [2], where the values of $\vec{\lambda}$ for squared, triangular and hexagonal oriented grids, are exactly determined. It is worth mentioning two other related works

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that are [11], dealing with the oriented $L(1, 1)$-labeling of Halin graphs, and [6], presenting a self-stabilizing algorithm for $L(2, 1)$-labeling ditrees.

In this paper we approach the problem of determining $\vec{\lambda}$ for the wide class of oriented planar graphs. Our work goes into two different directions. On the one hand we observe that when D is an oriented planar graph, $\vec{\lambda}(D)$ must be a constant. A side effect is a new bound on the chromatic number of the (unoriented) square graph of an oriented planar graph, $\chi(D^2)$. On the other hand, we improve the previous general bounds when focusing on some subclasses of oriented planar graphs, i.e. prisms, Halin graphs and cacti, proving for them very close upper and lower bounds on $\vec{\lambda}$ as follows:

- For the class $\mathcal{P}_n$ of oriented prisms on n vertices we show that $\vec{\lambda}(\mathcal{P}_n) = 5$ if $n \equiv 0 \pmod 3$ and $\vec{\lambda}(\mathcal{P}_n) = 6$ otherwise.
- For the class $\mathcal{H}$ of oriented Halin graphs we prove that $8 \leq \vec{\lambda}(\mathcal{H}) \leq 11$. We improve these bounds for two subclasses: for the class $\mathcal{W}$ of oriented wheels we found the exact result $\vec{\lambda}(\mathcal{W}) = 8$, and for the class $\mathcal{H}'$ of the oriented Halin graphs whose vertices on the cycle are at distance greater than two in the tree we get $7 \leq \vec{\lambda}(\mathcal{H}') \leq 8$.
- For the class $\mathcal{Y}$ of oriented cacti we prove that $6 \leq \vec{\lambda}(\mathcal{Y}) \leq 8$.

2. Preliminary results: background and upper bounds on $\vec{\lambda}$ for oriented planar graphs

For an oriented graph $\vec{G}$, the *underlying graph* G is obtained ignoring all edge orientations. Given an oriented graph $\vec{G}$ and a vertex v we denote by $N^+(v)$ and $N^-(v)$ the set of out- and in-neighbors of v. Let l be a fixed $L(2, 1)$-labeling of $\vec{G}$ and let L be the label set it uses; we denote by $L^+(v)$ and $L^-(v)$ the sets of labels used in $N^+(v)$ and $N^-(v)$, respectively.

A seemingly related concept to the $L(2, 1)$-labeling of an oriented graph is the oriented chromatic number. An *oriented coloring* of $\vec{G} = (V, A)$ is a function $f : V \to \mathbb{N}$ such that:

- $f(u) \neq f(v)$ if $(u, v) \in A$ and,
- for any two arcs (u, v) and (x, y), $\{f(x), f(v)\} \neq \{f(y), f(u)\}$.

The *oriented chromatic number* of $\vec{G}$, $\vec{\chi}(\vec{G})$, is the smallest integer κ such that $\vec{G}$ has an oriented coloring using κ colors.

A direct consequence of the previous definition is that $\vec{\lambda}(\vec{G}) \leq 2(\vec{\chi}(\vec{G}) - 1)$ and the bound is trivially attained for tournaments, i.e. orientations of complete graphs. Furthermore $\vec{G}$, $\vec{\lambda}(\vec{G}) \geq 2(\chi(G) - 1)$ [7]. Thus we have:

Proposition 1. *For any oriented graph $\vec{G}$ it holds:*

$$2\big(\chi(G) - 1\big) \leq \vec{\lambda}(\vec{G}) \leq 2\big(\vec{\chi}(\vec{G}) - 1\big).$$

It is worth mentioning that, in general, Proposition 1 is a quite unsatisfactory result as there can be a huge gap between the claimed lower and upper bounds on $\vec{\lambda}$. However, it will turn out to be helpful in case of oriented planar graphs. Indeed, there are many results concerning $\vec{\chi}(\vec{G})$ for an oriented planar graph $\vec{G}$ according to the maximum degree of its underlying graph G [14,15,12] or to the *girth* g of G [13] (the girth is defined as the length of the shortest cycle in a graph). Combining all these results with Proposition 1, we obtain the following:

Proposition 2. *Let $\vec{G}$ be an oriented planar graph with maximum degree Δ and girth g; then $\vec{\lambda}(\vec{G})$ is a constant and it is upper bounded as follows:*

a. $\vec{\lambda}(\vec{G}) \leq 8\Delta - 13$ *when* $\Delta \leq 8$; $\vec{\lambda}(\vec{G}) \leq 2\Delta + 35$ *when* $9 \leq \Delta \leq 61$ *and* $\vec{\lambda}(\vec{G}) \leq 158$ *otherwise;*
b. $\vec{\lambda}(\vec{G}) \leq 8$ *when* $g \geq 16$; $\vec{\lambda}(\vec{G}) \leq 12$ *when* $g \geq 11$; $\vec{\lambda}(\vec{G}) \leq 22$ *when* $g \geq 7$ *and* $\vec{\lambda}(\vec{G}) \leq 62$ *when* $g \geq 6$.

Observe that Proposition 2.b suggests that the smaller the girth of a graph the larger the value of $\vec{\lambda}$. However, in the following sections we will consider subclasses of planar graphs with small girth requiring also a small $\vec{\lambda}$.

The (unoriented) *square graph* $\vec{G}^2$ of an oriented graph $\vec{G}$ is obtained from $\vec{G}$ by adding new arcs joining all pairs of vertices at (directed) distance 2 in $\vec{G}$ and then eliminating the orientations of the arcs. It is immediate to see that $\chi(\vec{G}^2) \leq \vec{\lambda}(\vec{G})$. It straightly descends the following interesting fact:

Corollary 1. *Given an oriented planar graph $\vec{G}$, its (unoriented) square graph has chromatic number $\chi(\vec{G}^2)$ upper bounded by a constant.*

This result is quite surprising because it states a big difference between the behaviors of the square graphs of planar digraphs and of planar graphs. Indeed, it is well known that, given any planar graph G, $\chi(G^2)$ can be of the order of Δ and the tight upper bound on $\chi(G^2)$ is matter of a huge study (for example, see [15,16]).

3. $L(2, 1)$-labeling of oriented prisms

In this section we determine the exact value of $\vec{\lambda}$ for the planar oriented prism Pr_n isomorphic to the Cartesian product $C_n \times P_2$. It is well known [8] that $\lambda(Pr_n) = 5$ if $n \equiv 0 \pmod 3$ and $\lambda(Pr_n) = 6$ otherwise. The following theorem proves that the same result holds in the case of oriented prisms.

Theorem 1. *Let $\mathcal{P}_n$ be the set of all the orientations of the planar prism Pr_n. It holds $\vec{\lambda}(\mathcal{P}_n) = \lambda(\mathrm{Pr}_n)$.*

Proof. As $\vec{\lambda}(\vec{\mathrm{Pr}}_n) \geq \lambda(\mathrm{Pr}_n)$, to prove the theorem it is sufficient to show that there exists an orientation $\vec{\mathrm{Pr}}_n$ of the prism Pr_n which preserves the two length distances between vertices.

Given Pr_n, let C and C' be its two n length cycles. Cyclically orient all the edges of C clockwise and those of C' anti-clockwise. Then, for each edge $\{v, v'\}$, $v \in C$ and $v' \in C'$, orient it from v to v'. Clearly, two vertices that are connected by a path of length two in Pr_n are still connected by a dipath of length two in $\vec{\mathrm{Pr}}_n$. Thus, $\lambda(\mathrm{Pr}_n)$ and $\vec{\lambda}(\vec{\mathrm{Pr}}_n)$ coincide for each $n \geq 3$. □

4. $L(2, 1)$-labeling of oriented Halin graphs

Here we provide some very close bounds on $\vec{\lambda}$ for oriented Halin graphs and two specific subclasses: oriented wheels and leaf-distant oriented Halin graphs. In particular, for oriented wheels we find the exact value of $\vec{\lambda}$.

A *Halin graph H* is a planar graph constructed from a plane embedding of a tree with at least four vertices and with no vertices of degree 2, by connecting all the leaves with a cycle that passes around the tree in the natural cyclic order defined by the embedding of the tree.

