

PURE NASH EQUILIBRIA AND BEST-RESPONSE DYNAMICS IN RANDOM GAMES

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ABSTRACT. In finite games, mixed Nash equilibria always exist, but pure equilibria may fail to exist. To assess the relevance of this nonexistence, we consider games where the payoffs are drawn at random. In particular, we focus on games where a large number of players can each choose one of two possible strategies and the payoffs are i.i.d. with the possibility of ties. We provide asymptotic results about the random number of pure Nash equilibria, such as fast growth and a central limit theorem, with bounds for the approximation error. Moreover, by using a new link between percolation models and game theory, we describe in detail the geometry of pure Nash equilibria and show that, when the probability of ties is small, a best-response dynamics reaches a pure Nash equilibrium with a probability that quickly approaches one as the number of players grows. We show that various phase transitions depend only on a single parameter of the model, that is, the probability of having ties.

1. INTRODUCTION

1.1. Background and motivation. A *pure Nash equilibrium* (PNE) in a normal form game is a profile of strategies (one for each player) such that, given the choice of the other players, no player has an incentive to make a different choice. In other words, deviations from an equilibrium are not profitable for any player. This concept of equilibrium, although quite simple and powerful, has the drawback that not every game admits PNE. The idea of *mixed strategy*—i.e., a probability distribution over a player’s strategy set—goes back to [Borel \(1921\)](#) and [von Neumann \(1928\)](#). John Nash showed that, if mixed strategies are allowed and the choice criterion is the expected payoff with respect to their product, then any game with a finite number of players and strategies admits an equilibrium ([Nash, 1950, 1951](#)). As before, a mixed equilibrium is a profile of mixed strategies that does not allow profitable deviations.

Although the definition and properties of mixed strategies and mixed equilibria are clear, their interpretation is far from unanimous. Section 3.2 of [Osborne and Rubinstein \(1994\)](#), dedicated to the interpretation of mixed equilibria, has paragraphs individually signed by each of the two authors, since they could not reach an agreement. In general, PNE have a stronger epistemic foundation than mixed equilibria. As mentioned before, the main problem of PNE is existence.

Some authors have tried to frame this problem in a stochastic way: given a set of players and a set of strategies for each player, if payoffs are drawn at random, what is the probability that the game admits PNE? More precisely, what is the distribution of the number of PNE in a game with random payoffs? The answer to this question clearly depends on the way random payoffs are drawn. In any case, for a fixed number of players

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and strategies, the answer is computationally daunting. It is for this reason that some papers have chosen to investigate the problem from an asymptotic viewpoint; that is, they have looked at the limit distribution of the number of PNE as the number of either strategies or players increases to infinity.

The basic common assumption of much of this literature is that the distribution of the random payoffs is nonatomic and payoff profiles are independent. Under these hypotheses, the probability that two payoffs coincide is zero and, as a consequence, calculations are significantly simplified.

It is well-known that PNE are hard to compute (Daskalakis et al., 2009). One way to address the issue is to devise iterative procedures that converge to a PNE. For instance, some adaptive procedures start from a strategy profile and allow a single player (picked *at random*) to choose a different strategy. In better-response dynamics the chosen player picks uniformly among the strategies that guarantee a higher payoff than the one the player presently has, given the strategies of the other players. In *best-response dynamics* (BRD) the player chooses one of the strategies that give the highest possible payoff (there could be ties), given the strategies of the other players. When each player has only two available strategies, the two procedures coincide. In either scheme, if no such strategy exists, the player will not move and a different player is chosen at random. When a new strategy profile is reached, the process is repeated. If we reach a profile for which no player has a profitable deviation, then the process has reached a PNE. The question is whether, starting from any strategy profile, a PNE is reached. In general the answer is negative: first, because a game may fail to have PNE; second, due to the fact that even when PNE exist, players may be trapped in a subset of vertices that does not contain a PNE. One way to determine the severity of this failure to reach a PNE via best-response dynamics is to examine games with random payoffs.

1.2. Our contribution. In the present paper we consider games in which the number N of players is large, each player has two strategies, and payoffs are random. The main novelty of our approach is to show the strict relation between games with random payoffs and percolation, and to provide analytic results rather than simulations. In games with random payoffs, a significant part of the existing literature has focused on the number of PNE when the payoffs are i.i.d. with absolutely continuous distribution with respect to the Lebesgue measure. Here we extend this analysis to the case where ties are allowed. We also discuss the behavior of BRD. In particular, we provide results that concern not only the number of PNE, but also how easily they can be reached via BRD. The main tool for this analysis is a correspondence between a random oriented graph that represents our random game and a suitable percolation graph.

A main feature of the present work is that we dispense with the assumption of nonatomic distribution of the payoffs and therefore allow ties to exist. We show that the probability of ties plays a crucial role in many ways. For example, it determines the asymptotic distribution of the number of PNE. Moreover, we use tools from percolation theory to describe the geometry of the set of PNE, which also depends on the probability of ties. This description permits us to analyze the performance of BRD. This has been extensively done in the literature for the class of potential games. In our paper we can show

that, asymptotically in the number of players, with high probability BRD converges to a PNE, if the probability of ties is positive, but small (less than 0.68).

As mentioned before, the probability of ties in the payoffs, which we call α , is the fundamental parameter in this model. Different values of α produce different possible behaviors in the number of PNE. We will show that, for as long as α is positive, the game has typically around $(1 + \alpha)^N$ PNE. For all N large enough and α small enough we have that BRD converges to a PNE (Theorems 3.3 and 3.4 below). Moreover, for all large N , if α is strictly less than $1/2$, then all PNE are reachable via BRD from any deterministic starting point. Conversely, some of them are unreachable when α is at least $1/2$. Furthermore, when α is positive, Theorem 2.3 shows a concentration of the number of PNE around $(1 + \alpha)^N$ and establishes a central limit theorem (CLT). After a draft of this paper was ready, we learned that a similar CLT was obtained in Raič (2003). Our approach is different and relies on a Poisson approximation, which we believe is of independent interest.

To illustrate this phenomenon, we plot in Fig. 1 the case where the payoffs take only the values $\{-1, 1\}$ with equal probability (notice that $\alpha = 0.5$ in this context). The average number of PNE exactly fits the curve $(1.5)^N$, confirming our prediction. Moreover, we

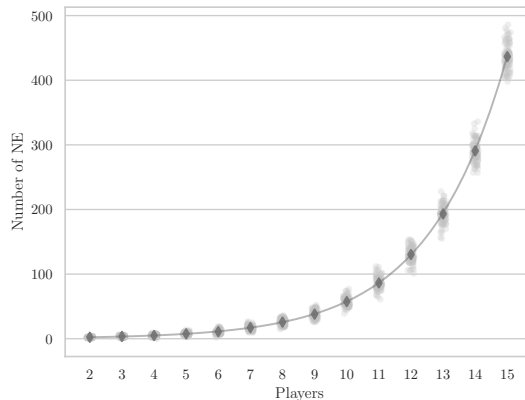


FIGURE 1. Number of PNE for $2 \leq N \leq 15$, $\alpha = 0.5$, with 100 trials per N . Diamond markers represent average number per value of N , and the curve $(1.5)^N$ is included for comparison.

are able to quantify the fluctuations (see Theorem 2.3) which are of the order $(1 + \alpha)^{N/2}$. Finally, the number of PNE, once properly rescaled, rapidly converges to a standard normal (see Fig. 2). We emphasize that our results depend on the payoff distributions only through the parameter α , and remain applicable even when the payoff distributions vary across players.

1.3. Related work. As mentioned before, several papers have considered aspects related to the number of PNE in games with random payoffs. In many of the papers that we consider below, and unless otherwise stated, the random payoffs are i.i.d. from a continuous distribution.

Goldman (1957) considered zero-sum two-person games and showed that the probability of having a PNE goes to zero as the number of strategies grows. He also briefly

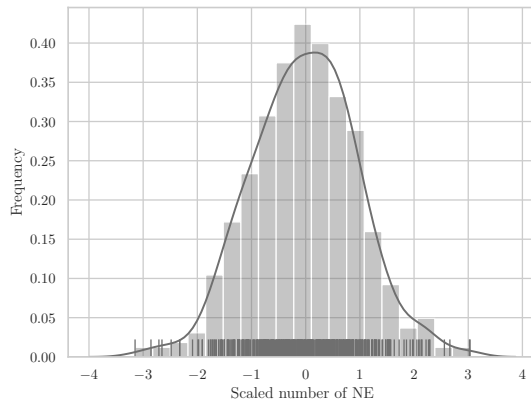


FIGURE 2. CLT result for $N = 15$, $\alpha = 0.9$, with 500 trials.

mentioned the case of payoffs with a Bernoulli distribution. Goldberg et al. (1968) considered general two-person games and showed that the probability of having at least one PNE converges to $1 - e^{-1}$ as the number of strategies diverges. Drescher (1970) generalized this result to the case of an arbitrary finite number of players.

More recent papers have looked at the asymptotic distribution of the number of PNE. Powers (1990) showed that, when the number of strategies of at least two players goes to infinity, the distribution of the number of PNE converges to a $\text{Poisson}(1)$.

Stanford (1995) derived an exact formula for the distribution of the number of PNE in random games and obtained the result in Powers (1990) as a corollary. Stanford (1996) dealt with the case of two-person symmetric games and obtained Poisson convergence for the number of both symmetric and asymmetric PNE. In all the above models, the expected number of PNE is in fact 1. Under different hypotheses, this expected number diverges. For instance, Stanford (1997, 1999) showed that this is the case for games with vector payoffs and for games of common interest, respectively. In Stanford (1999) both strictly and weakly ordinal preferences were studied. Raič (2003) used the Chen-Stein method to bound the distance between the distribution of the normalized number of PNE and a normal distribution. His result does not assume continuity of the payoff distributions.

Rinott and Scarsini (2000) weakened the hypothesis of i.i.d. payoffs; that is, they assumed that payoff vectors corresponding to different strategy profiles are i.i.d., but they allowed some dependence within the same payoff vector. In this setting, they proved asymptotic results when either the number of players or the number of strategies diverges. More precisely, if each payoff vector has a multinormal exchangeable distribution with correlation coefficient ρ , then the following hold: for ρ negative the number of PNE goes to zero in probability, for $\rho = 0$ it converges to a $\text{Poisson}(1)$, and for ρ positive it diverges and a CLT holds.

Takahashi (2008) considered the distribution of the number of PNE in a random game with two players, conditionally on the game having nondecreasing best-response functions. This assumption greatly increases the expected number of PNE. Daskalakis et al. (2011) extended the framework of games with random payoffs to graphical games. Strategy profiles are vertices of a graph and players' strategies are binary, like in our model.

Moreover, their payoff depends only on their strategy and the strategies of their neighbors. The authors studied how the structure of the graph affects existence of PNE and they examined both deterministic and random graphs.

The issue of solution concepts in games with random payoffs has been explored by various authors in different directions. For instance, Cohen (1998) studied the probability that Nash equilibria (both pure and mixed) in a finite random game maximize the sum of the players' payoffs. Pei and Takahashi (2019) devoted their attention to rationalizable strategies in two-person games with random payoffs and performed an asymptotic analysis in the number of strategies.

Finding a PNE in a game is PPAD-complete (Daskalakis et al. (2009)). Therefore, given this computational difficulty, several learning procedures have been proposed to reach an equilibrium by playing the game several times (see, e.g., Tardos and Vazirani, 2007, Blum and Mansour, 2007). Probably the simplest such procedure is BRD. This approach has been taken, among others, by Blume (1993), Young (1993), Friedman and Mezzetti (2001), Takahashi and Yamamori (2002) and Fabrikant et al. (2013). The main problem that arises is that BRD is guaranteed to converge to a PNE only when the game is of some specific type, for instance, a potential game (Monderer and Shapley (1996)). Bernheim (1984) introduced the concept of point rationalizable strategies, i.e., strategies that survive iterated elimination of never best responses against pure strategies.

