

Network-Based Optimal Control of Pollution Growth

Fausto Gozzi^{*1}, Marta Leocata^{†2}, and Giulia Pucci^{‡3}

¹Department of Economics and Finance, Luiss University, Rome, Italy.

²Department of Economics and Finance, Luiss University, Rome, Italy.

³Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden

June 24, 2024

Abstract

This paper studies a model for the optimal control (by a centralized economic agent which we call the planner) of pollution diffusion over time and space. The controls are the investments in production and depollution and the goal is to maximize an intertemporal utility function. The main novelty is the fact that the spatial component has a network structure. Moreover, in such a time-space setting we also analyze the trade-off between the use of green or non-green technologies: this also seems to be a novelty in such a setting. Extending methods of previous papers, we can solve explicitly the problem in the case of linear costs of pollution.

Key words: Optimal control problems; Value function; Graphs and networks; Pollution control; Transboundary pollution.

Contents

1	Introduction	2
2	The general model	3
2.1	Well posedness of the objective function	7
2.2	Rewriting the objective function	8

*fgozzi@luiss.it

†mleocata@luiss.it

‡pucci@kth.se

All the authors have equally contributed to the paper.

Fausto Gozzi and Marta Leocata are supported by the Italian Ministry of University and Research (MIUR), in the framework of PRIN projects 2017FKHBA8 001 (*The Time-Space Evolution of Economic Activities: Mathematical Models and Empirical Applications*) and 20223PNJ8K (*Impact of the Human Activities on the Environment and Economic Decision Making in a Heterogeneous Setting: Mathematical Models and Policy Implications*). Giulia Pucci is supported by the Swedish Research Council grant (2020-04697).

3	The model with $a_i^R \equiv 1 \forall i \in \mathcal{V}$.	10
3.1	Solution of the problem and Optimal paths	11
4	The model with Renewable Production	13
4.1	Explicit solution for Linear Cost function	13
4.2	Long time behaviour of the optimal state trajectory	18
4.3	Some investigations on the quadratic cost function	19

1 Introduction

Pollution is one of the most important problems of our times, both for the global consequences (especially climate change) and for the local ones (emission of poisonous substances which diffuse in the air, water, and soil). This problem is strongly connected with the economic dynamics of the various countries as, typically, pollution comes from the production/consumption processes. This creates a trade-off between advantages and disadvantages that have to be managed by the governments. Hence in the last years more and more researchers are putting efforts to develop and study mathematical models to help the institutions deal with this problem.

In particular, when one wants to study the management of the local consequences of pollution, a natural theoretical framework to do this is the one of optimal control or, in the case of more than one institutional agent, the one differential games. The state variables are typically the cumulated amount of pollutants, while the control variables are the production/consumption levels (and in some cases also the investments in pollution abatement). Since the key phenomenon here is the diffusion of the pollutants over space, it is again natural that the state and control variables depend on time and space.

In this direction, in the last years, various contributions studied problems of this type where the state variables follow a diffusion-like Partial Differential Equation (PDE) and the agents maximize an intertemporal utility function. In this way, the mathematical problem becomes the optimal control (or a differential game, in the case of more than one agent) of an infinite dimensional system. These problems in general are very difficult to study and solve, even from the numerical point of view (see on the books [LY95] and [Trö24]). However they become feasible when explicit solutions are available, and this allows to perform a deep analysis of the economic features of the model, see on this the papers [BFFG21, BFFG19] for the case of control and [BFFG22, dFMH19a, dFMH19b, dFLPMH21, dFMH20, dFGLPMH22, JMHZ10] for the case of games.

The aim of this paper is to contribute to this line of research by studying the explicit solutions in the case of one agent (hence optimal control) with the following modeling novelties:

- the space is modeled by a network of interconnected locations instead of being continuous¹;
- the agent has also the possibility, at some cost, to shift part of the production in a less polluting process (which we call renewable).

From the mathematical point of view such novelties requires some nontrivial changes in the approach with respect to the previous models, in particular, the closer ones of [BFFG21, BFFG19]:

¹We observe that also [dFMH19a, dFMH19b] treat the discrete space. However, in their case the discrete space is the result of the discretization of the PDE and is not a network with pollution flow over arches. In this respect we must mention here also the recent paper [XW24] on which we comment in the main text.

- Since the space is now modeled by a network, the state equation becomes a system of ODEs. On one side this is in principle easier than the case of continuous space, as the dimension of the state variable here is finite. On the other hand, this allows us to deal with more general diffusion operators \mathcal{L} which needs to be treated differently (see the recent paper [CGL⁺24] where the network structure is used in a different context).
- The explicit solutions are found exploiting the linearity of the state equation and of the costs to reduce the problem to a parametric static problem (see Theorem 2.4). However, the introduction of the less polluting production process R makes it more difficult, with respect to previous papers (see [BFFG22]) to find explicit solutions. For this reason we first present and study (in Section 3) the solution in the case when R is absent and then (in Section 4) discuss in detail the changes when R is present.

We have here to mention the very recent paper [XW24] which also studies the spatial diffusion of pollutants when the space is a network. Differently from our paper, such paper does not deal with the less polluting production mode R and also proposes a different method of solution based on HJB equations.

The main results of the paper are: Theorem 2.4 on the reduction to a parametric static problem; Theorem 3.3 on the solution of the case without R ; Theorem 4.3 on the solution of the case with R and linear cost of it; Theorem 4.4 on the asymptotic behavior of pollution in the particular case of time-independent coefficients.

The content of the paper is as follows: Section 2 presents the general model and examines its well-posedness. In Section 3 we address the case of non-renewable production only, with 3.1 focusing on the solution of the problem and corresponding optimal paths. In Section 4, we analyze the complete model with both renewable and non-renewable production: Section 4.1 provides explicit solutions for the scenario with linear costs associated with renewable production. In Section 4.2, we address the long-run optimal pollution distributions. Finally, in Section 4.3, we consider a quadratic cost for renewable technologies and focus on the analysis of some numerical simulations.

2 The general model

In this section, we provide an overview of the model. Our focus is on a central planning challenge within a spatially organized economy. Within this economy, a single commodity serves multiple roles: it is consumed, utilized as input in both renewable and non-renewable production (invested), employed in pollution control efforts, and produced at various locations. Additionally, it's important to note that this commodity is not subject to trade between different locations; however, pollution does cross geographical boundaries.

In our model, the space variable is described as a network of interconnected geographic locations. When we refer to a network, we are describing a graph with weights, where the nodes correspond to these locations (e.g., cities, regions, etc.), the edges represent the connections between select locations, and the weights signify the importance of each connection. This network structure enhances the realism of capital transportation within the model and aligns with the inherent network-like nature of pollutant data, which often exhibits a similar network pattern.

We model the network of $n \geq 2$ geographically distributed location, by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where \mathcal{V} is a set of vertices, that corresponds to locations and \mathcal{E} is a set of edges connecting vertices. The graph is simple, weighted, and finite. We identify \mathcal{V} with the set $\{1, \dots, n\}$ of sites, where pollution

is accumulated, capital input is invested and output is produced, consumed and locally re-invested and \mathcal{E} as a subset of $\{(i, j) \in \{1, \dots, n\}^2 \text{ s.t. } i \neq j\}$. We say that two vertices $i, j \in V$ are connected, and we write $i \sim j$ if there exists an edge connecting them, i.e.,

$$i \sim j \iff (i, j) \in \mathcal{E}.$$

We denote with $W = (w_{ij})_{i,j \in V}$ the matrix of the graph with $w_{i,j} \geq 0$. To ease the notation, we assume that vertices $i, j \in \mathcal{V}$ are not connected if and only if $w_{i,j} = 0$. We also assume that there are no self-loops i.e. $w_{ii} = 0$. We stress that $w_{i,j}$ represents the intensity of geographical connection from the node j to the node i .