An *n-wheel W_n* is formed by connecting a single vertex c (the central vertex), to all vertices of an n-cycle. Trivially, wheels are Halin graphs.

We define an oriented Halin graph $\vec{H}$ as *leaf distant* if any two vertices on the cycle are at a distance strictly greater than 2 in the tree. In other words, $\vec{H}$ is a leaf distant oriented Halin graph if all the leaves that are children of the same father v are either all in $N^+(v)$ or all in $N^-(v)$.

An *Out-segment* of a vertex v of an oriented Halin graph $\vec{G}$ is a maximal consecutive sequence of vertices in the cycle of $\vec{G}$ such that all its vertices belong to the set $N^+(v)$. An *In-segment* of v is defined in an analog way.

In [11] it is proved that the span of any $L(1, 1)$-labeling of oriented Halin graphs is upper bounded by 6 which implies that $\vec{\lambda}(\mathcal{H}) \leq 12$. In the following we will improve this bound first when dealing with orientations of wheels, and then with orientations of Halin graphs.

Theorem 2. *Let $\mathcal{W}$ be the set of all oriented wheels. It holds $\vec{\lambda}(\mathcal{W}) = 8$.*

The proof will directly follow by the next two lemmas.

Lemma 1. *For every oriented wheel $\vec{W}$ it holds $\vec{\lambda}(\vec{W}) \leq 8$.*

Proof. Consider an arbitrary oriented wheel $\vec{W}$ and let c be its central vertex and C_n its cycle. Observe that any vertex of $N^+(c)$ is connected by a dipath of length two (passing through c) to every vertex of $N^-(c)$ thus it must be $L^+(c) \cap L^-(c) = \emptyset$. We construct an $L(2, 1)$-labeling of the vertices of $\vec{W}$ as follows. Label the central vertex c with 8. Let S be a segment of maximum length and w.l.o.g. suppose it is an Out-segment. We label the vertices of the cycle using labels from 0 to 6 according to the following cases:

Case I: S contains all the vertices of the cycle. Then clearly we can label the cycle using labels $\{0, 1, 2, 3, 4\}$ (see [9,18]).

Case II: S consists of one vertex, meaning that the vertices in the cycle are alternatively from $N^+(c)$ and $N^-(c)$ (then n is even). Then we can easily label the cycle repeatedly using the sequence 0314 if n is a multiple of 4 and labeling the last two vertices with 25 when $n \equiv 2 \pmod 4$.

Case III: S consists of $m \geq 2$ vertices. Starting from S, we proceeding in a clockwise direction, labeling first all the extremes of the In-segments using alternatively labels 6,4 (we label 6 the first vertex following S). Then we label all the extremes of the Out-segments using alternatively labels 0, 2. It is easy to see that it is always possible to complete the labeling of In- and Out-segments using labels 1, 4, 6 and 0, 2, 5, respectively. It remains to label S and we will do it in a counterclockwise fashion. If $m = 2$, we use the sequence 31. If $m \geq 3$ then we label S by repeating the sequence 352, and we modify only the labels of the last few vertices. Namely, if $m \equiv 0 \pmod 3$ we substitute the last 2 by 0 if the second vertex before S is also labeled by 2. If $m \equiv 1 \pmod 3$ we re-label the last four vertices with 3052 or 3520, according to the fact that the second vertex before S is labeled 2 or not. Finally, if $m \equiv 2 \pmod 3$ we re-label the last five vertices with 20530 or 25302, according to the fact that the second vertex before S is labeled 2 or not.