The performance of BRD in randomly drawn potential games has been studied in Coucheney et al. (2014), Durand and Gaujal (2016) and Durand et al. (2019). To be able to deal also with games for which BRD does not converge to a PNE, Goemans et al. (2005) defined the concept of *sink equilibria* (which in this paper are called traps). A trap is a strongly connected set of two or more vertices with no edges leading out of the set. If players are selected at random and asked to choose a best response, the process may eventually reach a sink equilibrium and wander on the game's components forever. Christodoulou et al. (2012) considered a similar model, focusing on the rate of convergence to approximate solutions of the game. Dütting and Kesselheim (2017) considered BRD in the context of combinatorial auctions.

The idea of generating games at random to check properties of learning procedures was used by Galla and Farmer (2013) and, more recently, by Pangallo et al. (2019), who studied—mainly through simulations—the behavior of various learning procedures in games whose payoffs are drawn at random from a multinormal distribution.

Some of our results are proved by using a connection with percolation theory, a field introduced by Broadbent and Hammersley (1957). Since then, the theory has developed very quickly and has become very important in both the mathematics and physics communities. For a general account on percolation, see Grimmett (1999). We will focus on percolation on the hypercube, and will use classical results by Erdős and Spencer (1979) and Bollobás (2001), as well as more recent results by McDiarmid et al. (2020). The connection between percolation and random oriented graphs has been studied, for instance, in an unpublished manuscript by Linusson (2009), in the case where the graph is fully oriented, i.e., there are no ties in the payoffs.

1.4. Organization of the paper. Section 2 deals with the number of PNE in a random game. Section 3 studies the behavior of BRD in these games. The interaction between

games with random payoffs and percolation is expounded in Section 4. Section 5 contains the proofs.

2. NUMBER OF NASH EQUILIBRIA IN RANDOM GAMES

We first introduce some notation that will be adopted throughout the paper. We consider a game

$$\Gamma = ([N], (S_i)_{i \in [N]}, (g_i)_{i \in [N]}), \quad (2.1)$$

where $[N] := \{1, \dots, N\}$ is the set of players and S_i is the set of strategies of each player $i \in [N]$. We set $S = \times_{i \in [N]} S_i$, and we let $g_i: S \rightarrow \mathbb{R}$ be the payoff function of player i . For each $\mathbf{s} = (s_1, \dots, s_N) \in S$, we call \mathbf{s}_{-i} the strategy profile of all players except i .

Definition 2.1. A strategy profile \mathbf{s}^* is a PNE of the game Γ if for all $i \in [N]$ and for all $s_i \in S_i$ we have

$$g_i(\mathbf{s}^*) \geq g_i(s_i, \mathbf{s}_{-i}^*). \quad (2.2)$$

The set of PNE of Γ is denoted by $\text{PNE}(\Gamma)$.

A strategy profile \mathbf{s}^* is a *strict pure Nash equilibrium* (SPNE) if for all $i \in [N]$ and for all $s_i \in S_i$ we have

$$g_i(\mathbf{s}^*) > g_i(s_i, \mathbf{s}_{-i}^*). \quad (2.3)$$

The set of SPNE of Γ is denoted by $\text{SPNE}(\Gamma)$.

Our goal is to examine some generic properties of games with binary strategies and a large number of players. To achieve this, we assume that the payoffs of our game are drawn at random. To this end, consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, on which the following sequence of random games is defined. Let Ξ_N be a game with N players, $S_i = \{0, 1\}$ for each $i \in [N]$, and random i.i.d. payoffs. In particular, for each $\mathbf{s} \in S$, the payoff $g_i(\mathbf{s})$ is the realization of a random variable $Z_i^{\mathbf{s}}$, and the random variables $(Z_i^{\mathbf{s}})_{i \in [N], \mathbf{s} \in S}$ are i.i.d.. The symbol Z denotes a generic independent copy of $Z_i^{\mathbf{s}}$.

We also define

$$\alpha := \mathbf{P}(Z_1 = Z_2), \quad \beta := \mathbf{P}(Z_1 < Z_2) = \frac{1 - \alpha}{2}, \quad (2.4)$$

where Z_1 and Z_2 are i.i.d. copies of Z . As we will see, all of the results in the paper will depend on α . Most of the existing literature deals with the case $\alpha = 0$. In what follows we will show the role of α in various phase transitions. In particular, we are interested in the asymptotic behavior of $\text{PNE}(\Xi_N)$ and $\text{SPNE}(\Xi_N)$.

For any set A , the symbol $\text{card}(A)$ denotes its cardinality. Observe that the probability with which a strategy profile \mathbf{s} is a PNE is $(1 - \beta)^N$, and the probability that it is an SPNE is β^N . As a consequence, for any $N \geq 1$ we have

$$\mathbf{E}[\text{card}(\text{PNE}(\Xi_N))] = 2^N (1 - \beta)^N, \quad (2.5)$$

$$\mathbf{E}[\text{card}(\text{SPNE}(\Xi_N))] = 2^N \beta^N. \quad (2.6)$$

This implies that the expected number of PNE is always 1 when $\alpha = 0$ and diverges when $\alpha > 0$. The following theorems provide sharp description of the asymptotic behavior of the number of PNE and SPNE.

Theorem 2.2 (Behavior of strict pure Nash equilibria). *Consider a sequence of random games Ξ_N .*

(a) If $\alpha = 0$, then, for all $k \in \mathbb{N} \cup \{0\}$, we have

$$\lim_{N \rightarrow \infty} \mathbb{P}(\text{card}(\text{SPNE}(\Xi_N)) = k) = \frac{e^{-1}}{k!}. \quad (2.7)$$

(b) If $\alpha > 0$, then

$$\mathbb{P}\left(\lim_{N \rightarrow \infty} \text{card}(\text{SPNE}(\Xi_N)) = 0\right) = 1. \quad (2.8)$$

Remark 2.1. When $\alpha = 0$, the numbers of PNE and of SPNE are almost surely equal, as any two payoffs are almost surely different. In this case, convergence of the number of PNE to a Poisson distribution as the number of players increases was proved by [Arratia et al. \(1989\)](#), [Rinott and Scarsini \(2000\)](#) for any fixed number of strategies.

Conversely, when $\alpha > 0$, the number of both PNE and SPNE have radically different behavior. This fact will be better described in [Theorems 2.3](#) and [3.3](#) below.

Call Φ the cumulative distribution function of a standard normal random variable.

Theorem 2.3 (CLT for pure Nash equilibria). *Assume that the law of Z has atoms, i.e., $\alpha > 0$. Then there exists a constant $K_\alpha > 0$ which depends only on α , such that*

$$\sup_x \left| \mathbb{P}\left(\frac{\text{card}(\text{PNE}(\Xi_N)) - (1 + \alpha)^N}{(1 + \alpha)^{N/2}} \leq x\right) - \Phi(x) \right| \leq K_\alpha N \max\left(\frac{(1 + \alpha)}{2}, \frac{1}{(1 + \alpha)^{1/2}}\right)^N. \quad (2.9)$$

Remark 2.2. Call $\text{Norm}(\mu, \sigma^2)$ the normal distribution with mean μ and variance σ^2 . Define the Kolmogorov distance κ between two probability measures \mathbb{P}, \mathbb{Q} on \mathbb{R} as

$$\kappa(\mathbb{P}, \mathbb{Q}) := \sup_{x \in \mathbb{R}} |\mathbb{P}((-\infty, x]) - \mathbb{Q}((-\infty, x])|. \quad (2.10)$$

With an abuse of language, we will speak of Kolmogorov distance between two random variables to indicate the distance between their corresponding laws. Hence, [Eq. \(2.9\)](#) is equivalent to

$$\kappa(\text{card}(\text{PNE}(\Xi_N)), \text{Norm}((1 + \alpha)^N, (1 + \alpha)^N)) \leq K_\alpha N \max\left(\frac{(1 + \alpha)}{2}, \frac{1}{(1 + \alpha)^{1/2}}\right)^N. \quad (2.11)$$

A direct consequence of [Theorem 2.3](#) is that the number of PNE grows exponentially in N , whenever $\alpha > 0$. The following is a more precise statement.

Theorem 2.4. *If $\alpha > 0$, then*

$$\mathbb{P}\left(\lim_{N \rightarrow \infty} \frac{\text{card}(\text{PNE}(\Xi_N))}{(1 + \alpha)^N} = 1\right) = 1. \quad (2.12)$$

3. BEST-RESPONSE DYNAMICS

We focus on games with N players and 2 possible strategies per player. We associate to any such game Γ the graph $\mathcal{H}_N = (\mathcal{V}_N, \mathcal{E}_N)$, where the set of vertices is the set of strategy profiles, i.e., $\mathcal{V}_N = S$, and two vertices \mathbf{s}, \mathbf{t} are connected by an edge in \mathcal{E}_N if and only if

$$s_i \neq t_i \text{ for exactly one } i \in [N] \text{ and } s_j = t_j \text{ for all } j \neq i. \quad (3.1)$$

In this case we write $\mathbf{s} \sim_i \mathbf{t}$. Moreover, we write $\mathbf{s} \sim \mathbf{t}$ if $\mathbf{s} \sim_i \mathbf{t}$ for some $i \in [N]$ and say that \mathbf{s}, \mathbf{t} are *neighbors*. For each pair \mathbf{s}, \mathbf{t} of neighbors, call $[\mathbf{s}, \mathbf{t}]$ the edge connecting them. The vertex $(0, 0, \dots, 0) \in \mathcal{V}_N$ is denoted by $\mathbf{0}$.

Definition 3.1. Given a strategy profile $\mathbf{s} \in \mathcal{V}_N$, define

$$\mathcal{D}(\mathbf{s}) := \{\mathbf{t} \in \mathcal{V}_N : \exists i \in [N] \text{ such that } \mathbf{s} \sim_i \mathbf{t} \text{ and } g_i(\mathbf{t}) > g_i(\mathbf{s})\} \quad (3.2)$$

to be the set of strategy profiles that are *strictly profitable deviations* from \mathbf{s} .

We define recursively a discrete-time Markov chain $\text{Brd}(\Gamma, \mathbf{s}) = (\text{Brd}_k(\Gamma, \mathbf{s}))_{k \in \mathbb{N} \cup \{0\}}$ on \mathcal{V}_N as follows. The process starts at $\mathbf{s} \in \mathcal{V}_N$, i.e., $\text{Brd}_0(\Gamma, \mathbf{s}) = \mathbf{s}$. At time $k + 1$, independently of the history of the process, pick an element uniformly at random from the random set $\mathcal{D}(\text{Brd}_k(\Gamma, \mathbf{s}))$, say \mathbf{w} , and set $\text{Brd}_{k+1}(\Gamma, \mathbf{s}) = \mathbf{w}$. Define

$$\text{BR}^{(N)} := \text{Brd}(\Xi_N, \mathbf{0}). \quad (3.3)$$

Finally, we say that $\text{Brd}(\Gamma, \mathbf{s})$ converges to a PNE if there exists some $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $\mathcal{D}(\text{Brd}_k(\Gamma, \mathbf{s})) = \emptyset$.