The transboundary nature of pollution is represented by the action of a linear operator $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$ on the nodes of the graph. This represents the fact that pollution may enter or exit any locations as a result of interaction between them. And as much as the operator L tends to transport pollution across the locations, we also consider on each node the effect of nature's self-cleaning mechanisms through a parameter δ_i that goes to limit its spread.

At time t and at any location $i \in \mathcal{V}$, there is a single individual consuming $C_i(t)$, investing $I_i(t)$ in non-renewable production and $R_i(t)$ in renewable one, depolluting $B_i(t)$ and producing $Y_i(t)$. Production is given by

$$Y_i(t) = a_i^I(t)I_i(t) + a_i^R(t)R_i(t), \quad (1)$$

where $a_i^I(t) \geq 1$ and $a_i^R(t) \geq 1$ are productivity or technological levels for respectively non-renewable and renewable productions, at location i in time t . They can represent possible technological spillovers across sites, disparities in technological advancement across space (illustrating obstacles to technological diffusion), and similar dynamics. Notice that we distinguish between productivity coming from non-renewable and renewable investments, specifically the productivity coming from traditional non-renewable sources is generally greater than the renewable one. Inspired by the work [BFFG21] we assume, for simplification, that capital inputs do not accumulate over time nor are they exchanged across space.

As previously said, at any location, the output is produced, consumed, used in depollution, and locally invested (no trade across locations), implying the following resource constraints:

$$C_i(t) + I_i(t) + R_i(t) + B_i(t) = Y_i(t). \quad (2)$$

Which, together with (1) yields to

$$C_i(t) = (a_i^I(t) - 1)I_i(t) + (a_i^R(t) - 1)R_i(t) - B_i(t).$$

We consider the following control problem with an infinite time horizon on \mathcal{V} .

Let $\forall i \in \mathcal{V} \ p_i, \delta_i, \varepsilon_i \in \mathbb{R}$ and $\varphi_i: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a given measurable function. At each time $t \in \mathbb{R}^+$ and location $i \in \mathcal{V}$, the planner chooses the control variables: i.e. the investment in traditional or brown production $I_i(t)$, the investment in green production $R_i(t)$ and the investment in pollution abatement $B_i(t)$. All of them contribute to the dynamics of the pollution stock $P_i(t)$.

In each node $i \in \mathcal{V}$, the pollution's dynamics evolve according to the following ODE:

$$\begin{cases} \frac{d}{dt}P_i(t) = \sum_{j=1}^n w_{ij}P_j(t) - \sum_{j=1}^n w_{ji}P_i(t) - \delta_i P_i(t) + I_i(t) + \varepsilon R_i(t) - \varphi_i(t)B_i(t)^\theta & t \geq 0 \\ P_i(0) = p_i \in \mathbb{R}_+. \end{cases} \quad (3)$$

It is reasonable to assume that, for all $i \in \mathcal{V}$, $\varepsilon_i < 1$ given that renewable energy sources, such as solar, wind, and hydroelectric power, produce electricity with lower emissions in comparison to

non-renewable sources.

We use vector notation to describe the pollution stock $P(t) := (P_1(t), \dots, P_n(t))$, the consumption level $C(t) := (C_1(t), \dots, C_n(t))$, the investments $I(t) = (I_1(t), \dots, I_n(t))$, $R(t) = (R_1(t), \dots, R_n(t))$, the depollution effort $B(t) = (B_1(t), \dots, B_n(t))$ and the diffusion matrix $L = (\ell_{i,j})$ is defined as

$$\ell_{i,j} = \begin{cases} w_{i,j}, & i \neq j \\ -\sum_{k=1, k \neq i}^n w_{k,i}, & i = j. \end{cases}$$

Calling $p = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ the vector of pollution initial distribution, the dynamics of P in (3) can be rewritten in vector form as

$$\begin{cases} \frac{d}{dt}P(t) = (L - \delta)P(t) + I(t) + \varepsilon R(t) - \varphi(t)B(t)^\theta, & t \geq 0 \\ P(0) = p \in \mathbb{R}_+^n. \end{cases} \quad (4)$$

The pollution stock variation at location i depends on

- the action of the linear operator $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$ modelling transboundary pollution mobility on the networks,
- the natural self cleaning capacity given by $\delta P(t) \geq 0$, where the diagonal matrix $\delta = \text{diag}(\delta_1, \dots, \delta_n)$ represents location-specific decay parameters for each location,
- the inputs $I(t) \in \mathbb{R}_+^n$ and $R(t) \in \mathbb{R}_+^n$, here $\varepsilon = \text{diag}(\varepsilon_1, \dots, \varepsilon_n)$ denotes the pollution intensity factor associated with the investment in renewable energy,
- the abatement $\varphi(t)B(t)^\theta$, where $B(t) \in \mathbb{R}_+^n$, $\varphi(t) = \text{diag}(\varphi_1, \dots, \varphi_n)$ is the efficiency of abatement and $\theta \in (0, 1)$ is the return to scale of abatement (the power function is carried elementwise).

Consider a social planner, who aims at controlling investment levels (I, R, B) to maximize the following social welfare function (the set of admissible controls (I, R, B) which will be specified later)

$$J(p, (I, R, B)) := \int_0^{+\infty} e^{-\rho t} \left(\sum_{i=1}^n \left(\frac{C_i(t)^{1-\gamma}}{1-\gamma} - \omega_i P_i(t) - f_i(R_i(t)) \right) \right) dt, \quad (5)$$

where $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}_+^n$ measures, for instance, local environmental awareness for each location, $\rho \geq 0$ is a given discount factor, $\gamma \in (0, 1) \cup (1, \infty)$, $a_i^I, a_i^R: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \forall i \in \mathcal{V}$ are given measurable functions with $a_i^I(t) \geq 1, a_i^R(t) \geq 1$ and $f_i: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \forall i \in \mathcal{V}$ are a convex functions such that $f(0) = 0$ representing maintenance and operational costs related to renewable investment. Although we recognize that investments in nonrenewable energy also incur costs, in the context of transitioning toward greener energy, we have chosen to focus exclusively on the running costs coming from renewable energy. This decision highlights the critical need to shift towards sustainable energy solutions to address environmental issues. By using eq (1) and (2), we can explicitly rewrite the functional in terms of the investment and abatement controls

$$J(p, (I, R, B)) := \int_0^{+\infty} e^{-\rho t} \left(\sum_{i=1}^n \left(\frac{((a_i^I(t) - 1)I_i(t) + (a_i^R(t) - 1)R_i(t) - B_i(t))^{1-\gamma}}{1-\gamma} - \omega_i P_i(t) - f_i(R_i(t)) \right) \right) dt. \quad (6)$$

The objective functional to be maximized represents the social benefit of a community resulting in a trade-off between different interests, namely technological production and local awareness or sensitivity to environmental problems: in simple terms, $I(t)$ and $R(t)$ represent the two investments in production in each of the n locations. This investment increases utility through consumption but also increases pollution and can cause costs, which in turn decrease utility. On the other hand, allocating funds to depollution efforts through $B(t)$ helps mitigate pollution growth but reduces consumption.