We have $L^+(c) = \{0, 2, 3, 5\}$ and $L^-(c) = \{1, 4, 6\}$ and this is a feasible labeling, indeed: (i) the first vertex after S is always labeled 6; (ii) the second vertex after S is labeled either 4 or 1 or 0, according to the kind of sequence it belongs to; (iii) the vertex rightly before S is labeled either 6 or 4 and (iv) the second vertex before S is labeled with a label among 0, 1, 2, 4, 6. This concludes the proof. □

Lemma 2. *There exists an oriented wheel $\vec{W}$ with $\vec{\lambda}(\vec{W}) \geq 8$.*

Proof. Consider the oriented wheel $\vec{W}$ in Fig. 1. Suppose by contradiction that there is an $L(2, 1)$-labeling l of $\vec{W}$ that uses only labels $0, 1, \ldots, 7$. W.l.o.g. we assume $l(c) = 7$. Therefore we have to label the vertices of the cycle using the labels from 0 to 5. First note that in order to label a dipath of at least 4 vertices we have to use at least three labels at mutual distance at
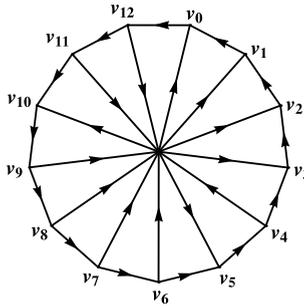
Fig. 1. An oriented wheel $\vec{W}$ for which $\vec{\lambda}(\vec{W}) = 8$.

least two. Observe that $v_3 \rightarrow v_2 \rightarrow v_1 \rightarrow v_0$ is a dipath of vertices of $N^+(c)$ and similarly $v_9 \rightarrow v_8 \rightarrow v_7 \rightarrow v_6$ is a dipath of vertices of $N^-(c)$. Furthermore, recall that the sets of labels $L^+(c)$ and $L^-(c)$ must be disjoint. Consequently, the only way to satisfy all these requirements is to use the two set of labels $\{0, 2, 4\}$ and $\{1, 3, 5\}$ for $N^+(c)$ and $N^-(c)$, and without losing of generality, we assume $L^+(c) = \{0, 2, 4\}$.

Consider now the vertex v_0. As we are forced to use only three labels for the path $v_3 \rightarrow v_2 \rightarrow v_1 \rightarrow v_0$, then $l(v_3) = l(v_0)$. Similarly $l(v_6) = l(v_9)$. If $l(v_0) = l(v_3) = 0$ then $l(v_4) \in \{3, 5\}$. If $l(v_4) = 3$ then there is no feasible label in $\{0, 2, 4\}$ to be assigned to v_5. So let $l(v_4) = 5$, then $l(v_5) = 2$ but then we cannot assign a label to v_6. If $l(v_0) = l(v_3) = 2$ then we are forced to have $l(v_4) = 5$, $l_{(v_5)} = 0$, $l(v_6) = l(v_9) = 3$, $l(v_{10}) = 0$ and $l(v_{11}) = 5$ but then we cannot assign a label to v_{12}. If, finally, $l(v_0) = l(v_3) = 4$ then $l(v_4) = 1$ but then we cannot assign a label to v_5. Thus, in all the cases it is not possible to complete the labeling using only labels from 0 to 7, and this concludes the proof.   □

Now, we consider oriented Halin graphs that are not necessarily wheels. We remind that $\vec{\lambda}(\mathcal{H}) \leq 12$ from the result in [11]. We slightly improve this bound as shown by the next theorem.

Theorem 3. *Let $\mathcal{H}$ and $\mathcal{H}'$ be the sets of all the oriented Halin graphs and leaf distant Halin graphs, respectively. It hold:*

$$8 \leq \vec{\lambda}(\mathcal{H}) \leq 11 \quad and \quad 7 \leq \vec{\lambda}(\mathcal{H}') \leq 8.$$

The proof follows immediately using Theorem 2 and the next three lemmas.

Lemma 3. *For every Halin digraph $\vec{H}$ it holds $\vec{\lambda}(\vec{H}) \leq 11$.*