We associate to each random game Ξ_N a partially oriented random graph $\overrightarrow{\mathcal{H}}_N^\beta = (\mathcal{V}_N, \overrightarrow{\mathcal{E}}_N)$ as follows. Let $\mathbf{s} \sim_i \mathbf{t}$; then the oriented edge $[\mathbf{s}, \mathbf{t}]$ from \mathbf{s} towards \mathbf{t} is in $\overrightarrow{\mathcal{E}}_N$ if and only if $Z_i^{\mathbf{s}} < Z_i^{\mathbf{t}}$. An unoriented edge connects \mathbf{s} and \mathbf{t} in $\overrightarrow{\mathcal{H}}_N^\beta$ if and only if $Z_i^{\mathbf{s}} = Z_i^{\mathbf{t}}$. A similar representation has been used by Young (1993) and Candogan et al. (2011) in the setting of deterministic games. Obviously, in our context, where the game Ξ_N has random payoffs, the partial orientation of $\overrightarrow{\mathcal{H}}_N^\beta = (\mathcal{V}_N, \overrightarrow{\mathcal{E}}_N)$ is itself random. If the law of Z is nonatomic, then the probability that two payoffs coincide is zero. Therefore, $\overrightarrow{\mathcal{H}}_N^{1/2}$ is a ‘proper’ random orientation of \mathcal{H}_N , where each edge is independently oriented in one direction or the other with probability 1/2. If, on the other hand, the law of Z has atoms, then $\text{P}(Z_i^{\mathbf{s}} = Z_i^{\mathbf{t}}) > 0$, so with positive probability some edges have no orientation.

Definition 3.2. We say that \mathbf{t} is *directly accessible* from \mathbf{s} if the oriented edge $[\mathbf{s}, \mathbf{t}] \in \overrightarrow{\mathcal{E}}_N$. We say that \mathbf{t} is *accessible* from \mathbf{s} if there exists a finite sequence $\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_k$ such that $\mathbf{s} = \mathbf{s}_0$, $\mathbf{t} = \mathbf{s}_k$ and, for all $i \in \{0, \dots, k-1\}$, we have $[\mathbf{s}_i, \mathbf{s}_{i+1}] \in \overrightarrow{\mathcal{E}}_N$.

Notice that \mathbf{t} is directly accessible from \mathbf{s} if and only if \mathbf{t} is a profitable deviation from \mathbf{s} for some player i . Definition 3.2 has a natural interpretation in terms of $\text{BR}^{(N)}$. Recall that this process starts at $\mathbf{0}$. Then \mathbf{t} is accessible from $\mathbf{0}$ if and only if there is a positive probability that $\text{BR}^{(N)}$ reaches \mathbf{t} .

An example with three players is given in Fig. 3, where the orientation of the edges is induced by the payoffs in the accompanying table. The absence of ties produces a complete orientation of the hypercube. The black edges are the possible paths of a BRD starting at the vertex $(0, 0, 0)$. The two vertices represented as filled circles, i.e., $(1, 1, 1)$ and $(0, 0, 1)$, are the two PNE of the game. Each of them can be reached, with positive probability, by $\text{BR}^{(N)}$. Theorem 3.4 below proves that for all large enough N this is always the case when α is small enough.

Our next result shows the existence of a sharp phase transition in the accessibility of PNE. Roughly speaking, as the mass of the atoms in the distribution of Z grows, so too does the number of PNE, though some may not be accessible from $\mathbf{0}$. Hence, in this case,

- (a) for the case $0 \leq \alpha < 1/2$, we can use the first Borel-Cantelli lemma and conclude that there exists a finite random N^* such that for all $N \geq N^*$ each of the PNE in Ξ_N is potentially reachable by $\text{BR}^{(N)}$;
- (b) for $\alpha = 1/2$, with positive probability there exist PNE that are not reachable by $\text{BR}^{(N)}$;
- (c) for $\alpha > 1/2$, the number of PNE that are not reachable by $\text{BR}^{(N)}$ grows to infinity, with probability approaching 1.

Therefore, we have an interesting phase transition at $\alpha = 1/2$.

To complete the picture, we give a result about the convergence of $\text{BR}^{(N)}$. We use the notation $\lfloor x \rfloor$ for the greatest integer less than or equal to x and $\lceil x \rceil$ for the least integer greater than or equal to x . The following theorem shows that $\text{BR}^{(N)}$ converges to a PNE if α is positive but smaller than a threshold.

Theorem 3.4 (Convergence of $\text{BR}^{(N)}$). *If $\alpha > 0$ satisfies*

$$\left\lfloor -\frac{1}{\log_2(1/2 + \alpha/2)} \right\rfloor \leq 3, \quad (3.8)$$

i.e., if $0 < \alpha < 2^{3/4} - 1 \approx 0.68$, then

$$\sum_{N=1}^{\infty} \mathbb{P} \left(\text{BR}^{(N)} \text{ does not converge to a PNE} \right) < \infty. \quad (3.9)$$

Using the first Borel-Cantelli lemma we conclude that there exists a finite random N^* such that for all $N \geq N^*$ the process $\text{BR}^{(N)}$ converges to a PNE. In the proof, we use the fact that $\text{BR}^{(N)}$ fails to converge to a PNE if and only if it enters a trap, which is defined as follows.

Definition 3.5. An oriented graph is *strongly connected* if every vertex is accessible from every other vertex. A *trap* is a strongly connected subgraph $\vec{\mathcal{T}}$ of $\vec{\mathcal{H}}_N^\beta$ with two or more vertices, such that, for all $\mathbf{s} \in \mathcal{V}(\vec{\mathcal{T}})$ and all $\mathbf{t} \notin \mathcal{V}(\vec{\mathcal{T}})$, we have that \mathbf{t} is not accessible from \mathbf{s} , where $\mathcal{V}(\vec{\mathcal{T}})$ is the vertex set of $\vec{\mathcal{T}}$.

For any trap $\vec{\mathcal{T}}$, the following holds:

$$\mathbb{P} \left(\exists j \geq k \text{ such that } \text{BR}_j^{(N)} \notin \mathcal{V}(\vec{\mathcal{T}}) \mid \text{BR}_k^{(N)} \in \mathcal{V}(\vec{\mathcal{T}}) \right) = 0. \quad (3.10)$$

Notice that a trap contains at least four vertices, by a simple parity argument, which is described as follows. Define the Hamming distance $h: S \times S$ as

$$h(\mathbf{s}, \mathbf{t}) := \text{card}\{i: s_i \neq t_i\}, \quad (3.11)$$

which, for binary vectors is equal to $\sum_{i \in [N]} |s_i - t_i|$. Each vertex \mathbf{s} of \mathcal{H}_N is called even if $h(\mathbf{s}, \mathbf{0})$ is even, and is called odd otherwise. If \mathbf{s} is even, all its neighbors are odd, and vice versa. Hence, the smallest cycles in this graph are of length 4. As a trap must include a cycle because it is strongly connected in $\vec{\mathcal{H}}_N^\beta$, a trap contains at least four vertices.

The simulation result in Fig. 4 illustrates the phenomenon described by Theorem 3.4 for $\alpha < 0.68$.

Remark 3.1. This paper deals with PNE of games in normal form. The definition of this concept requires the knowledge of players' preferences over strategy profiles. For any two

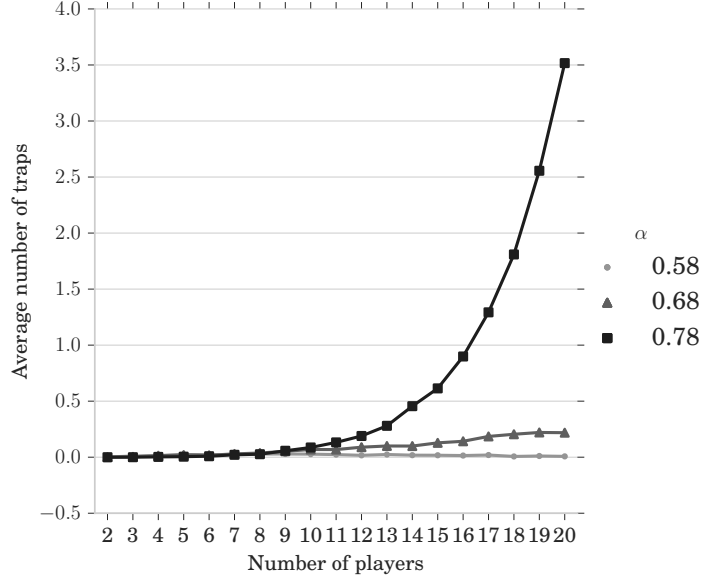


FIGURE 4. Average number of traps found in games ranging from $N = 2$ to 20 and $\alpha = 2^{3/4} - 1.1, 2^{3/4} - 1, 2^{3/4} - 0.9$, with 1500 trials per combination of α and N .

profiles \mathbf{s}, \mathbf{t} , with $\mathbf{s} \sim_i \mathbf{t}$, player i either prefers \mathbf{s} to \mathbf{t} or prefers \mathbf{t} to \mathbf{s} or is indifferent between the two. Any numerical representation of these preferences is invariant with respect to strictly increasing transformations. When dealing with random games, given the symmetry induced by the hypothesis of i.i.d. payoffs, what is relevant to determine the behavior of PNE is the probability that a certain preference exists between two strategy profiles that the player can choose from. This implies that payoff distributions that are quite different lead to the same behavior of PNE if they have the same parameter α . Consider the following examples.

- If the distribution of Z is a mixture of an absolutely continuous distribution and a degenerate distribution with weights $(1 - \gamma)$ and γ , respectively, then $\alpha = \gamma^2$.
- If $Z \sim \text{Bernoulli}(\theta)$, then $\alpha = \theta^2 + (1 - \theta)^2$.
- If $Z \sim \text{Poisson}(\lambda)$, then

$$\alpha = \sum_{k=0}^{\infty} \frac{e^{-2\lambda} \lambda^{2k}}{(k!)^2} = e^{-2\lambda} I_0(2\lambda), \quad (3.12)$$

where I_0 is the modified Bessel function of the first kind (Watson, 1995, Eq. 2, p. 77).

For each value of $\theta \in [0, 1]$, there exist γ and λ that produce the same value of α . Notice that if $Z \sim \text{Bernoulli}(\theta)$, then the possible values of α are the interval $[1/2, 1]$, whereas both the mixture and the Poisson distribution can cover the whole interval $(0, 1]$ of values of α . In the case of mixture, also the case $\alpha = 0$ can be covered, and the proof follows directly from the definition of the model.

For the Poisson case, we reason as follows. Using the integral representation of the

modified Bessel function (Watson, 1995, Eq. 4, p. 181), we obtain

$$\frac{d}{d\lambda} e^{-2\lambda} I_0(2\lambda) = \frac{1}{\pi} \frac{d}{d\lambda} \int_0^\pi e^{2\lambda(\cos(t)-1)} dt = \frac{1}{\pi} \int_0^\pi 2(\cos(t) - 1) e^{2\lambda(\cos(t)-1)} dt. \quad (3.13)$$

Clearly the contents of this integral are negative for $0 < \lambda < \pi$, so $e^{-2\lambda} I_0(2\lambda)$ is decreasing in λ . If $\lambda = 0$, then $\alpha = 1$, while for λ diverging to infinity we have that α converges to 0. By a simple continuity argument we see that all values of α can be covered, and to each $\alpha \in (0, 1]$ there exists exactly one value of λ satisfying Eq. (3.12).

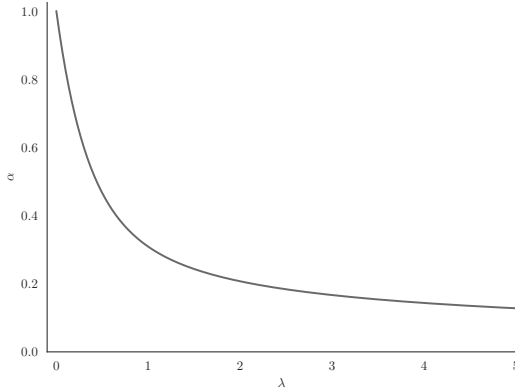


FIGURE 5. α as a function of λ .

4. BOND PERCOLATION

Most of the proofs of the results in Sections 2 and 3 require some tools from percolation theory. The main step to establish the connection between random games and percolation is to construct a coupling with the following property. The set of strategies that are accessible by $\text{BR}^{(N)}$ coincides with the connected component containing $\mathbf{0}$ of a percolation on the hypercube. We then use results about the geometry of this cluster to infer limit theorems for the number of Nash equilibria and the BRD. Our results rely on McDiarmid et al. (2020). Whenever we mention a connected component of the percolation that has a certain property, we refer to the *largest* connected component that satisfies that property. Moreover, we use the term *giant* component to refer to the connected component with the largest number of vertices. Of course, this definition makes sense only when such a component is unique, which is the case in the sequel, for all N large enough.