The set of admissible controls $\mathcal{A}(p)$ is defined as

$$\mathcal{A}(p) := \left\{ (I, R, B): \mathbb{R}^+ \rightarrow \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n : \right. \\ \left. \int_0^\infty e^{-\rho t} \left(\sum_{i=1}^n |f_i(R_i(t))|^2 \right)^{\frac{1}{2}} dt < \infty, \int_0^\infty e^{-\rho t} \left(\sum_{i=1}^n |I_i(t) + \varepsilon R_i(t) - \varphi_i(t) B_i(t)^\theta|^2 \right)^{\frac{1}{2}} dt < \infty, \right. \\ \left. C_i(t) = (a_i^I(t) - 1)I_i(t) + (a_i^R(t) - 1)R_i(t) - B_i(t) \geq 0 \forall t \in \mathbb{R}^+, \forall i \in \mathcal{V} \text{ and } P^p(t) \geq 0 \forall t \in \mathbb{R}^+ \right\}.$$

We call **(P)** the problem

$$\text{maximize } J(p, (I, R, B)) \quad \text{over } (I, R, B) \in \mathcal{A}(p) \quad \textbf{(P)}$$

and we define the value function

$$v(p) = \sup_{(I, R, B) \in \mathcal{A}(p)} J(p, (I, R, B)).$$

Notice that the problem is a state constraint optimal control problem. However, in the next sections, we will show that we can deal with this technical difficulty, see Theorem 2.3 and Theorem 2.4.

The following assumptions will be in force throughout the paper.

Remark 2.1. *Observe that the matrix L satisfies the following properties:*

- (i) L is a Metzler matrix, namely $\ell_{ij} \geq 0$ for all $i \neq j$. This ensures that $(e^{tL})_{t \geq 0}$ is a positive linear system, that is, for every non-negative $p \in \mathbb{R}_+^n$, we have that $e^{tL}p \in \mathbb{R}_+^n$, for all $t \geq 0$, see [FR11, Theorem 2].
- (ii) $\zeta = 0$ is an eigenvalue and the vector $(1, \dots, 1)$ is an eigenvector associated to it. All the other eigenvalues ζ are such that $2 \min_i \ell_{ii} \leq \text{Re}(\zeta) < 0$. This ensures that L is a dissipative operator and that L generates a strongly continuous contraction semigroup, see [ENB00, Chapter II].
- (iii) If we assume L to be also symmetric, the previous property implies L to be negative semidefinite. Notice that this case coincides with the discrete Laplacian with the opposite sign.

Consider the finite-dimensional operator $\mathcal{L}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ where

$$\mathcal{L}(\psi) = (L - \delta)\psi, \quad \psi \in \mathbb{R}^n.$$

Notice that since δ is diagonal, L Metzler matrix implies $L - \delta$ to be also a Metzler matrix, i.e.

$(e^{(L-\delta)})_{t \geq 0}$ is a positive linear system. This implies that, if we choose a zero-investment and abatements path, i.e. if $I(t) = 0, R(t) = 0, B(t) = 0$, for all $t \geq 0$, then the solution to (4) is non-negative for every non-negative initial pollution data.

The operator

$$\zeta \mathbb{1} - \mathcal{L}: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is invertible with bounded inverse $(\zeta \mathbb{1} - \mathcal{L})^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and the resolvent formula (see Theorem 1.10 in Chapter II of [ENB00]) holds for every $\zeta > 0$:

$$(\zeta \mathbb{1} - \mathcal{L})^{-1}h = \int_0^\infty e^{-(\zeta \mathbb{1} - \mathcal{L})t} h dt, \quad \forall h \in \mathbb{R}^n. \quad (7)$$

2.1 Well posedness of the objective function

By defining the net emissions function $N: \mathbb{R}^+ \rightarrow \mathbb{R}^n$:

$$N(t) := I(t) + \varepsilon R(t) - \varphi(t)B(t)^\theta,$$

We rewrite the problem in a vectorial form. The equation (4) is rewritten as

$$\begin{cases} \frac{d}{dt}P(t) = (L - \delta)P(t) + N(t) \\ P(0) = p \in \mathbb{R}_+^n. \end{cases} \quad (8)$$

The set $\mathcal{A}(p)$ is rewritten as

$$\mathcal{A}(p) := \left\{ (I, R, B): \mathbb{R}^+ \rightarrow \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n: \int_0^\infty e^{-\rho t} \|N(t)\| dt < \infty, \int_0^\infty e^{-\rho t} \|f(R(t))\| dt < \infty, \right. \\ \left. (a_i^I(t) - 1)I_i(t) + (a_i^R(t) - 1)R_i(t) - B_i(t) \geq 0 \forall t \in \mathbb{R}^+, i \in \mathcal{V} \text{ and } P^p(t) \geq 0 \forall t \in \mathbb{R}^+ \right\}.$$

The functional (6) can be reformulated as

$$J(p, (I, R, B)) := \int_0^{+\infty} e^{-\rho t} \left(\left\langle \frac{((A^I(t) - \mathbb{1})I(t) + (A^R(t) - \mathbb{1})R(t) - B(t))^{1-\gamma}}{1-\gamma}, \mathbf{1} \right\rangle - \langle \omega, P(t) \rangle - \langle f(R(t)), \mathbf{1} \rangle \right) dt, \quad (9)$$

where $\mathbf{1}$ represents the vector of ones in \mathbb{R}^n , $A^I(t) = \text{diag}(a_1^I(t), \dots, a_n^I(t))$, $A^R(t) = \text{diag}(a_1^R(t), \dots, a_n^R(t))$ and $\omega = (\omega_1, \dots, \omega_n)$. Standard results ensure that, for every admissible control, ODE (8) has a unique solution which verifies

$$P(t) = e^{t\mathcal{L}}p + \int_0^t e^{(t-s)\mathcal{L}}N(s)ds, \quad t \geq 0. \quad (10)$$

Proposition 2.2. *$J(p, (I, R, B))$ is well defined for all $p \in \mathbb{R}_+^n$ and $(I, R, B) \in \mathcal{A}(p)$, possibly equal to $+\infty$ or $-\infty$ (depending, respectively, on the occurrences $\gamma \in (0, \infty)$ and $\gamma \in (1, \infty)$, respectively).*

Proof. The term $\frac{((A^I(t)-1)I(t)+(A^R(t)-1)R(t)-B(t))^{1-\gamma}}{1-\gamma}$ in (9) is always either positive (if $\gamma \in (0, 1)$) or negative (if $\gamma > 1$). Since the map $t \rightarrow e^{-\rho t} \|f(R_t)\|$ is integrable, it suffices to show that $\int_0^t e^{-\rho t} \langle \omega, P(t) \rangle dt$ is well defined and finite. We have

$$\int_0^\infty e^{-\rho t} \langle \omega, P(t) \rangle dt = \int_0^\infty e^{-\rho t} \langle \omega, e^{t\mathcal{L}} p \rangle + \int_0^t e^{(t-s)\mathcal{L}} N(s) ds \rangle dt.$$