Proof. Let be given an oriented Halin graph $\vec{H}$ constituted by an oriented tree $\vec{T}$ with at least two inner vertices and an oriented cycle $\vec{C}$ connecting all the leaves of $\vec{T}$. For any oriented tree there is an $L(2, 1)$-labeling of its vertices according to the algorithm in [5] using labels 0, 2, 4. We label the inner vertices of the tree considering only its directed edges, ignoring the arcs of the cycle $\vec{C}$, and using labels 0, 4, 8 instead of 0, 2, 4. To completely define a feasible labeling it remains to set only the labels of the vertices on the cycle. We label these vertices using labels from the set $A = \{0, \ldots, 11\} \setminus \{0, 4, 8\}$. Note that in this way there is no conflict between the label of any vertex on the cycle and the label of an inner vertex of the tree that is at distance two from it. Let us now consider the following six sets of labels: $I_0 = \{2, 5, 9\}$, $I_4 = \{1, 6, 9\}$, $I_8 = \{1, 5, 10\}$, $O_0 = \{3, 6, 10\}$, $O_4 = \{2, 7, 10\}$, $O_8 = \{2, 6, 11\}$. The names of these sets derive from the fact that we will use labels from set I_x (O_x) to label the In(Out)-segments of an inner vertex of the tree labeled with x. Note that this definition of the label sets is consistent with the labeling of the tree and for any inner vertex v $L^+(v) \cap L^-(v) = \emptyset$ as $I_x \cap O_x = \emptyset$ for each $x \in \{0, 4, 8\}$. It is not difficult to check that these six sets have the following properties:

Property 1. *The intersection of any pair of label sets not corresponding to the same label (i.e. either $I_x \cap I_y$, or $I_x \cap O_y$, or $O_x \cap O_y$ with $x \neq y$), consists of exactly one label.*

Property 2. *For any label $x \in A$ the intersection of the set $\{x - 1, x, x + 1\}$ with any of the label sets consists of exactly one label.*

Let S be a segment of maximum length. It is not restrictive to assume that S is an In-segment of a vertex labeled 0. (Otherwise rename the labels in $\vec{T}$ and, if S is an Out-segment, exchange the role of I_0 and O_0.)

Let $P_1, \ldots, P_r = S$ be the sequence of segments in clockwise order starting from the segment that follows S and ending with S. Label the segments in order from P_1 to P_{r-1}, using labels from I_x or O_x according to their definition. Observe that, from Properties 1 and 2, once the segments $P_1, \ldots, P_{i-1}$ have been labeled, it is always possible to label P_i avoiding conflicts between labels and guaranteeing the feasibility of the $L(2, 1)$-labeling by simply alternating the three labels of the corresponding set and starting from a label that does not create a conflict with the labeling of P_{i-1}. Hence, we concentrate our attention on the labeling of S whose sequence of vertices is $s_1, s_2 \ldots, s_m$.
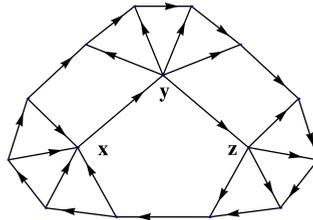
Fig. 2. An oriented leaf distant Halin graph for which $\vec{\lambda} = 7$.

Suppose first $m \geq 3$ and let v, k and w, u be the two vertices that follow and precede S clockwise, respectively. We label the vertices of S once v, k, w and u are labeled, and use labels from I_0 except for s_{m-1} and s_m for which we may use labels from the set $B = I_0 \cup \{7, 11\}$. Observe that $B \cap O_0 = \emptyset$ and this is necessary as we must always have $L^+(0) \cap L^-(0) = \emptyset$.

We start by labeling s_1. In view of Properties 1 and 2 $l(w)$ and $l(u)$ together exclude at most two labels in I_0 and thus it is always possible to label s_1. Also, for any vertex s_i with $i \leq m - 2$ there is at least one possible label in I_0. Thus, we focus on labeling the last two vertices of S. When arriving at s_{m-1} it may be the case that exactly one label a of I_0 is available to label s_{m-1}. If $a \neq l(v)$ then we set $l(s_{m-1}) = a$, otherwise we set $l(s_{m-1}) = 11$. Note that in this latter case if $|S| = 3$ we could have $l(u) = 11$ but then we would have two choices in I_0 for s_{m-1} and thus we can label it using the one different from $l(v)$.

Finally, we arrive at s_m. If $l(s_{m-1}) = 11$ in the worst case, all four vertices s_{m-1}, v, k and s_{m-2} have labels from B and hence are forbidden for s_m. We label s_m with the remaining color. If $l(s_{m-1}) \in I_0$ and $l(v) \neq 10$ then we set $l(v) = 11$, sure that it is feasible. Finally, if $l(s_{m-1}) \in I_0$ and $l(v) = 10$ then we set $l(s_m) = 7$ if $l(k) \neq 7$, otherwise we set $l(s_m) = l(s_{m-1})$ and change the label of s_{m-1} to 11 or 7 according to the label of the vertex preceding s_{m-2}.