Independent bond percolation on \mathcal{H}_N is defined as follows. For each edge in \mathcal{H}_N , flip an independent coin having probability β of showing heads. If the toss shows heads, then declare the edge to be *open*; otherwise the edge is *closed*. The random subgraph of \mathcal{H}_N obtained by deleting the closed edges is called percolation; it includes all vertices in \mathcal{V}_N , but could be disconnected. This model allows us to give a detailed description of the geometry of PNE.

The next result relates $\vec{\mathcal{L}}_N^{\beta, \mathbf{0}}$ to a percolation on the hypercube.

Proposition 4.1. *For any $\beta \in [0, 1/2]$ there exists a percolation \mathcal{B}_N^β that satisfies the following property: the vertex set of its connected component that contains $\mathbf{0}$, denoted by $\mathcal{L}_N^{\beta, \mathbf{0}}$, coincides with $\vec{\mathcal{L}}_N^{\beta, \mathbf{0}}$.*

Proof. First, we define the event

$$\{\mathbf{r} \rightarrow \mathbf{t}\} := \{[\overrightarrow{\mathbf{r}, \mathbf{t}}] \in \overrightarrow{\mathcal{E}}_N\}. \quad (4.1)$$

Since each player has only two strategies, we have that $\{\mathbf{r} \rightarrow \mathbf{t}\}$ is independent of $\{\mathbf{u} \rightarrow \mathbf{w}\}$ for every $\{\mathbf{u} \rightarrow \mathbf{w}\} \notin \{\{\mathbf{r} \rightarrow \mathbf{t}\}, \{\mathbf{t} \rightarrow \mathbf{r}\}\}$.

For any subset $\mathcal{U} \subseteq \mathcal{V}_N$, we call $\overrightarrow{\partial\mathcal{U}}$ the set of vertices in $\mathcal{U}^c = \mathcal{V}_N \setminus \mathcal{U}$ that are out-neighbors of some elements in \mathcal{U} , that is,

$$\overrightarrow{\partial\mathcal{U}} := \{\mathbf{w} \in \mathcal{U}^c : \exists \mathbf{u} \in \mathcal{U} \text{ such that } \{\mathbf{u} \rightarrow \mathbf{w}\} \text{ is true}\} \quad (4.2)$$

and we call $\partial\mathcal{U}$ the set of vertices in \mathcal{U}^c that are neighbors of some elements in \mathcal{U} , that is,

$$\partial\mathcal{U} := \{\mathbf{w} \in \mathcal{U}^c : \exists \mathbf{u} \in \mathcal{U} \text{ such that } \mathbf{u} \sim \mathbf{w}\}. \quad (4.3)$$

Define the process

$$\mathcal{Q}_1 = \{\mathbf{0}\} \text{ and, for each } k \in \mathbb{N}, \mathcal{Q}_{k+1} = \mathcal{Q}_k \cup \overrightarrow{\partial\mathcal{Q}_k}. \quad (4.4)$$

We also need to construct a finite sequence of unoriented random graphs such that each graph of the sequence is a bond percolation with parameter β . The last percolation in this finite sequence has the property that we want, that is, the vertex set of its connected component containing $\mathbf{0}$ is equal to $\overrightarrow{\mathcal{L}}_N^{\beta, \mathbf{0}}$. Start with a bond percolation on \mathcal{H}_N with parameter β that is independent of the random variables $\{Z_i^t : i \in [N], t \in \mathcal{V}_N\}$. Call the resulting graph \mathcal{B}_1 . For every $k \geq 1$ we will update \mathcal{B}_k by changing the status of some edges at each stage, in such a way that \mathcal{B}_{k+1} is still a bond percolation with parameter β . For each edge $e \in \mathcal{E}_N$, we define $\mathcal{B}_k\{e\}$ the status (open or closed) of edge e in \mathcal{B}_k . We obtain \mathcal{B}_{k+1} from \mathcal{B}_k , by updating *all and only* the edges in \mathcal{E}_N that connect an element of \mathcal{Q}_k to an element of $\partial\mathcal{Q}_k$. More precisely, for any $\mathbf{u} \in \mathcal{Q}_k$ and any $\mathbf{w} \in \partial\mathcal{Q}_k$, with $\mathbf{u} \sim \mathbf{w}$, set

$$\mathcal{B}_{k+1}\{[\mathbf{u}, \mathbf{w}]\} = \begin{cases} \text{open,} & \text{if } \{\mathbf{u} \rightarrow \mathbf{w}\}, \\ \text{closed,} & \text{otherwise;} \end{cases} \quad (4.5)$$

for all other edges $e \in \mathcal{E}_N$, we have $\mathcal{B}_{k+1}\{e\} = \mathcal{B}_k\{e\}$.

Since the status of edges is updated independently of the original configuration and with i.i.d. Bernoulli(β) random variables, we have that \mathcal{B}_{k+1} is still a bond percolation with parameter β .

Notice that, in the worst-case scenario, each of these processes explores the entirety of \mathcal{H}_N in 2^N iterations. That is $\mathcal{B}_{k+1} = \mathcal{B}_k$ and $\mathcal{Q}_{k+1} = \mathcal{Q}_k$ for all $k \geq 2^N$. By construction, \mathcal{Q}_{2^N} is exactly the set of vertices in the connected component of the percolation graph \mathcal{B}_{2^N} that contains $\mathbf{0}$. In this context, we have $\overrightarrow{\mathcal{L}}_N^{\beta, \mathbf{0}} = \mathcal{Q}_{2^N}$. Set $\mathcal{B}_N^\beta = \mathcal{B}_{2^N}$. Recall that $\mathcal{L}_N^{\beta, \mathbf{0}}$ is the set of vertices in the connected component that contains $\mathbf{0}$ in the percolation graph \mathcal{B}_N^β . By construction, we have $\mathcal{L}_N^{\beta, \mathbf{0}} = \overrightarrow{\mathcal{L}}_N^{\beta, \mathbf{0}}$. \square

The case $\beta = 1/2$ of [Proposition 4.1](#) was studied by [Linusson \(2009, Lemma 2.1\)](#) in an unpublished manuscript, using different techniques.

Remark 4.1. We constructed a percolation \mathcal{B}_N^β that is coupled with the random partial orientation of the cube, which in turn is induced by a random game. This will allow us to identify vertices in the percolation \mathcal{B}_N^β with reference to properties of both the random

partially oriented hypercube and the random game. For instance, we will talk about PNE in the percolation.

Define \mathcal{L}_N^β to be the set of vertices in the giant component of the percolation \mathcal{B}_N^β introduced in [Proposition 4.1](#). We call the complement of the giant component, denoted by \mathcal{M}_N^β , the *fragment of the percolation*. With an abuse of notation, we use \mathcal{M}_N^β also to denote the set of vertices of the fragment, i.e., $\mathcal{V}_N \setminus \mathcal{L}_N^\beta$.

Proposition 4.2. *For any $\beta \in (0, 1/2]$, we have*

$$\sum_{N=1}^{\infty} \mathbb{P}\left(\mathcal{L}_N^\beta \neq \overrightarrow{\mathcal{L}}_N^{\beta, \mathbf{0}}\right) < \infty. \quad (4.6)$$

Proof. We first prove the proposition for $\beta \in (0, 1/2)$. In virtue of [Proposition 4.1](#), we have $\overrightarrow{\mathcal{L}}_N^{\beta, \mathbf{0}} = \mathcal{L}_N^{\beta, \mathbf{0}}$, and in order to prove [Eq. \(4.6\)](#) it is enough to prove

$$\sum_{N=1}^{\infty} \mathbb{P}\left(\mathcal{L}_N^\beta \neq \mathcal{L}_N^{\beta, \mathbf{0}}\right) < \infty. \quad (4.7)$$

By symmetry, we have

$$\mathbb{P}\left(\mathbf{0} \in \mathcal{M}_N^\beta \mid \text{card}\left(\mathcal{M}_N^\beta\right)\right) = \frac{\text{card}\left(\mathcal{M}_N^\beta\right)}{2^N}. \quad (4.8)$$

Hence,

$$\mathbb{P}\left(\mathcal{L}_N^\beta \neq \mathcal{L}_N^{\beta, \mathbf{0}} \mid \text{card}\left(\mathcal{M}_N^\beta\right)\right) = \mathbb{P}\left(\mathbf{0} \in \mathcal{M}_N^\beta \mid \text{card}\left(\mathcal{M}_N^\beta\right)\right) = \frac{\text{card}\left(\mathcal{M}_N^\beta\right)}{2^N}. \quad (4.9)$$

Given two sequences $(a_n)_n$ and $(b_n)_n$, we say that $a_n = O(b_n)$ if $\sup_{n \in \mathbb{N}} a_n/b_n < \infty$. Moreover, we say $a_n = o(b_n)$ if $\lim_{n \rightarrow \infty} a_n/b_n = 0$. We make use of the following lemma about upper and lower bounds for the cardinality of the fragment.

Lemma 4.3 ([McDiarmid et al. \(2020, Theorem 1\(a\)\)](#)). *For each $\varepsilon > 0$ and for a fixed $\beta \in (0, 1/2)$, the percolation \mathcal{B}_N^β satisfies*

$$\sum_{N=1}^{\infty} \mathbb{P}\left(\left|\text{card}\left(\mathcal{M}_N^\beta\right) - \mathbb{E}\left[\text{card}\left(\mathcal{M}_N^\beta\right)\right]\right| \geq \varepsilon \sqrt{N(2(1-\beta))^N}\right) < \infty, \quad (4.10)$$

where $\mathbb{E}\left[\text{card}\left(\mathcal{M}_N^\beta\right)\right] = (2(1-\beta))^N(1 + O(N(1-\beta)^N))$.

Set

$$\zeta_N := \mathbb{E}\left[\text{card}\left(\mathcal{M}_N^\beta\right)\right] + \varepsilon \sqrt{N(2(1-\beta))^N}. \quad (4.11)$$

Using [Eqs. \(4.9\)](#) and [\(4.10\)](#), we obtain

$$\begin{aligned} \sum_{N=1}^{\infty} \mathbb{P}\left(\mathcal{L}_N^\beta \neq \mathcal{L}_N^{\beta, \mathbf{0}}\right) &\leq \sum_{N=1}^{\infty} \mathbb{P}\left(\left\{\mathcal{L}_N^\beta \neq \mathcal{L}_N^{\beta, \mathbf{0}}\right\} \cap \left\{\text{card}\left(\mathcal{M}_N^\beta\right) < \zeta_N\right\}\right) \\ &\quad + \sum_{N=1}^{\infty} \mathbb{P}\left(\text{card}\left(\mathcal{M}_N^\beta\right) \geq \zeta_N\right) \\ &\leq \sum_{N=1}^{\infty} \frac{\zeta_N}{2^N} + \sum_{N=1}^{\infty} \mathbb{P}\left(\text{card}\left(\mathcal{M}_N^\beta\right) \geq \zeta_N\right) < \infty. \end{aligned} \quad (4.12)$$

Next, we prove the case $\beta = 1/2$ of [Proposition 4.2](#). It is well known that there exists a coupling such that if $\beta < \beta'$, then $\mathcal{L}_N^\beta \subseteq \mathcal{L}_N^{\beta'}$ (see, e.g., [Bollobás, 2001](#), Theorem 2.1, page 36). In other words, \mathcal{L}_N^β is monotone in β . As

$$\mathcal{L}_N^\beta \neq \mathcal{L}_N^{\beta, \mathbf{0}} \iff \mathbf{0} \notin \mathcal{L}_N^\beta, \quad (4.13)$$

it follows that

$$\mathbb{P}\left(\mathcal{L}_N^\beta \neq \mathcal{L}_N^{\beta, \mathbf{0}}\right) \quad (4.14)$$

is nonincreasing in β . For any $\beta \in (0, 1/2)$, we have that

$$\sum_{N=1}^{\infty} \mathbb{P}\left(\mathcal{L}_N^{1/2} \neq \mathcal{L}_N^{1/2, \mathbf{0}}\right) \leq \sum_{N=1}^{\infty} \mathbb{P}\left(\mathcal{L}_N^\beta \neq \mathcal{L}_N^{\beta, \mathbf{0}}\right) < \infty. \quad \square$$

The following lemma will be used several times in our proofs.