Now since ω a constant in \mathbb{R}^n and $e^{t\mathcal{L}}$ is a contraction, the integral $\int_0^\infty e^{-\rho t} \langle \omega, e^{t\mathcal{L}} p \rangle dt$ is finite. Moreover for $T > 0$ we get, by Fubini-Tonelli's theorem:

$$\begin{aligned} \int_0^T \left(\int_0^t e^{-\rho t} \langle \omega, e^{(t-s)\mathcal{L}} N(s) \rangle ds \right) dt &= \int_0^T \left(\int_0^t e^{-\rho s} \langle \omega, e^{-(\rho\mathbb{1}-\mathcal{L})(t-s)} N(s) \rangle ds \right) dt \\ &= \int_0^T e^{-\rho s} \langle \omega, \int_s^T e^{-(\rho\mathbb{1}-\mathcal{L})(t-s)} N(s) dt \rangle ds. \end{aligned}$$

Using again that $e^{(t-s)\mathcal{L}}$ is a contraction, we have

$$\left\| \int_s^T e^{-(\rho\mathbb{1}-\mathcal{L})(t-s)} N(s) dt \right\| \leq \int_s^\infty e^{-\rho(t-s)} \|N(s)\| dt \leq \frac{1}{\rho} \|N(s)\|.$$

And the claim follows by sending T to $+\infty$. \square

2.2 Rewriting the objective function

The planner aims at solving the optimization problem

$$v(p) := \sup_{(I,R,B) \in \mathcal{A}(p)} J(p, (I, R, B)).$$

The function v denotes the value function of the optimization problem and a triple (I^*, R^*, B^*) such that $J(p; (I^*, R^*, B^*)) = v(p)$ is said to be an optimal control for the problem starting at p .

We now define a vector α (which can also be seen as a function of the nodes), which we use to rewrite the objective functional in a convenient way. Set

$$\alpha := (\rho\mathbb{1} - \mathcal{L}^\top)^{-1} \omega = \int_0^\infty e^{-(\rho\mathbb{1}-\mathcal{L}^\top)t} \omega dt. \quad (11)$$

By definition, α is the unique solution in H of the abstract equation

$$(\rho\mathbb{1} - \mathcal{L}^\top) \alpha = \omega,$$

more explicitly

$$(\rho\mathbb{1} - (L - \delta)^\top) \alpha = \omega.$$

Proposition 2.3. *We have, for all $p \in \mathbb{R}_+^n$ and $(I, R, B) \in \mathcal{A}(p)$,*

$$\begin{aligned} J(p, (I, R, B)) &= -\langle \alpha, p \rangle + \int_0^{+\infty} e^{-\rho t} \left[\left\langle \frac{((A^I(t)-1)I(t)+(A^R(t)-1)R(t)-B(t))^{1-\gamma}}{1-\gamma}, \mathbf{1} \right\rangle \right. \\ &\quad \left. - \langle \alpha, I(t) + \varepsilon R(t) - \varphi(t)B(t)^\theta \rangle - \langle f(R(t)), \mathbf{1} \rangle \right] dt. \end{aligned} \quad (12)$$

Proof. Using (10), we can rewrite the second term of the functional (9) in a more convenient way. Set

$$e^{-(\rho\mathbb{1}-\mathcal{L})} := e^{-\rho t} e^{t\mathcal{L}}, \quad t \geq 0,$$

and rewrite

$$\begin{aligned} \int_0^t e^{-\rho t} \langle \omega, P(t) \rangle dt &= \int_0^\infty e^{-\rho t} \langle \omega, e^{t\mathcal{L}} p + \int_0^t e^{(t-s)\mathcal{L}} N(s) ds \rangle dt \\ &= \langle \omega, \int_0^\infty e^{-(\rho\mathbb{1}-\mathcal{L})t} p dt \rangle + \int_0^\infty e^{-\rho t} \langle \omega, \int_0^t e^{(t-s)\mathcal{L}} N(s) ds \rangle dt. \end{aligned} \quad (13)$$

Note that the first term of the right-hand side is the only one which depends on the initial datum p . By (7), the first term can be rewritten as

$$= \langle \omega, \int_0^\infty e^{-(\rho\mathbb{1}-\mathcal{L})t} p dt \rangle = \langle \omega, (\rho\mathbb{1} - \mathcal{L})^{-1} p \rangle = \langle (\rho\mathbb{1} - \mathcal{L}^\top)^{-1} \omega, p \rangle = \langle \alpha, p \rangle.$$

The second term in (13) can be rewritten by exchanging the integrals as:

$$\begin{aligned} \int_0^\infty e^{-\rho t} \langle \omega, \int_0^t e^{(t-s)\mathcal{L}} N(s) ds \rangle dt &= \int_0^\infty \left(\int_0^t e^{-\rho t} \langle \omega, e^{(t-s)\mathcal{L}} N(s) \rangle ds \right) dt \\ &= \int_0^\infty \left(\int_0^t e^{-\rho s} \langle \omega, e^{-(\rho\mathbb{1}-\mathcal{L})(t-s)} N(s) \rangle ds \right) dt \\ &= \int_0^\infty e^{-\rho s} \left\langle \omega, \int_s^\infty e^{-(\rho\mathbb{1}-\mathcal{L})(t-s)} N(s) dt \right\rangle ds \\ &= \int_0^\infty e^{-\rho s} \langle \omega, (\rho\mathbb{1} - \mathcal{L})^{-1} N(s) \rangle ds \\ &= \int_0^\infty e^{-\rho s} \langle (\rho\mathbb{1} - \mathcal{L}^\top)^{-1} \omega, N(s) \rangle ds. \end{aligned}$$

□

As a consequence of (12) we get the following useful result.

Theorem 2.4. *Let $(I^*(t), R^*(t), B^*(t))$ be an admissible strategy, i.e. $(I^*(t), R^*(t), B^*(t)) \in \mathcal{A}(p)$. Assume moreover that, for a.e. $t \in \mathbb{R}^+$, and for each $i = 1, \dots, n$, the triplet $(I_i^*(t), R_i^*(t), B_i^*(t))$ is a maximum point for the function*

$$F_{it} : D_i(t) \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$$

where

$$D_i(t) := \{(a_i^I(t) - 1)I_i + (a_i^R(t) - 1)R_i - B_i \geq 0\}$$

and

$$F_{it}(I_i, R_i, B_i) = \frac{((a_i^I(t) - 1)I_i + (a_i^R(t) - 1)R_i - B_i)^{1-\gamma}}{1-\gamma} - \alpha_i(I_i + \varepsilon_i R_i - \varphi_i B_i) - f_i(R_i), \quad (14)$$

then $(I^*(t), R^*(t), B^*(t))$ is optimal for the problem **(P)**.

Proof. This is a straightforward consequence of Proposition 2.3. Indeed, according to the reformulation presented in (12) the problem is reduced to a static one because the integral in (12) can be optimized pointwise, fixed time $t \in \mathbb{R}$ and fixed $i \in cV$. Indeed, the objective function can be rewritten as

$$J(p, (I, R, B)) = -\langle \alpha, p \rangle + \sum_{i=1}^n \int_0^{+\infty} e^{-\rho t} F_i(I_i(t), R_i(t), B_i(t)) dt, \quad (15)$$

where F_i is defined in (14). If $(I_i^*(t), R_i^*(t), B_i^*(t))$ is a maximum of the function F_i that is integrated in time, then $(I^*(t), R^*(t), B^*(t))$ is a maximum for the control problem without any constraint on the state variable and on the control. If moreover $(I^*(t), R^*(t), B^*(t))$ belong to $\mathcal{A}(p)$, then it is also optimal for **(P)**. \square

Remark 2.5. From Theorem 2.4 it is clear that the optimal control I_i, R_i, B_i are interlaced, and they depend on the other nodes, only through the parameter α_i which depends on the matrix L .