Observe that the case $|S| = 2$ can be handled easily using the same argument of above with $s_{m-2} = u$. Finally, if $|S| = 1$ then simply label inner vertices of the tree using 0, 2, 4, label the cycle using 6, 9, 7, 10 alternatively and label its last two vertices with 8, 11 when $n \equiv 2 \pmod 4$.

All the described $L(2, 1)$-labelings are feasible and use the labels from 0 to 11. This concludes the proof. □

As wheels are Halin graphs, from Lemma 2 we have that $\vec{\lambda}(\mathcal{H}) \geq 8$. However, it is not difficult to see that the same result holds for oriented Halin graphs which are not wheels.

Lemma 4. *If $\vec{H}$ is a leaf distant oriented Halin graph then it holds $\vec{\lambda}(\vec{H}) \leq 8$.*

Proof. Observe that the distance greater than two in the tree of any two vertices on the cycle implies that all the leaves that are children of the same father v are either all in $N^+(v)$ or all in $N^-(v)$. Thus, it is sufficient to use only one set of labels for the neighbors of a vertex that are on the cycle. The proof follows using the same labeling technique as in the previous lemma. We label the inner vertices of the tree using labels 0, 3, 6 and set $I_0 = O_0 = \{2, 5, 8\}$, $I_3 = O_3 = \{1, 5, 7\}$, $I_6 = O_6 = \{1, 4, 8\}$. □

Lemma 5. *There exists a leaf distant oriented Halin graph $\vec{H}$ with $\vec{\lambda}(\vec{H}) \geq 7$.*

Proof. Consider the oriented Halin graph $\vec{H}$ in Fig. 2. Suppose by contradiction that labels from 0 to 6 are sufficient. First observe that one between the labels of x, y, z must be different from 0 and 6. Without losing generality suppose this is y. The four neighbors of y in the cycle, must have labels distinct from $l(y) - 1$, $l(y)$, $l(y) + 1$ and $l(x)$. Thus, there are only three labels left for these vertices. Moreover, as they are on a directed path in the cycle, these 3 labels must have pairwise distance at least 2. Consequently, y and its four neighbors must get 4 labels with pairwise difference at least two. The only fourtuple that satisfies this property is 0, 2, 4, 6. This means that $l(x)$ is different from 0, 2, 4 and 6 and again x and its four neighbors must get 4 labels with pairwise difference at least 2. By the uniqueness of the fourtuple we arrive at a contradiction and this concludes the proof. □

5. $L(2, 1)$-labeling of oriented cacti

A *cactus* is a connected graph in which very block (maximal subgraph without a cut-vertex) is an edge or a cycle. Here we provide very close upper and lower bounds for $\vec{\lambda}$ of oriented cacti.

Theorem 4. *Let $\mathcal{Y}$ be the set of all the oriented cacti. It holds $6 \leq \vec{\lambda}(\mathcal{Y}) \leq 8$.*

Lemma 6. *For every oriented cactus $\vec{Y}$ it holds $\vec{\lambda}(\vec{Y}) \leq 8$.*

Proof. The proof is by induction on the number of vertices of the oriented cactus. The base cases $n \leq 3$ are trivially true. So, suppose $n > 3$ and consider an oriented cactus $\vec{Y}$ on n vertices. If $\vec{Y}$ is a unique block then the proof is trivial as it is either an edge or a cycle. Otherwise, if it has at least two blocks then there exists a cut-vertex c of $\vec{Y}$, belonging to at least a block not containing further cut-vertices. Let B_0, B_1, . . . , B_r be the set of all the blocks containing only c as a cut-vertex.
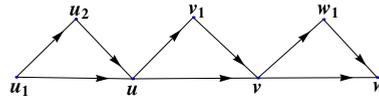
Fig. 3. An oriented cactus for which $\vec{\lambda} = 6$.