Lemma 4.4 ([McDiarmid et al. \(2020, Theorem 2\(a\)\)](#)). *Fix $\beta \in (0, 1/2]$. For any $r \in \mathbb{N}$ and any $\mathbf{s} \in \mathcal{V}_N$, call*

$$B_r(\mathbf{s}) := \{\mathbf{t}: h(\mathbf{s}, \mathbf{t}) \leq r\}, \quad (4.15)$$

where h is the Hamming distance. Set

$$m_\beta := \left\lfloor \frac{1}{-\log_2(1 - \beta)} \right\rfloor. \quad (4.16)$$

Then there exists $\bar{\delta} > 0$ such that, for any $\delta < \bar{\delta}$, we have

$$\sum_{N=1}^{\infty} \mathbb{P}\left(\exists \mathbf{t}: \text{card}\left(B_{\lceil \delta N \rceil}(\mathbf{t}) \setminus \vec{\mathcal{L}}_N^{\beta, \mathbf{0}}\right) > m_\beta\right) < \infty. \quad (4.17)$$

Remark 4.2. Notice that [McDiarmid et al. \(2020, Theorem 2\(a\)\)](#) is actually formulated in terms of giant component of \mathcal{B}_N^β , i.e., \mathcal{L}_N^β instead of $\vec{\mathcal{L}}_N^{\beta, \mathbf{0}}$. This substitution holds in virtue of [Proposition 4.1](#) and [Proposition 4.2](#). Moreover, [McDiarmid et al. \(2020, Theorem 2\(a\)\)](#) covers only the case $\beta \in (0, 1/2)$. On the other hand, the case $\beta = 1/2$ follows immediately using monotonicity of percolation with respect to β . In fact, choose $\beta < 1/2$ such that $m_\beta = m_{1/2} = 1$. By monotonicity, we have that $\mathcal{L}_N^{\beta, \mathbf{0}}$ is stochastically smaller than $\mathcal{L}_N^{1/2, \mathbf{0}}$. Hence

$$\mathbb{P}\left(\exists \mathbf{t}: \text{card}\left(B_{\lceil \delta N \rceil}(\mathbf{t}) \setminus \vec{\mathcal{L}}_N^{1/2, \mathbf{0}}\right) > 1\right) \leq \mathbb{P}\left(\exists \mathbf{t}: \text{card}\left(B_{\lceil \delta N \rceil}(\mathbf{t}) \setminus \vec{\mathcal{L}}_N^{\beta, \mathbf{0}}\right) > 1\right). \quad (4.18)$$

The next result focuses on the nonatomic case, that is, $\beta = 1/2$ (or equivalently, $\alpha = 0$). This corresponds to the classical bond percolation with parameter $1/2$. A vertex is *isolated* if it has degree zero in the graph induced by the percolation.

[Erdős and Spencer \(1979\)](#) analyzed the asymptotic behavior of $\mathcal{B}_N^{1/2}$, and showed that the random graph is connected with probability tending to one. Upon further inspection of their proof, it is evident that, with probability tending to one, the giant component of this percolation contains all the vertices in \mathcal{V}_N with the exception of some isolated vertices in the random graph $\mathcal{B}_N^{1/2}$. As the following result was not explicitly stated in [Erdős and Spencer \(1979\)](#), we include a proof for the sake of clarity.

Proposition 4.5 (Erdős and Spencer (1979)). *Let $\alpha = 0$, i.e., $\beta = 1/2$, let Υ_N be the set of isolated vertices in $\mathcal{B}_N^{1/2}$, and recall that $\mathcal{M}_N^{1/2}$ is its fragment. Then*

$$\sum_{N=1}^{\infty} \mathbb{P} \left(\mathcal{M}_N^{1/2} \neq \Upsilon_N \right) < \infty. \quad (4.19)$$

Moreover,

$$\lim_{N \rightarrow \infty} \mathbb{P}(\text{card}(\Upsilon_N) = k) = \frac{e^{-1}}{k!}, \quad k \in \mathbb{N}. \quad (4.20)$$

Proof. Choose δ as in Lemma 4.4. If $\{\mathcal{M}_N^{1/2} \neq \Upsilon_N\}$ then there exists \mathbf{s} such that $\text{card}(B_{[\delta N]}(\mathbf{s}) \setminus \mathcal{L}_N^{1/2, \mathbf{0}}) > 1$. Combining Lemma 4.4 (case $\beta = 1/2$) with Proposition 4.1 we have

$$\sum_N \mathbb{P} \left(\exists \mathbf{t} : \text{card}(B_{[\delta N]}(\mathbf{t}) \setminus \mathcal{L}_N^{1/2, \mathbf{0}}) > 1 \right) < \infty. \quad (4.21)$$

This proves Eq. (4.19). To prove Eq. (4.20) we reason as follows. The PNE are the vertices of $\overrightarrow{\mathcal{H}}_N$ that are incident only to incoming edges. When $\alpha = 0$, the number of PNE has the same distribution as the number of vertices in $\overrightarrow{\mathcal{H}}_N$ which are incident only to outgoing edges. Call $\overrightarrow{\Upsilon}_N$ the set of such vertices. It is well-known that, when $\alpha = 0$, the number of PNE converges in distribution to a Poisson(1) (see Arratia et al. (1989), Rinott and Scarsini (2000)). Hence,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\text{card}(\overrightarrow{\Upsilon}_N) = k \right) = \frac{e^{-1}}{k!}, \quad k \in \mathbb{N}. \quad (4.22)$$

We will prove that $\overrightarrow{\Upsilon}_N = \Upsilon_N$ for all N large enough. If $\mathbf{0} \notin \overrightarrow{\Upsilon}_N$ then all the vertices in $\overrightarrow{\Upsilon}_N$ are not accessible from $\mathbf{0}$. As

$$\sum_{N=1}^{\infty} \mathbb{P}(\mathbf{0} \in \overrightarrow{\Upsilon}_N) = \sum_{N=1}^{\infty} \frac{1}{2^N} < \infty, \quad (4.23)$$

we can use the first Borel-Cantelli lemma to conclude that for all N large enough, $\overrightarrow{\Upsilon}_N \subseteq \overrightarrow{\mathcal{M}}_N^{1/2, \mathbf{0}}$. Using Proposition 4.1, we have that $\overrightarrow{\Upsilon}_N \subseteq \mathcal{V}_N \setminus \mathcal{L}_N^{1/2, \mathbf{0}}$. Using Proposition 4.2 and the first Borel-Cantelli lemma, we have that for N large enough, $\overrightarrow{\Upsilon}_N \subseteq \mathcal{M}_N^{1/2}$. Conversely, each isolated point \mathbf{s} in $\mathcal{B}_N^{1/2}$ whose neighbors are all in $\mathcal{L}_N^{1/2, \mathbf{0}}$ satisfies $\mathbf{s} \in \overrightarrow{\Upsilon}_N$. Hence, $\overrightarrow{\Upsilon}_N \neq \mathcal{M}_N^{1/2}$ only if there exist two elements $\mathbf{s}, \mathbf{t} \in \mathcal{M}_N^{1/2}$, such that $h(\mathbf{s}, \mathbf{t}) = 1$, i.e., \mathbf{s} and \mathbf{t} are neighbors in \mathcal{H}_N . Using Eq. (4.21), we have

$$\sum_{N=1}^{\infty} \mathbb{P} \left(\overrightarrow{\Upsilon}_N \neq \mathcal{M}_N^{1/2} \right) < \infty. \quad (4.24)$$

Combining Eq. (4.24) with Eq. (4.19), we have

$$\sum_{N=1}^{\infty} \mathbb{P} \left(\overrightarrow{\Upsilon}_N \neq \Upsilon_N \right) < \infty. \quad (4.25)$$

Using the first Borel-Cantelli lemma and Eq. (4.22) we prove Eq. (4.20). \square

5. PROOFS

Proof of Section 2.

Proof of Theorem 2.2. (a) As we mentioned before, when $\alpha = 0$, convergence of the number of PNE to a Poisson(1) was proved by Arratia et al. (1989), Rinott and Scarsini (2000). Moreover, since almost surely no two payoffs are equal, we have that each PNE is also an SPNE.

(b) Now we focus on the case $\alpha > 0$ and prove that the number of SPNE in Ξ_N is zero for all large N , P-a.s.. Notice that $\alpha > 0$ implies that $\beta < 1/2$.

We have

$$\text{card}(\text{SPNE}(\Xi_N)) = \sum_{\mathbf{s} \in \mathcal{V}_N} \mathbf{1}_{\mathbf{s} \in \text{SPNE}(\Xi_N)}, \quad (5.1)$$

where $\mathbf{1}_A$ denotes the indicator function of the event A . Since $\mathbb{P}(\mathbf{s} \in \text{SPNE}(\Xi_N)) = \beta^N$ for every $\mathbf{s} \in \mathcal{V}_N$, we have

$$\mathbb{E}[\text{card}(\text{SPNE}(\Xi_N))] = (2\beta)^N. \quad (5.2)$$

Markov's inequality implies

$$\mathbb{P}(\text{card}(\text{SPNE}(\Xi_N)) \geq 1) \leq \mathbb{E}[\text{card}(\text{SPNE}(\Xi_N))] = (2\beta)^N. \quad (5.3)$$

Since $2\beta < 1$, the upper bound goes to zero geometrically fast. Hence,

$$\sum_{N=1}^{\infty} \mathbb{P}(\text{card}(\text{SPNE}(\Xi_N)) \geq 1) < \infty. \quad (5.4)$$

Using the first Borel-Cantelli Lemma, we have that the event $\{\text{card}(\text{SPNE}(\Xi_N)) < 1\}$ holds true for all N large enough. As the cardinality is a non-negative integer, it must be zero for all large N . \square

Given two probability measures \mathbb{P}, \mathbb{Q} on \mathbb{N} , their total variation distance, ρ_{TV} is defined as

$$\rho_{\text{TV}}(\mathbb{P}, \mathbb{Q}) := \sup_{A \subseteq \mathbb{N}} |\mathbb{P}(A) - \mathbb{Q}(A)|. \quad (5.5)$$

With an abuse of language, we will often speak of total variation distance of two random variables to indicate the total variation distance of their laws.

Remark 5.1. Recall the definition of the Kolmogorov distance κ , given in Eq. (2.10). We have that

$$\kappa(\mathbb{P}, \mathbb{Q}) \leq \rho_{\text{TV}}(\mathbb{P}, \mathbb{Q}). \quad (5.6)$$

In fact, the first can be seen as the supremum over the collection of sets $A = [0, k]$ for $k \in \mathbb{N}$ while the variational distance is a supremum over a richer collection of sets.

To prove Theorem 2.3 we will make use of the following lemma.

Lemma 5.1 (McDiarmid et al. (2020, Theorem 4(c))). *Call C_j the number of connected components with exactly j vertices in the percolation \mathcal{B}_N^β and let $\mu_j := \mathbb{E}[C_j]$. Then for $\beta \in (0, 1/2)$, we have*

$$\rho_{\text{TV}}(C_j, \text{Poisson}(\mu_j)) = O(N^j(1 - \beta)^{jN}). \quad (5.7)$$

Proof of Theorem 2.3. Consider the random partially oriented hypercube $\overrightarrow{\mathcal{H}}_N^\beta$ and invert the orientation of each edge, keeping the unoriented edges unoriented. The new random partially oriented hypercube $\overleftarrow{\mathcal{H}}_N^\beta$ has the same law as the original one. Moreover, a vertex is a PNE in $\overrightarrow{\mathcal{H}}_N^\beta$ if and only if it is an isolated point in $\overleftarrow{\mathcal{H}}_N^\beta$.