Remark 2.6. If the value function is finite, from the rewriting of the functional (5) presented in (12) and (15) we can deduce some monotonic relationships between the value function and the various parameters of the model. For simplicity, by assuming that parameters are equals in all nodes, namely $a_i^I = a^I$, $a_i^R = a^R$, $\varepsilon_i = \varepsilon$, $\varphi_i = \varphi$ and $\omega_i = \omega \forall i \in \mathcal{V}$ we deduce that

- the value function v is increasing with respect to technological productivities, a^I , $a_i^R = a^R$, to the efficient of abatement φ .
- the value function v is decreasing with respect to pollution intensity ε , with respect to pollution awareness ω (because α is a linear function of ω).

By assuming quadratic or linear costs, $f_i(R_i) = \lambda_i R_i$ or $f_i(R_i) = \lambda_i R_i^2$ and by assuming that $\lambda_i = \lambda \forall i \in \mathcal{V}$ we can deduce that the value function is decreasing also with respect to the cost parameter λ .

3 The model with $a_i^R \equiv 1 \forall i \in \mathcal{V}$.

We will now address the case in which on every node, the non-renewable productivity factor is greater than one, while the renewable one equals one, namely $a_i^R \equiv 1$ and $a_i^I > 1 \forall i \in \mathcal{V}$. In this setting, any investment in renewable energy sources is economically unfeasible, and thus the entire production relies on a single (traditional) energy source. So, to simplify our setting, we will directly consider as the only possible investment I . With this simplification, the production takes the form

$$Y_i(t) = a_i^I(t)I_i(t),$$

and the resource constraint implies that the consumption is simply given by

$$C_i(t) = (a_i^I(t) - 1)I_i(t) - B_i(t).$$

The dynamics of P can be written as

$$\begin{cases} \frac{d}{dt}P(t) = (L - \delta)P(t) + I(t) - \varphi(t)B(t)^\theta \\ P(0) = p \in \mathbb{R}_+^n, \end{cases}$$

and the social welfare to be maximised

$$J(p, (I, R, B)) := \int_0^{+\infty} e^{-\rho t} \left(\sum_{i=1}^n \left(\frac{C_i(t)^{1-\gamma}}{1-\gamma} - \omega_i P_i(t) \right) \right) dt. \quad (16)$$

Finally, the set of admissible controls

$$\mathcal{A}(p) := \left\{ (I, B) : \mathbb{R}^+ \rightarrow \mathbb{R}_+^n \times \mathbb{R}_+^n : \int_0^\infty e^{-\rho t} \left(\sum_{i=1}^n |I_i(t) - \varphi_i(t) B_i(t)^\theta|^2 \right)^{\frac{1}{2}} dt < \infty, \right. \\ \left. C_i(t) = (a_i^I(t) - 1)I_i(t) - B_i(t) \geq 0 \forall t \in \mathbb{R}^+, i \in \mathcal{V} \text{ and } P^p(t) \geq 0 \forall t \in \mathbb{R}^+ \right\}.$$

3.1 Solution of the problem and Optimal paths

To ensure the existence of a solution we will make the additional assumptions:

Assumption 3.1. (i) *There exist $C \geq 0, g \geq 0$ such that*

$$(a_i^I(t) - 1)^{\frac{1-\gamma}{\gamma}} + \varphi_i(t)^{\frac{1}{1-\theta}} (a_i^I(t) - 1)^{\frac{\theta}{1-\theta}} \leq C e^{gt}, \quad \forall t \in \mathbb{R}^+, \quad \forall i \in \mathcal{V},$$

(ii) $\rho > g$,

$$(iii) \alpha_i^{-\frac{1}{\gamma}} (a_i^I(t) - 1)^{\frac{1-\gamma}{\gamma}} + \theta^{\frac{1}{1-\theta}} (1 - \theta^{-1}) \varphi_i(t)^{\frac{1}{1-\theta}} (a_i^I(t) - 1)^{\frac{\theta}{1-\theta}} \geq 0.$$

Remark 3.2. *Assumptions 3.1-(i)-(ii) guarantee that the value function is finite. Given this, to solve the problem we use the alternative form (12) of the objective functional. In such form, we take the control which, for every t maximizes the integrand in (12). This is a candidate optimal control. To show that it is indeed optimal, using Theorem 2.4 we need to show that it is admissible: Assumption 3.1-(iii) ensures that such candidate optimal control is admissible since it leads to positive net emissions, hence to positive state trajectories.*

Theorem 3.3. *The couple (I^*, B^*) given by*

$$B^*(t) = (\theta \varphi(t) (A^I(t) - \mathbb{1}))^{\frac{1}{1-\theta}}, \\ I^*(t) = \alpha^{-\frac{1}{\gamma}} (A^I(t) - \mathbb{1})^{\frac{1-\gamma}{\gamma}} + (\theta \varphi(t))^{\frac{1}{1-\theta}} (A^I(t) - \mathbb{1})^{\frac{\theta}{1-\theta}},$$

belongs to $\mathcal{A}(p)$ in (3) and is optimal for (16) starting at each p . The optimal emissions flow is

$$N^*(t) := I^*(t) - \varphi(t) B^*(t)^\theta = \alpha^{-\frac{1}{\gamma}} (A^I(t) - \mathbb{1})^{\frac{1-\gamma}{\gamma}} + \theta^{\frac{1}{1-\theta}} (1 - \theta^{-1}) \varphi(t)^{\frac{1}{1-\theta}} (A^I(t) - \mathbb{1})^{\frac{\theta}{1-\theta}}, \quad (17)$$

and the optimal consumption flow is

$$C^*(t) = (A^I(t) - \mathbb{1}) I^*(t) - B^*(t) = \left(\frac{A^I(t) - \mathbb{1}}{\alpha} \right)^{\frac{1}{\gamma}}.$$

The optimal pollution flow is

$$P^*(t) := e^{t\mathcal{L}} p + \int_0^t e^{(t-s)\mathcal{L}} N^*(s) ds, \quad t \geq 0.$$

The value function is affine in p :

$$v(p) = J(p; (I^*, B^*)) = -\langle \alpha, p \rangle + \int_0^\infty e^{-\rho t} \left(\sum_{i=1}^n \frac{\gamma}{1-\gamma} \left(\frac{a_i^I(t) - 1}{\alpha_i} \right)^{\frac{1-\gamma}{\gamma}} \right) dt \\ - \theta^{\frac{1}{1-\theta}} (1 - \theta^{-1}) \int_0^\infty e^{-\rho t} \left(\sum_{i=1}^n \alpha_i \varphi_i^{\frac{1}{1-\theta}} ((a_i^I(t) - 1))^{\frac{\theta}{1-\theta}} \right) dt.$$

Proof. As suggested by Theorem 2.4, we look for a control that are admissible and that are optimal for the function F , defined in (14), where the control R is zero, namely

$$F(I, B) = \left\langle \left(\frac{((A^I(t) - 1)I - B)^{1-\gamma}}{1-\gamma}, \mathbf{1} \right), \alpha \right\rangle - \langle \alpha, I - \varphi B^\theta \rangle \\ = \sum_{i=1}^n \frac{((a_i^I(t) - 1)I_i - B_i)^{1-\gamma}}{1-\gamma} - \alpha_i (I_i - \varphi_i B_i^\theta) = \sum_{i=1}^n F_i(I_i, R_i, B_i).$$

First, we need to check that $(I^*, B^*) \in \mathcal{A}(p)$. We have

$$(A^I(t) - \mathbb{1})I^*(t) - B^*(t) = \left(\frac{A^I(t) - \mathbb{1}}{\alpha} \right)^{\frac{1}{\gamma}} \geq 0, \quad \forall t \in \mathbb{R}^+, \forall i \in \mathcal{V}.$$

Moreover, considering $N^*(t)$ as in (17) and Assumption 3.1, we get the existence of some constant $C_0 > 0$ such that

$$0 \leq N_i^*(t) \leq C_0 e^{gt}, \quad \forall t \in \mathbb{R}^+, \forall i \in \mathcal{V}.$$

We conclude that $(I^*, B^*) \in \mathcal{A}(p)$ by Assumption 3.1-(iii).