If some B_i, $1 \leq i \leq r$ is a cycle of length at least four, then choose in B_i a vertex y not adjacent to c. By induction, we can label $\vec{Y} - \{y\}$ properly with labels from 0 to 8. As the two (labeled) adjacent vertices of y forbid at most three labels each, and the two (labeled) vertices at distance two from y (possibly coinciding, if B_i is a length 4 cycle) forbid at least one color each, we have that $9 - 3 - 3 - 1 - 1 = 1$ label is always available for y. So we can extend the labeling to $\vec{Y}$.

If all B_i are K_2 or triangles for $1 \leq i \leq r$, then we can label the cactus $\vec{Y} - (\cup_{1 \leq i \leq r} B_i) \cup \{c\}$ exploiting the induction. It remains to label all the vertices of $B_i - \{c\}$, for each $1 \leq i \leq r$. Observe that, for all these vertices, the (labeled) cut-vertex c forbids at least three labels, and the two (labeled) adjacent vertices of c in $\vec{Y} - (\cup_{1 \leq i \leq r} B_i) \cup \{c\}$ forbid at most other two labels. It follows that at least $9 - 3 - 1 - 1 = 4$ labels are available for the vertices of $(\cup_{1 \leq i \leq r} B_i) - \{c\}$. Let $c_1 < c_2 < c_3 < c_4$ be these four colors. We can use c_1 and c_3 for the out-neighbors of c, and c_2 and c_4 for the in-neighbors of c. We only have to be careful on the triangle blocks B_i in which the two vertices of $B_i - \{c\}$ must be labeled with labels at distance at least two.

It is easy to see that the produced labeling is feasible and uses labels from 0 to 8. □

Lemma 7. *There exists an oriented cactus $\vec{Y}$ for which $\vec{\lambda}(\vec{Y}) \geq 6$.*

Proof. We exhibit an oriented cactus $\vec{Y}$ for which $\vec{\lambda}$ is at least 6. Consider the cactus in Fig. 3. Suppose, by contradiction, that there is an $L(2, 1)$-labeling l with labels $0, 1, \ldots, 5$. Clearly, if one from u, v is different from 0 and 5 then we arrive at a contradiction as each of them has four neighbors and there are only three possible labels left. Otherwise, without loss of generality we can suppose $l(u) = 0$ and $l(v) = 5$. Then we have $l(v_1) = 2$ or $l(v_1) = 3$, but then both of the pairs u_1, u_2 and w_1, w will get labels differing by just one. This leads to a contradiction and thus $\vec{\lambda} \geq 6$. □

6. Concluding remarks and open problems

In this paper we approach the problem of determining $\vec{\lambda}$ for the wide class of oriented planar graphs providing some lower and upper bounds on $\vec{\lambda}$. For the particular subclass of oriented prisms we completely determine the value of $\vec{\lambda}$, while for the subclasses of oriented Halin graphs and cacti we provide nearly tight bounds for $\vec{\lambda}$. Determining the exact value of $\vec{\lambda}$ for these two subclasses still remains an open problem as well as improving the general bounds for the whole class of oriented planar graphs.

It is worth noting that in the unoriented case there is a strong relationship between λ and the graph's maximum degree Δ, and it efficiently expresses the value of λ (for a wide list of bounds having Δ as a parameter, see for example [3]). However the $L(2, 1)$-labeling problem on oriented graphs presents different issues with respect to the unoriented case and Δ is not an appropriate parameter anymore. The case of oriented planar graphs attests this fact as $\vec{\lambda}$ is bounded by a constant, independently on the maximum degree.

Due to the fact that very few classes of oriented graphs have been investigated, it is not possible yet to identify the most natural graph parameter that efficiently expresses the value of $\vec{\lambda}$ for arbitrary oriented graphs. However, for the class of oriented planar graphs it seems reasonable to suggest that the girth of the underlying graph is in some relation with $\vec{\lambda}$. This is suggested by Proposition 2.b and by the following Conjecture in [2]:

Conjecture 1. *Every oriented planar graph D whose underlying graph has girth $g \geq 5$ has $\vec{\lambda}(D) \leq 5$.*

Nevertheless, this relation does not seem to be as strong as in the unoriented case between λ and Δ. Investigating in this direction is an interesting open problem.

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