Consider the percolation associated to $\overleftarrow{\mathcal{H}}_N^\beta$ (see Proposition 4.1). In virtue of our reasoning above, it is enough to establish a limit theorem for the number of isolated vertices of this percolation. Resorting to Lemma 5.1 with $j = 1$, we obtain that there exists a constant $K_{1,\alpha}$, which depends on α , but not on N , such that

$$\rho_{\text{TV}}\left(\text{card}(\text{PNE}(\Xi_N)), \text{Poisson}(2^N(1-\beta)^N)\right) \leq K_{1,\alpha}(N(1-\beta)^N), \quad (5.8)$$

which implies (see Remark 5.1)

$$\kappa\left(\text{card}(\text{PNE}(\Xi_N)), \text{Poisson}(2^N(1-\beta)^N)\right) \leq K_{1,\alpha}(N(1-\beta)^N). \quad (5.9)$$

Notice that $\text{Poisson}(2^N(1-\beta)^N)$ can be expressed as the sum of $\lfloor 2^N(1-\beta)^N \rfloor$ independent Poisson with parameter one, plus a small remainder. In this context, we can use the Berry-Esseen theorem (see, e.g., Durrett, 2019, Theorem 3.4.17), which implies that

$$\kappa\left(\text{Poisson}(2^N(1-\beta)^N), \text{Norm}(2^N(1-\beta)^N, 2^N(1-\beta)^N)\right) \leq \frac{K_{2,\alpha}}{2^{N/2}(1-\beta)^{N/2}}. \quad (5.10)$$

For simplicity, we make the substitution $2(1-\beta) = 1+\alpha$. Using the triangular inequality we obtain

$$\begin{aligned} & \kappa(\text{card}(\text{PNE}(\Xi_N)), \text{Norm}((1+\alpha)^N, (1+\alpha)^N)) \\ & \leq \kappa\left(\text{card}(\text{PNE}(\Xi_N)), \text{Poisson}((1+\alpha)^N)\right) \\ & \quad + \kappa\left(\text{Poisson}((1+\alpha)^N), \text{Norm}((1+\alpha)^N, (1+\alpha)^N)\right) \\ & \leq K_{1,\alpha}\left(N\left(\frac{1+\alpha}{2}\right)^N\right) + \frac{K_{2,\alpha}}{(1+\alpha)^{N/2}} \\ & \leq NK_\alpha \max\left(\frac{(1+\alpha)}{2}, \frac{1}{(1+\alpha)^{1/2}}\right)^N. \end{aligned} \quad (5.11)$$

□

Proof of Theorem 2.4. We first prove that

$$\mathbb{P}\left(\liminf_{N \rightarrow \infty} \frac{\text{card}(\text{PNE}(\Xi_N))}{(1+\alpha)^N} \geq 1\right) = 1. \quad (5.12)$$

Notice that Theorem 2.3 implies that the number of PNE grows geometrically when $\alpha > 0$. More precisely, we can show that, for any $\varepsilon > 0$ small enough, Eq. (2.11) implies that

$$\sum_{N=1}^{\infty} \mathbb{P}(\text{card}(\text{PNE}(\Xi_N)) < (1+\alpha)^N - (1+\varepsilon)^N(1+\alpha)^{N/2}) < \infty. \quad (5.13)$$

Fix $\varepsilon > 0$ such that

$$(1+\varepsilon) < (1+\alpha)^{1/2}.$$

With this choice of ε , the right-hand side of Eq. (5.13) is $(1 + \alpha)^N(1 + o(1))$. Hence, by establishing Eq. (5.13) we would prove that the number of PNE grows like $(1 + \alpha)^N$. Recall the standard inequality $\Phi(-x) \leq \phi(x)/x$, valid for $x > 0$, where ϕ is the density of a standard normal. We then have that

$$\Phi(-(1 + \varepsilon)^N) \leq K_3(1 + \varepsilon)^{-N}$$

for some constant $K_3 > 0$. Hence,

$$\begin{aligned} & \sum_{N=1}^{\infty} \mathbb{P}(\text{card}(\text{PNE}(\Xi_N)) \leq (1 + \alpha)^N - (1 + \varepsilon)^N(1 + \alpha)^{N/2}) \\ &= \sum_{N=1}^{\infty} [\mathbb{P}(\text{card}(\text{PNE}(\Xi_N)) \leq (1 + \alpha)^N - (1 + \varepsilon)^N(1 + \alpha)^{N/2}) - \Phi(-(1 + \varepsilon)^N)] \\ & \quad + \sum_{N=1}^{\infty} \Phi(-(1 + \varepsilon)^N) \\ &\leq \sum_{N=1}^{\infty} NK_{\alpha} \max\left(\frac{(1 + \alpha)}{2}, \frac{1}{(1 + \alpha)^{1/2}}\right)^N + \sum_{N=1}^{\infty} \frac{K_3}{(1 + \varepsilon)^N} < \infty. \end{aligned} \tag{5.14}$$

Similarly, we can prove the following

$$\sum_{N=1}^{\infty} \mathbb{P}(\text{card}(\text{PNE}(\Xi_N)) > (1 + \alpha)^N + (1 + \varepsilon)^N(1 + \alpha)^{N/2}) < \infty. \tag{5.15}$$

This is achieved by repeating the argument above with the choice of $x_N = (1 + \varepsilon)^N$ and using the symmetry of normal distribution, i.e., $1 - \Phi(x) = \Phi(-x)$. In turn, Eq. (5.15) implies that

$$\mathbb{P}\left(\limsup_{N \rightarrow \infty} \frac{\text{card}(\text{PNE}(\Xi_N))}{(1 + \alpha)^N} \leq 1\right) = 1, \tag{5.16}$$

which, together with Eq. (5.12), ends the proof. \square

Proofs of Section 3.

Proof of Theorem 3.3. (a) First we deal with the case $\alpha = 0$, which corresponds to $\beta = 1/2$, i.e., $m_{\beta} = 1$. In this case, there are no unoriented edges in $\vec{\mathcal{H}}_N^{\beta}$. If a PNE belongs to $\vec{\mathcal{M}}_N^{1/2, \mathbf{0}}$, then all its neighbors must also belong to $\vec{\mathcal{M}}_N^{1/2, \mathbf{0}}$. In fact, if one of these neighbors were accessible from $\mathbf{0}$, then also the PNE would be accessible from $\mathbf{0}$, i.e., it would belong to $\vec{\mathcal{Z}}_N^{1/2, \mathbf{0}}$. This means that there exists a ball of radius 1 with $N + 1$ vertices in $\vec{\mathcal{M}}_N^{1/2, \mathbf{0}}$. Hence, by Lemma 4.4 (case $\beta = 1/2$), we have

$$\sum_{N=1}^{\infty} \mathbb{P}(\text{PNE}(\Xi_N) \notin \vec{\mathcal{Z}}_N^{1/2, \mathbf{0}}) \leq \sum_{N=1}^{\infty} \mathbb{P}(\exists \mathbf{t}: \text{card}(B_{\lceil \delta N \rceil}(\mathbf{t}) \setminus \vec{\mathcal{Z}}_N^{1/2, \mathbf{0}}) > N) < \infty. \tag{5.17}$$

We now consider the case $0 < \alpha < 1/2$. Let Y_N be the number of vertices that are incident to at least $N - m_{\beta}$ unoriented edges. Markov's inequality yields

$$\mathbb{P}(Y_N \geq 1) \leq 2^N \sum_{k=N-m_{\beta}}^N \binom{N}{k} \alpha^k (1 - \alpha)^{N-k} \leq K_{4, \alpha} N^{m_{\beta}} 2^N \alpha^{N-m_{\beta}}, \tag{5.18}$$

for some constant $K_{4, \alpha}$.

Hence, as $\alpha < 1/2$,

$$\sum_{N=1}^{\infty} \mathbf{P}(Y_N \geq 1) < \infty. \quad (5.19)$$

Suppose that \mathbf{s} is a PNE and belongs to $\overrightarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}$. Then, either:

- (i) the vertex \mathbf{s} is adjacent to more than m_β oriented edges, which implies that the size of the connected component of $\overrightarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}$ containing \mathbf{s} is larger than m_β , or
- (ii) the vertex \mathbf{s} is adjacent to at most m_β oriented edges.

When (ii) holds, we necessarily have that $Y_N \geq 1$. Fix $\delta > 0$ as in Lemma 4.4. Hence,

$$\begin{aligned} \sum_{N=1}^{\infty} \mathbf{P}\left(\text{PNE}(\Xi_N) \not\subseteq \overrightarrow{\mathcal{L}}_N^{\beta, \mathbf{0}}\right) \\ \leq \sum_{N=1}^{\infty} \mathbf{P}\left(\exists \mathbf{t}: \text{card}\left(B_{\lceil \delta N \rceil}(\mathbf{t}) \setminus \overrightarrow{\mathcal{L}}_N^{\beta, \mathbf{0}}\right) > m_\beta\right) + \sum_{N=1}^{\infty} \mathbf{P}(Y_N \geq 1) < \infty. \end{aligned}$$

(b) For this case, we introduce a different percolation $\tilde{\mathcal{B}}_N^\beta$ on \mathcal{H}_N which is defined below. This percolation is related to \mathcal{B}_N^β , as defined in Proposition 4.1. For any pair of vertices $\mathbf{r}, \mathbf{t} \in \mathcal{V}_N$, we declare the edge $[\mathbf{r}, \mathbf{t}]$ open in $\tilde{\mathcal{B}}_N^\beta$ if $\{\mathbf{r} \rightarrow \mathbf{t}\} \cup \{\mathbf{t} \rightarrow \mathbf{r}\}$ holds true, that is, the edge connecting the two profiles \mathbf{r} and \mathbf{t} is oriented in $\overrightarrow{\mathcal{H}}_N^\beta$. Otherwise the edge $[\mathbf{r}, \mathbf{t}]$ is declared closed in $\tilde{\mathcal{B}}_N^\beta$. Since $\alpha = 1/2$, the parameter of the percolation $\tilde{\mathcal{B}}_N^\beta$ is also $1/2$ and we are in the framework studied in Erdős and Spencer (1979), so we can apply Proposition 4.5.

Call $\tilde{\mathcal{L}}_N^{\beta, \mathbf{0}}$ the connected component of $\tilde{\mathcal{B}}_N^\beta$ that contains $\mathbf{0}$. Any isolated vertex in $\tilde{\mathcal{B}}_N^\beta$ is a PNE in $\overrightarrow{\mathcal{H}}_N^\beta$, as it is incident only to non-oriented edges, which in turn implies that each player has no incentive to deviate. Using Proposition 4.5, we have that the number of PNE outside $\tilde{\mathcal{L}}_N^{\beta, \mathbf{0}}$ is asymptotically a Poisson(1) random variable. Notice that $\mathcal{L}_N^{\beta, \mathbf{0}} \subseteq \tilde{\mathcal{L}}_N^{\beta, \mathbf{0}}$. This is because any edge that is open in \mathcal{B}_N^β is also open in $\tilde{\mathcal{B}}_N^\beta$. Using Proposition 4.1, we have that $\overrightarrow{\mathcal{L}}_N^{\beta, \mathbf{0}} \subseteq \tilde{\mathcal{L}}_N^{\beta, \mathbf{0}}$. Hence, the number of PNE outside $\overrightarrow{\mathcal{L}}_N^{\beta, \mathbf{0}}$ is stochastically larger than a Poisson(1) random variable.