Concerning optimality, as stressed in Theorem 2.4 the integrals in (12) can be optimized pointwisely, indeed fix $t \in \mathbb{R}^+$, $i \in \mathcal{V}$.

By strict concavity of F_i with respect to $\tilde{i} := I_i(t)$ and $\tilde{b} := B_i(t)$, the unique maximum point can be found just by first-order optimality conditions. The resulting system is

$$\begin{cases} ((a_i^I(t) - 1)\tilde{i} - \tilde{b})^{-\gamma} (a_i^I(t) - 1) - \alpha_i = 0 \\ -((a_i^I(t) - 1)\tilde{i} - \tilde{b})^{-\gamma} + \alpha_i \varphi_i(t) \theta b^{\theta-1} = 0. \end{cases}$$

The claim on the optimal control then follows by solving the above system and all the remaining claims immediately follow from straightforward computations. \square

In our model, local pollution reduction efforts are determined by local productivity, taking into account both production and de-pollution activities. This aspect is independent of the transboundary nature of pollution. However, when it comes to investments, they do depend on this transboundary aspect. The regulator must consider not only the local technological factors but also the potential impact of making investments in a specific location on the neighbouring areas in terms of pollution. It's worth noting that local investments may not necessarily increase with local productivity production, denoted as $A^I(t)$. In some cases, higher local productivity might result in lower investments ($I(t)$), which leads to reduced local emissions, albeit at the cost of a slight reduction in production.

4 The model with Renewable Production

In this section, we will consider the general model which includes the possibility of also investing in renewable energy sources in each location, in particular, we assume the technological level for both non-renewable and renewable production to be greater than one in each node, namely $a_i^I(t) > 1$, $a_i^R(t) > 1$, $\forall i \in \mathcal{V}$.

We recall that investing in renewable production has a twofold effect: it increases the total production Y and influences pollution dynamics, even with a lower impact than non-renewable production.

4.1 Explicit solution for Linear Cost function

To ensure the existence of an optimal solution for (6) in the set of admissible controls, we should consider a slightly modified version of Assumption 3.1.

Assumption 4.1. (i) There exist $C \geq 0, g \geq 0$ such that $\forall t \in \mathbb{R}^+, \forall i \in \mathcal{V}$,

$$(a_i^I(t) - 1)^{\frac{1-\gamma}{\gamma}} + (a_i^R(t) - 1)^{\frac{1-\gamma}{\gamma}} + \varphi_i(t)^{\frac{1}{1-\theta}} \left((a_i^I(t) - 1)^{\frac{\theta}{1-\theta}} + (a_i^R(t) - 1)^{\frac{\theta}{1-\theta}} \right) \leq C e^{gt},$$

(ii) $\rho > g$,

(iii)

$$\begin{aligned} & \min \left((a_i^I(t) - 1)^{\frac{1-\gamma}{\gamma}} \alpha_i^{-\frac{1}{\gamma}} + \theta^{\frac{1}{1-\theta}} (1 - \theta^{-1}) \varphi_i(t)^{\frac{1}{1-\theta}} (a_i^I(t) - 1)^{\frac{\theta}{1-\theta}} \right. \\ & \left. - (a_i^R(t) - 1)^{-1} \left((a_i^I(t) - 1)^{\frac{1}{\gamma}} \alpha_i^{-\frac{1}{\gamma}} + ((a_i^I(t) - 1) \varphi_i \theta)^{\frac{1}{1-\theta}} \right) \frac{\lambda_i}{\alpha_i}, \varepsilon_i (a_i^R(t) - 1)^{\frac{1-\gamma}{\gamma}} (\lambda_i + \varepsilon_i \alpha_i)^{-\frac{1}{\gamma}} + \right. \\ & \left. (a_i^R(t) - 1)^{\frac{\theta}{1-\theta}} \varphi_i^{\frac{1}{1-\theta}} \left(\frac{\theta \alpha_i}{\lambda_i + \varepsilon_i \alpha_i} \right)^{\frac{1}{1-\theta}} \left(\varepsilon_i - \left(\frac{\theta \alpha_i}{\lambda_i + \varepsilon_i \alpha_i} \right)^{-1} \right) \right) \geq 0. \end{aligned}$$

Remark 4.2. Just as in Section 3, Assumptions 4.1-(i)-(ii) guarantee that the value function is finite. Assumption 4.1-(iii) ensures that the optimal control leads to positive net emissions, hence that the associated state trajectory remains positive.

Theorem 4.3. Consider as cost function $f(R) = \Lambda(R(t))$ where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$.

- If $\lambda_i < \left[\frac{(a_i^R - 1)}{(a_i^I - 1)} - \varepsilon_i \right] \alpha_i$ for some $i \in \{1, \dots, n\}$,

$$\begin{cases} I_i^* = 0 \\ B_i^* = \left(\frac{\lambda_i + \varepsilon_i \alpha_i}{\theta (a_i^R - 1) \varphi_i \alpha_i} \right)^{\frac{1}{\theta-1}} \\ R_i^* = (a_i^R - 1)^{\frac{1-\gamma}{\gamma}} (\lambda_i + \varepsilon_i \alpha_i)^{-\frac{1}{\gamma}} + (a_i^R - 1)^{-1} \left(\frac{\lambda_i + \varepsilon_i \alpha_i}{\theta (a_i^R - 1) \varphi_i \alpha_i} \right)^{\frac{1}{\theta-1}} \end{cases}$$

belongs to $\mathcal{A}(p)$ and is optimal starting at each p . The optimal emissions flow is

$$N^*(t) := \varepsilon R^*(t) - \varphi(t) B^*(t)^\theta,$$

and the optimal consumption flow is

$$C^*(t) = (A^R(t) - \mathbb{1})R^*(t) - B^*(t) = \left(\frac{A^R(t) - \mathbb{1}}{\lambda + \varepsilon\alpha} \right)^{\frac{1}{\gamma}}.$$

- If $\lambda_i > \left[\frac{(a_i^R - 1)}{(a_i^I - 1)} - \varepsilon_i \right] \alpha_i$ for some $i \in \{1, \dots, n\}$,

$$\begin{cases} I_i^* = (a_i^I - 1)^{\frac{1-\gamma}{\gamma}} \alpha_i^{-\frac{1}{\gamma}} + (a_i^I - 1)^{\frac{\theta}{1-\theta}} (\theta \varphi_i \alpha_i)^{\frac{1}{1-\theta}} \\ B_i^* = ((a_i^I - 1) \varphi_i \theta)^{\frac{1}{1-\theta}} \\ R_i^* = 0 \end{cases}$$

belongs to $\mathcal{A}(p)$ and is optimal starting at each p . The optimal emissions flow is

$$N^*(t) := I^*(t) - \varphi(t)B^*(t)^\theta,$$

and the optimal consumption flow is

$$C^*(t) = (A^I(t) - \mathbb{1})I^*(t) - B^*(t) = \left(\frac{A^I(t) - \mathbb{1}}{\alpha} \right)^{\frac{1}{\gamma}}.$$

- If $\lambda_i = \left[\frac{(a_i^R - 1)}{(a_i^I - 1)} - \varepsilon_i \right] \alpha_i$ for some $i \in \{1, \dots, n\}$ and $\gamma \neq 1$,

$$\begin{cases} (a_i^I - 1)I_i^* + (a_i^R - 1)R_i^* = (a_i^I - 1)^{\frac{1}{\gamma}} \alpha_i^{-\frac{1}{\gamma}} + ((a_i^I - 1) \varphi_i \theta)^{\frac{1}{1-\theta}} \\ B_i^* = ((a_i^I - 1) \varphi_i \theta)^{\frac{1}{1-\theta}} \\ 0 < R_i^* < (a_i^R - 1)^{-1} \left((a_i^I - 1)^{\frac{1}{\gamma}} \alpha_i^{-\frac{1}{\gamma}} + ((a_i^I - 1) \varphi_i \theta)^{\frac{1}{1-\theta}} \right) \end{cases}$$

belongs to $\mathcal{A}(p)$ and is optimal starting at each p . The optimal emissions flow is

$$N^*(t) := I^*(t) + \varepsilon R^*(t) - \varphi(t)B^*(t)^\theta,$$

and the optimal consumption flow is

$$C^*(t) = (A^I(t) - \mathbb{1})I^*(t) - B^*(t) = \left(\frac{A^I(t) - \mathbb{1}}{\alpha} \right)^{\frac{1}{\gamma}}.$$

Proof. As suggested by Theorem 2.4, we look for a control that are admissible and that are optimal for the function F_i , defined in (14) with $f_i(R_i) = \lambda_i R_i$. Consider the set

$$\mathcal{D}_i = \{I_i, B_i, R_i \geq 0 : (a_i^I - 1)I_i + (a_i^R - 1)R_i - B_i \geq 0\}.$$

First, we observe that F_i is coercive on \mathcal{D}_i . Indeed, if $\gamma \in (0, 1)$,

$$F_i(I_i, R_i, B_i) \leq c_1 \frac{(I_i + R_i)^{1-\gamma}}{1-\gamma} - \alpha_i I_i - (\varepsilon_i + \lambda_i) R_i + c_2 (I_i + R_i)^\theta,$$

while if $\gamma \in (1, +\infty)$,

$$F_i(I_i, R_i, B_i) \leq -\alpha_i I_i - (\varepsilon_i + \lambda_i) R_i + c_2 (I_i + R_i)^\theta.$$

Since F_i is coercive on the set \mathcal{D}_i , F_i admits a global maximum on \mathcal{D}_i . We look for the maximum between points that satisfy the Karush-Kuhn-Tucker condition and points where the function is not derivable. Points satisfying KKT conditions are solutions of the following system:

$$\begin{cases} ((a_i^I - 1)I_i + (a_i^R - I_i)R_i - B_i)^{-\gamma} (a_i^I - 1) - \alpha_i = -\mu_{1,i} \\ -((a_i^I - 1)I_i + (a_i^R - I_i)R_i - B_i)^{-\gamma} - \theta \varphi_i \alpha_i B_i^{\theta-1} = -\mu_{2,i} \\ ((a_i^I - 1)I_i + (a_i^R - I_i)R_i - B_i)^{-\gamma} \cdot (a_i^R - 1) - \varepsilon_i \alpha_i - \lambda_i = -\mu_{3,i} \\ \mu_{1,i}, \mu_{2,i}, \mu_{3,i} \geq 0 \\ I_i \cdot \mu_{1,i} = 0 \\ B_i \cdot \mu_{2,i} = 0 \\ R_i \cdot \mu_{3,i} = 0. \end{cases} \quad (18)$$

Case 1: $I_i = B_i = R_i = 0$. Since the origin is a point where the function is not derivable, we exclude this point and study the behaviour of the function in the origin separately.

Case 2: $I_i = B_i = 0, R_i \neq 0$. Same argument here. $B_i = 0$ is a set of points where the function is not derivable. So this case will be studied separately.

Case 3: $I_i = R_i = 0, B_i \neq 0$. For this values, the system (18) becomes

$$\begin{cases} (-B_i)^{-\gamma} \cdot (a_i^I - 1) - \alpha_i = -\mu_{1,i} \\ -(-B_i)^{-\gamma} - \theta \varphi_i \alpha_i B_i^{\theta-1} = 0 \\ (-B_i)^{-\gamma} (a_i^R - 1) - \alpha_i \varepsilon_i - \lambda_i = -\mu_{3,i} \\ \mu_{1,i}, \mu_{3,i} > 0. \end{cases}$$

If $\gamma \in (0, 1)$ the system does not admit any solutions. If $\gamma > 1$, it is an integer even number, the system does not admit any solution. Indeed, the second equation becomes

$$\alpha_i \varphi_i \theta B_i^{\theta-1} = -\frac{1}{(-B_i)^\gamma},$$

where for $B_i > 0$, the left-hand side is a positive quantity, while the right-hand side is a negative quantity. In conclusion for $\gamma > 1$, different from an integer even number, we have that the system admits solution but the constraint $(a_i^I - 1)I_i + (a_i^R - 1)R_i - B_i > 0$ is not satisfied.

Case 4: $I_i = 0, B_i \neq 0, R_i \neq 0$. If $\lambda_i < -\alpha_i \varepsilon_i + \alpha_i \frac{(a_i^R - 1)}{(a_i^I - 1)}$ the system (18) admits as solution the point

$$\begin{cases} I_i^* = 0 \\ B_i^* = \left(\frac{\lambda_i + \alpha_i \varepsilon_i}{(a_i^R - 1) \alpha_i \varphi_i \theta} \right)^{\frac{1}{\theta-1}} \\ R_i^* = \frac{1}{(a_i^R - 1)} \left(\frac{\lambda_i + \alpha_i \varepsilon_i}{(a_i^R - 1)} \right)^{-\frac{1}{\gamma}} + \frac{1}{(a_i^R - 1)} \left(\frac{\lambda_i + \alpha_i \varepsilon_i}{(a_i^R - 1) \alpha_i \varphi_i \theta} \right)^{\frac{1}{\theta-1}}. \end{cases}$$

Notice that this point satisfies the constraint $(a_i^I - 1)I_i + (a_i^R - 1)R_i - B_i > 0$.

Case 5: $I_i \neq 0, B_i = R_i = 0$. Since $B_i = 0$ is a set of points where the function is not derivable, we study the behavior of the function F on these points separately.

Case 6: $I_i \neq 0, B_i = 0, R_i \neq 0$. This case is similar to the previous one.