(c) For $\mathbf{t} \in \mathcal{V}_N$, define $\Theta^{\mathbf{t}}$ to be the event that vertex \mathbf{t} is incident only to unoriented edges in $\overrightarrow{\mathcal{H}}_N^\beta$, and

$$\Theta_N = \sum_{\mathbf{t} \in \mathcal{V}_N} \mathbf{1}_{\Theta^{\mathbf{t}}}. \quad (5.20)$$

As we showed after Definition 3.5, \mathcal{H}_N can be decomposed as

$$\mathcal{V}_N = \mathcal{V}_N^{\text{even}} \dot{\cup} \mathcal{V}_N^{\text{odd}}, \quad (5.21)$$

where $\mathcal{V}_N^{\text{even}}$ is the set of vertices for which the sum of coordinates is even and $\mathcal{V}_N^{\text{odd}}$ is the set of vertices for which the sum of coordinates is odd. Edges connect only vertices from different components, so no pair of vertices in $\mathcal{V}_N^{\text{even}}$ (or in $\mathcal{V}_N^{\text{odd}}$) can be neighbors.

Obviously $\text{card}(\mathcal{V}_N^{\text{even}}) = \text{card}(\mathcal{V}_N^{\text{odd}}) = 2^{N-1}$. Our first goal is to prove the following result.

Lemma 5.2. *The class $\{\Theta^{\mathbf{t}} : \mathbf{t} \in \mathcal{V}_N^{\text{even}}\}$ is a collection of independent events.*

Proof. The event $\Theta^{\mathbf{s}}$ depends only on the payoffs at \mathbf{s} and at each of its neighbors. It is enough to prove that, for every subset $I \subseteq \mathcal{V}_N^{\text{even}}$, we have

$$\mathbb{P}\left(\bigcap_{\mathbf{s} \in I} \Theta^{\mathbf{s}}\right) = \prod_{\mathbf{s} \in I} \mathbb{P}(\Theta^{\mathbf{s}}). \quad (5.22)$$

Fix I and $\mathbf{t} \in I$ and define $I_{-\mathbf{t}} := I \setminus \{\mathbf{t}\}$. We need to prove that

$$\mathbb{P}\left(\bigcap_{\mathbf{s} \in I} \Theta^{\mathbf{s}}\right) = \mathbb{P}\left(\Theta^{\mathbf{t}} \bigcap_{\mathbf{s} \in I_{-\mathbf{t}}} \Theta^{\mathbf{s}}\right) = \mathbb{P}(\Theta^{\mathbf{t}}) \mathbb{P}\left(\bigcap_{\mathbf{s} \in I_{-\mathbf{t}}} \Theta^{\mathbf{s}}\right). \quad (5.23)$$

The set of profiles in $I_{-\mathbf{t}}$ that share a neighbor with \mathbf{t} has cardinality at most $\binom{N}{2}$. If this set is empty, then Eq. (5.23) trivially holds. Otherwise, for $i \in [N]$, let $\mathbf{s}^{ij} \in I_{-\mathbf{t}}$ and \mathbf{w}^i be such that $\mathbf{w}^i \sim_i \mathbf{t}$ and $\mathbf{s}^{ij} \sim_j \mathbf{w}^i$, with $i \neq j$. If, for some i , the event $\Theta^{\mathbf{s}^{ij}}$ is true, then $Z_j^{\mathbf{s}^{ij}} = Z_j^{\mathbf{w}^i}$, and this event is independent of $Z_i^{\mathbf{w}^i}$. Therefore the class of events $\{\Theta^{\mathbf{s}^{ij}}\}_{i \in [N]}$ is independent of the class of random variables $\{Z_i^{\mathbf{w}^i}\}_{i \in [N]}$. Since the event $\Theta^{\mathbf{t}}$ depends only on $\{Z_i^{\mathbf{w}^i}\}_{i \in [N]}$ and $Z_i^{\mathbf{t}}$, we have that $\Theta^{\mathbf{t}}$ is independent of $\{\Theta^{\mathbf{s}^{ij}}\}_{i \in [N]}$. Moreover, $\Theta^{\mathbf{t}}$ is independent of $\Theta^{\mathbf{s}}$ for all $\mathbf{s} \in I_{-\mathbf{t}}$. This ends the proof of Lemma 5.2. \square

As $\Theta_N \geq \sum_{\mathbf{t} \in \mathcal{V}_N^{\text{even}}} \mathbf{1}_{\Theta^{\mathbf{t}}}$, Lemma 5.2 implies that Θ_N is stochastically larger than a Binomial($2^{N-1}, \alpha^N$). Each vertex \mathbf{t} that is incident only to unoriented edges has the following properties:

- it is a PNE; and
- it lies in $\overrightarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}$, unless $\mathbf{t} = \mathbf{0}$.

Hence, we have that for any fixed $K > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(\text{card}\left(\text{PNE}(\Xi_N) \cap \overrightarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}\right) > K\right) = 1. \quad \square$$

Recall the definition of trap, given in Definition 3.5. Notice that a trap does not contain any PNE. This is because it is strongly connected whereas no vertex is accessible from a PNE.

Proof of Theorem 3.4. The process $\text{BR}^{(N)}$ does not converge to a PNE if and only if it visits a trap. Hence, it is enough to prove the following stronger result about the existence of traps

$$\sum_{N=1}^{\infty} \mathbb{P}(\exists \text{ a trap in } \overrightarrow{\mathcal{H}}_N^{\beta}) < \infty. \quad (5.24)$$

To prove Eq. (5.24), we first study some properties of traps. Fix a trap $\overrightarrow{\mathcal{F}}$. Each edge connecting $\overrightarrow{\mathcal{F}}$ to its boundary $\partial \overrightarrow{\mathcal{F}}$ is either unoriented or points towards $\overrightarrow{\mathcal{F}}$ (see the left of Fig. 6).

Invert the orientation of the partially oriented hypercube $\overrightarrow{\mathcal{H}}_N^{\beta}$, while leaving the unoriented edges unchanged, to obtain $\overleftarrow{\mathcal{H}}_N^{\beta}$. This is illustrated in Fig. 6. The random partially oriented graphs $\overrightarrow{\mathcal{H}}_N^{\beta}$ and $\overleftarrow{\mathcal{H}}_N^{\beta}$ share the same distribution. Consider the partition

$$\mathcal{V}_N = \overleftarrow{\mathcal{L}}_N^{\beta, \mathbf{0}} \dot{\cup} \overleftarrow{\mathcal{M}}_N^{\beta, \mathbf{0}} \quad (5.25)$$

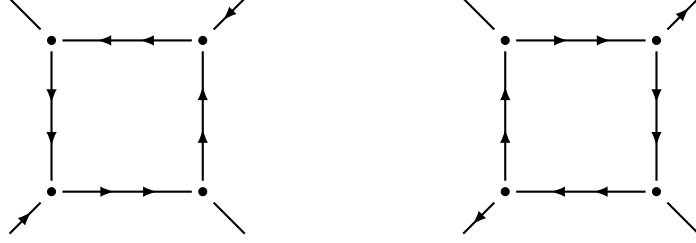


FIGURE 6. A trap of size 4 (left) and the corresponding subgraph after reversing the orientation on the edges (right). Notice that, after reversing the edges, there is no way for the process to enter the cycle from the outside.

in such a way that $\overleftarrow{\mathcal{L}}_N^{\beta, \mathbf{0}}$ is the set that contains $\mathbf{0}$ as well as all vertices \mathbf{t} that are accessible from $\mathbf{0}$ in the oriented graph $\overleftarrow{\mathcal{H}}_N^\beta$. Conversely, all vertices that are *not* accessible from $\mathbf{0}$ are contained in $\overleftarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}$ in the oriented graph $\overleftarrow{\mathcal{H}}_N^\beta$. Let $\overleftarrow{\mathcal{T}}$ be the subgraph in $\overleftarrow{\mathcal{H}}_N^\beta$ corresponding to the trap $\overrightarrow{\mathcal{T}} \subseteq \overrightarrow{\mathcal{H}}_N^\beta$. More precisely, $\overleftarrow{\mathcal{T}}$ and $\overrightarrow{\mathcal{T}}$ share the same vertex set and edges, either unoriented or with opposite orientation. We call $\overleftarrow{\mathcal{T}}$ a reversed trap.

Note that all edges in $\partial \overleftarrow{\mathcal{T}}$ are either unoriented or oriented away from $\overleftarrow{\mathcal{T}}$. Hence, for all large N , either:

- (i) $\mathbf{0} \notin \mathcal{V}(\overleftarrow{\mathcal{T}})$, which implies that $\mathcal{V}(\overleftarrow{\mathcal{T}})$ is contained in $\overleftarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}$, or
- (ii) $\mathbf{0} \in \mathcal{V}(\overleftarrow{\mathcal{T}})$, which implies (using Eq. (3.10)) that $\mathcal{V}(\overleftarrow{\mathcal{T}}) = \overleftarrow{\mathcal{L}}_N^{\beta, \mathbf{0}}$.

We consider the two cases separately, and focus on case (i) first. We prove that

$$\sum_{N=1}^{\infty} \mathbb{P}(\exists \text{ a reversed trap } \overleftarrow{\mathcal{T}} \text{ such that } \mathcal{V}(\overleftarrow{\mathcal{T}}) \subseteq \overleftarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}) < \infty. \quad (5.26)$$

Recall that any (reversed) trap has at least four vertices and notice that $\overleftarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}$ and $\overrightarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}$ are equally distributed. By Lemma 4.4, there exists $\delta > 0$ such that Eq. (4.17) holds. By Eq. (3.8), we have

$$m_\beta = \left\lfloor \frac{1}{-\log_2(1-\beta)} \right\rfloor = \left\lfloor -\frac{1}{\log_2(1/2 + \alpha/2)} \right\rfloor \leq 3.$$

Hence,

$$\sum_{N=1}^{\infty} \mathbb{P}(\exists \text{ a connected component of size at least 4 whose vertices are in } \overleftarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}) < \infty, \quad (5.27)$$

which implies Eq. (5.26).

We now focus on case (ii) and prove that

$$\sum_{N=1}^{\infty} \mathbb{P}(\exists \text{ a trap } \overrightarrow{\mathcal{T}} \text{ such that } \mathcal{V}(\overrightarrow{\mathcal{T}}) = \overrightarrow{\mathcal{L}}_N^{\beta, \mathbf{0}}) < \infty. \quad (5.28)$$

As a trap cannot contain any PNE, to show that Eq. (5.28) holds it is enough to prove that

$$\sum_{N=1}^{\infty} \mathbb{P}(\vec{\mathcal{L}}_N^{\beta, \mathbf{0}} \text{ contains no PNE}) < \infty. \quad (5.29)$$

To this end, we introduce a new class of events. For every strategy \mathbf{s} , define the event

$$G_{\mathbf{s}} := \left\{ \mathbf{s} \text{ is incident to at most 2 oriented edges in } \vec{\mathcal{H}}_N^{\beta} \right\}. \quad (5.30)$$

Moreover, set

$$H_{\mathbf{s}} := G_{\mathbf{s}}^c \cap \left\{ \mathbf{s} \in \vec{\mathcal{M}}_N^{\beta, \mathbf{0}} \right\} \cap \{ \mathbf{s} \text{ is a PNE} \}. \quad (5.31)$$

Fix $\varepsilon < \sqrt{1 + \alpha} - 1$ and set

$$\ell_N = (1 + \alpha)^N - (1 + \varepsilon)^N (1 + \alpha)^{N/2}. \quad (5.32)$$

We have that

$$\left\{ \vec{\mathcal{L}}_N^{\beta, \mathbf{0}} \text{ contains no PNE} \right\} \subseteq \left\{ \text{card}(\text{PNE}(\Xi_N)) < \ell_N \right\} \cup \left\{ \sum_{\mathbf{s} \in \mathcal{V}_N} \mathbf{1}_{G_{\mathbf{s}}} \geq \ell_N \right\} \cup \left\{ \bigcup_{\mathbf{s} \in \mathcal{V}_N} H_{\mathbf{s}} \right\}. \quad (5.33)$$

To see Eq. (5.33), notice that the event in which all the PNE are in $\vec{\mathcal{M}}_N^{\beta, \mathbf{0}}$ can be decomposed into two disjoint subevents:

- (a) the number of PNE is smaller than ℓ_N ; and
- (b) the number of PNE is at least ℓ_N .

Then, under case (b), at least one of the following must hold:

- (i) there exists a PNE in $\vec{\mathcal{M}}_N^{\beta, \mathbf{0}}$ which is incident to at least three directed edges so $\bigcup_{\mathbf{s} \in \mathcal{V}_N} H_{\mathbf{s}}$ holds; or
- (ii) all the PNE are incident to at most two directed edges, which implies $\sum_{\mathbf{s} \in \mathcal{V}_N} \mathbf{1}_{G_{\mathbf{s}}} \geq \ell_N$.

Next, we show how to use Eq. (5.33) to prove Eq. (5.29).

Bound for (a): We recall Eq. (5.13), which can be rewritten as

$$\sum_{N=1}^{\infty} \mathbb{P}(\text{card}(\text{PNE}(\Xi_N)) < \ell_N) < \infty. \quad (5.34)$$

Bound for (b)(i): For any $\mathbf{s} \in \mathcal{V}_N$, we prove that

$$H_{\mathbf{s}} \subseteq \left\{ \exists \text{ a connected component of size at least 4 whose vertices are in } \vec{\mathcal{M}}_N^{\beta, \mathbf{0}} \right\}. \quad (5.35)$$

In fact, under the event $H_{\mathbf{s}}$, the vertex \mathbf{s} is in $\vec{\mathcal{M}}_N^{\beta, \mathbf{0}}$ and is incident to at least three oriented edges. Owing to \mathbf{s} being a PNE, each of these edges points toward \mathbf{s} . Hence, there are (at least) three other vertices in the connected component where \mathbf{s} belongs, proving Eq. (5.35). As a consequence, we have

$$\bigcup_{\mathbf{s} \in \mathcal{V}_N} H_{\mathbf{s}} \subseteq \left\{ \exists \text{ a connected component of size at least 4 whose vertices are in } \vec{\mathcal{M}}_N^{\beta, \mathbf{0}} \right\}. \quad (5.36)$$

Therefore,

$$\begin{aligned} & \sum_{N=1}^{\infty} \mathbb{P} \left(\bigcup_{s \in \mathcal{V}_N} H_s \right) \\ & \leq \sum_{N=1}^{\infty} \mathbb{P} \left(\exists \text{ a connected component of size at least 4 whose vertices are in } \overleftarrow{\mathcal{M}}_N^{\beta, \mathbf{0}} \right) < \infty. \end{aligned} \quad (5.37)$$

Bound for (b)(ii): We have that

$$\mathbb{P}(G_s) = \sum_{k=0}^2 \binom{N}{k} \alpha^{N-k} (1-\alpha)^k \leq K_{5,\alpha} N^2 \alpha^N, \quad (5.38)$$

for some constant $K_{5,\alpha}$, which depends only on α . Using Markov's inequality, we have

$$\sum_{N=1}^{\infty} \mathbb{P} \left(\sum_s \mathbb{1}_{G_s} \geq \ell_N \right) \leq \sum_{N=1}^{\infty} K_{5,\alpha} N^2 \frac{(2\alpha)^N}{\ell_N} < \infty. \quad (5.39)$$

The convergence of the sum is a consequence of $1 + \alpha > 2\alpha$, and the fact that $\ell_N = (1 + \alpha)^N (1 + o(1))$.

Eqs. (5.34), (5.37) and (5.39) show that the subevents in Eq. (5.33) are summable, proving Eq. (5.29). Furthermore, Eqs. (5.26) and (5.29) imply Eq. (5.24). \square

6. CONCLUSIONS AND OPEN PROBLEMS

Large random games have many PNE, as long as the probability of ties is nonzero. We identified the limiting distribution of the number of PNE and their position with respect to the starting point of a BRD. More specifically, in Theorems 2.3 and 2.4 we have proved that the number of PNE grows geometrically when $\alpha > 0$. In Theorem 3.3 we have described the set of PNE and have proved that, when α is larger than a certain threshold, some of them are not accessible. In Theorem 3.4, instead, we have established that, when α is small enough, the BRD converges to a PNE. The relevance of our approach is that it creates a link between different subjects.

The next important question is the following. How long does it take for a BRD to reach a PNE? This is equivalent to studying the path-length of a non-backtracking random walk on the percolation cluster of the hypercube. Fig. 7 shows the results of a simulation exploring this problem.

Then, it is important to study the geometry of PNE when more strategies are available, and when the payoffs are weakly dependent.

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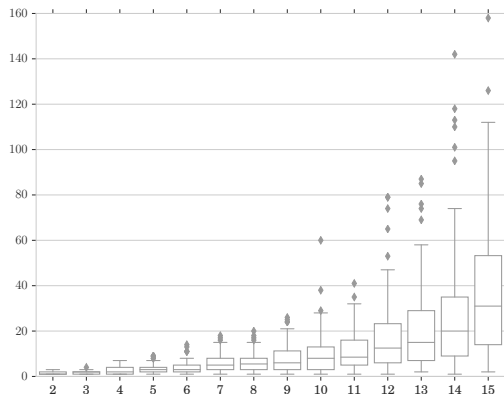


FIGURE 7. Iterations needed for BRD to reach a PNE for $\alpha = 0.5$, with 100 trials per N .

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7. LIST OF SYMBOLS

$B_r(\mathbf{s})$	ball of radius r centred at \mathbf{s} , introduced in Lemma 4.4
\mathcal{B}_k	percolation process at time k , defined in Eq. (4.5)
\mathcal{B}_N^β	edge percolation coupled with random game
$\tilde{\mathcal{B}}_N$	percolation process, defined in proof of Theorem 3.3 (b)
$\text{BR}^{(N)}$	BRD on \mathcal{H}_N
C_j	number of components of size j in the percolation \mathcal{B}^*
$\text{card}(A)$	cardinality of the set A
$\mathcal{D}(\mathbf{s})$	set of profitable deviations from s , define in Eq. (3.2)
\mathcal{E}_N	edge set of \mathcal{H}_N
$\vec{\mathcal{E}}_N$	edge set of $\vec{\mathcal{H}}_N^\beta$
\mathcal{F}	σ -algebra associated with the probability space, introduced after Definition 2.1
g_i	payoff function for player i , introduced in Eq. (2.1)
$G_{\mathbf{s}}$	event defined in Eq. (5.30)
h	Hamming distance on \mathcal{H}_N , defined in Eq. (3.11)
\mathcal{H}_N	N -cube
$\vec{\mathcal{H}}_N^\beta$	partially oriented hypercube
$\overleftarrow{\mathcal{H}}_N^\beta$	the graph obtained by reversing the oriented edges in $\vec{\mathcal{H}}_N^\beta$, introduced in the proof of Theorem 2.3
I	a subset of $\mathcal{V}_N^{\text{even}}$, introduced in Eq. (5.22)
I_{-t}	equal to $I \setminus \{t\}$, defined in Eq. (5.23)
K_α	a constant dependent on α , introduced in Theorem 2.3
ℓ_N	defined in Eq. (5.32)
$\vec{\mathcal{L}}_N^{\beta, \mathbf{0}}$	set of all vertices in $\vec{\mathcal{H}}_N^\beta$ accessible from $\mathbf{0}$, introduced in Eq. (3.4)
$\overleftarrow{\mathcal{L}}_N^{\beta, \mathbf{0}}$	set of all vertices in $\overleftarrow{\mathcal{H}}_N^\beta$ accessible from $\mathbf{0}$, introduced in Eq. (5.25)
\mathcal{L}_N^β	giant component of \mathcal{B}_N^β
$\mathcal{L}_N^{\beta, \mathbf{0}}$	connected component of \mathcal{B}_N^β containing $\mathbf{0}$
m_β	constant, introduced in Lemma 4.4

\mathcal{M}_N^β	the fragment of the percolation, introduced in the proof of Proposition 4.2
$\overrightarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}$	set of all vertices in $\overrightarrow{\mathcal{H}}_N^\beta$ not accessible from $\mathbf{0}$, introduced in Eq. (3.4)
$\overleftarrow{\mathcal{M}}_N^{\beta, \mathbf{0}}$	set of all vertices in $\overleftarrow{\mathcal{H}}_N^\beta$ not accessible from $\mathbf{0}$, introduced in Eq. (5.25)
N	number of players, introduced in Eq. (2.1)
$[N]$	set of players, introduced in Eq. (2.1)
$\text{Norm}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2 , as introduced after Theorem 2.3
$\text{PNE}(\Gamma_N)$	set of PNE in Γ_N , introduced in Definition 2.1
\mathcal{Q}_k	exploration process at time k , defined in Eq. (4.4)
$\{\mathbf{s} \rightarrow \mathbf{t}\}$	event in which $[\mathbf{s}, \mathbf{t}] \in \overrightarrow{\mathcal{E}}$, introduced in Eq. (4.1)
\mathbf{s}	a strategy profile, introduced after Eq. (2.1)
\mathbf{s}_{-i}	strategy profile \mathbf{s} for all players except i , introduced after Eq. (2.1)
S_i	set of strategies for player i , introduced in Eq. (2.1)
S	set of all possible strategy profiles, introduced after Eq. (2.1)
$[\mathbf{s}, \mathbf{t}]$	edge connecting vertices \mathbf{s} and \mathbf{t}
$\text{SPNE}(\Gamma_N)$	set of SPNE in Γ_N , introduced in Definition 2.1
$\overrightarrow{[\mathbf{s}, \mathbf{t}]}$	edge $[\mathbf{s}, \mathbf{t}]$ oriented from \mathbf{s} to \mathbf{t}
\mathcal{T}	subgraph of \mathcal{H}_N corresponding to $\overrightarrow{\mathcal{F}}$, defined in the proof of Theorem 3.4
$\overrightarrow{\mathcal{T}}$	a trap, as defined in Definition 3.5
$\overleftarrow{\mathcal{T}}$	subgraph of $\overleftarrow{\mathcal{H}}_N^\beta$ corresponding to $\overrightarrow{\mathcal{F}}$, defined in the proof of Theorem 3.4
\mathcal{V}_N	vertex set of \mathcal{H}_N
$\mathcal{V}_N^{\text{even}}$	set of vertices whose sum of coordinates is even, introduced in Eq. (5.21)
$\mathcal{V}_N^{\text{odd}}$	set of vertices whose sum of coordinates is odd, introduced in Eq. (5.21)
Y_N	number of vertices incident to at least 2^{N-m_β} unoriented vertices
Z_i^s	random variable dictating the payoff for player i of strategy profile \mathbf{s}
$\alpha = 1 - 2\beta$	probability of payoffs being equal, introduced in Eq. (2.4)
Γ_N	game with N players, introduced in Eq. (2.1)
$\partial\mathcal{U}$	set of vertices which are neighbors of vertex set \mathcal{U} , introduced in Eq. (4.3)
ζ_N	a bounding constant, defined in Eq. (4.11)
$\overrightarrow{\partial\mathcal{U}}$	set of vertices which are out-neighbors of vertex set \mathcal{U} , introduced in Eq. (4.2)
Θ^t	the indicator of the event that the vertex \mathbf{t} is incident only to unoriented edges in $\overrightarrow{\mathcal{H}}_N^\beta$, introduced in Eq. (5.20)
Θ_N	defined in Eq. (5.20)
κ	Kolmogorov distance, defined in Eq. (2.10)
μ_j	$\mathbb{E}[C_j]$
Ξ_N	random game with N players
ρ_{TV}	total variation distance, defined in Eq. (5.5)
Υ_N	the set of all isolated vertices in \mathcal{B}_N^β , introduced in Proposition 4.5
$\mathbf{0}$	strategy profile $(0, 0, \dots, 0)$

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