Case 7: $I_i \neq 0, B_i \neq 0, R_i = 0$. If $\lambda_i > -\alpha_i \varepsilon_i + \alpha_i \frac{(a_i^R - 1)}{(a_i^I - 1)}$ the system (18) admits as solution the point

$$\begin{cases} I_i^* = \frac{1}{(a_i^I - 1)} \left(\frac{\alpha_i}{(a_i^I - 1)} \right)^{-\frac{1}{\gamma}} + \frac{1}{(a_i^I - 1)} \left(\frac{1}{(a_i^I - 1) \alpha_i \varphi_i \theta} \right)^{\frac{1}{\theta - 1}} \\ B_i^* = \left(\frac{\alpha_i}{(a_i^I - 1) \alpha_i \varphi_i \theta} \right)^{\frac{1}{\theta - 1}} \\ R_i^* = 0. \end{cases}$$

Notice that this point satisfies the constraint $(A_i^I - 1)I_i + (A_i^R - 1)R_i - B_i > 0$.

Case 8: $I_i \neq 0, B_i \neq 0, R_i \neq 0$. If $\lambda_i = -\alpha_i \varepsilon_i + \alpha_i \frac{(a_i^R - 1)}{(a_i^I - 1)}$, the system (18) admits as solutions the points of the line

$$\begin{cases} B_i^* = (\theta \varphi_i (a_i^I - 1))^{-\frac{1}{\theta - 1}} \\ (a_i^I - 1)I_i + (a_i^R - 1)R_i = (\alpha_i \cdot (a_i^I - 1)^{-1})^{-\frac{1}{\gamma}} + (\theta \varphi_i (a_i^I - 1))^{-\frac{1}{\theta - 1}}. \end{cases}$$

We will denote with (P_0^*) the set of points belonging to this line.

Let us investigate if there exists any maximum at the boundary.

Case Boundary I: First, we consider the points (P_1) satisfying the equation

$$B_i = (a_i^I - 1)I_i + (a_i^R - 1)R_i.$$

On this set, the function F becomes,

$$F(P_1) = -\alpha_i(I_i + \varepsilon R_i - \varphi_i((a_i^I - 1)I_i + (a_i^R - 1)R_i)^\theta) - \lambda_i R_i + \Xi,$$

where Ξ contains all the terms that do not depend on the node i , and in particular it does not depend on the triplets (I_i, R_i, B_i) . Candidate maximum of F on the restriction $B = (a_i^I - 1)I_i + (a_i^R - 1)R_i$, are solution of the system

$$\begin{cases} \partial_{I_i} F(P_1) = -(\alpha_i - \alpha_i \varphi_i \theta ((a_i^I - 1)I_i + (a_i^R - 1)R_i)^{\theta - 1} (a_i^I - 1)) = 0 \\ \partial_{R_i} F(P_1) = -(\alpha_i \varepsilon_i - \alpha_i \varphi_i \theta ((a_i^I - 1)I_i + (a_i^R - 1)R_i)^{\theta - 1} (a_i^R - 1)) - \lambda_i = 0. \end{cases}$$

If $\lambda_i = -\alpha_i \varepsilon_i + \alpha_i \frac{(a_i^R - 1)}{(a_i^I - 1)}$, the system admits as solution the points (P_1^*) belonging to the line

$$\begin{cases} (a_i^I - 1)I_i + (a_i^R - 1)R_i = \left(\frac{1}{\varphi_i \theta (a_i^I - 1)} \right)^{\frac{1}{\theta - 1}} \\ B_i = \left(\frac{1}{\varphi_i \theta (a_i^I - 1)} \right)^{\frac{1}{\theta - 1}}. \end{cases}$$

Otherwise, the maxima does not belong to this portion of the boundary. Indeed, if the maximum would have stayed on the boundary, it would have been a critical point on the restriction of the function F on the boundary.

Case Boundary II: Then, we consider the points (P_2) such that

$$B_i = 0.$$

On this set, the function F becomes

$$F(P_2) = \frac{((a_i^I - 1)I_i + (a_i^R - 1)R_i)^{1-\gamma}}{1-\gamma} - \alpha_i (I_i + \varepsilon_i R_i) - \lambda_i R_i + \Xi.$$

The candidate maxima are solution of the system,

$$\begin{cases} \partial_{I_i} F(P_2) = (a_i^I - 1)I_i + (a_i^R - 1)R_i)^{-\gamma}(a_i^I - 1) - \alpha_i = 0 \\ \partial_{R_i} F(P_2) = (a_i^I - 1)I_i + (a_i^R - 1)R_i)^{-\gamma}(a_i^R - 1) - \alpha_i \varepsilon_i - \lambda_i = 0. \end{cases}$$

If $\lambda_i = -\alpha_i \cdot \varepsilon_i + \alpha_i \cdot (a^R - 1) \cdot (a^I - 1)^{-1}$, the system admits as solution the points (P_2^*) belonging to the line

$$\begin{cases} (a^I - 1)I_i + (a_i^R - 1)R_i = (\alpha_i \cdot (a_i^I - 1)^{-1})^{-\frac{1}{\gamma}} \\ B_i = 0. \end{cases}$$

Otherwise, the maxima does not belong to this portion of the boundary. Notice that all the points where both $F(P_1)$ and $F(P_2)$ is not differentiable, are points where I_i, B_i, R_i are not positive, so they are excluded as candidate maxima.

In summary,

- If $\lambda < -\alpha \cdot \varepsilon + \alpha \cdot (A^R - 1) \cdot (A^I - 1)^{-1}$ the maximum could be one the critical points listed above or it could be one of the points where the function is not differentiable. However, the latter are excluded, so the maximum is attained at the point

$$\begin{cases} I_i^* = 0 \\ B_i^* = \left(\frac{\lambda_i + \alpha_i \varepsilon_i}{(A_i^R - 1) \alpha_i \varphi_i \theta} \right)^{\frac{1}{\theta - 1}} \\ R_i^* = \frac{1}{(A_i^R - 1)} \left(\frac{\lambda_i + \alpha_i \varepsilon_i}{(A_i^R - 1)} \right)^{-\frac{1}{\gamma}} + \frac{1}{(A_i^R - 1)} \left(\frac{\lambda_i + \alpha_i \varepsilon_i}{(A_i^R - 1) \alpha_i \varphi_i \theta} \right)^{\frac{1}{\theta - 1}}. \end{cases}$$

- If $\lambda > -\alpha \cdot \varepsilon + \alpha \cdot (A^R - 1) \cdot (A^I - 1)^{-1}$ the maximum could be one the critical points listed above or it could be one of the points where the function is not differentiable. However, the latter are excluded, so the maximum is attained at the point

$$\begin{cases} I_i^* = \frac{1}{(A_i^I - 1)} \left(\frac{\alpha}{(A_i^I - 1)} \right)^{-\frac{1}{\gamma}} + \frac{1}{(A_i^I - 1)} \left(\frac{1}{(A_i^I - 1) \alpha_i \varphi_i \theta} \right)^{\frac{1}{\theta - 1}} \\ B_i^* = \left(\frac{1}{(A_i^I - 1) \varphi_i \theta} \right)^{\frac{1}{\theta - 1}} \\ R_i^* = 0. \end{cases}$$

- If $\lambda = -\alpha \cdot \varepsilon + \alpha \cdot (A^R - 1) \cdot (A^I - 1)^{-1}$ there are no critical points, thus the maximum stays on the boundary. In particular, it could be attained on the set of points P_1^*, P_2^* or P_0^* . To understand where the maximum is attained, we need to compare $F(P_1^*), F(P_2^*)$ and $F(P_0^*)$.

If $\gamma < 1$ we evaluate the function F in the three different lines, we get

$$F(P_0^*) = \mathbf{A} + \mathbf{B} + \mathbf{C} + \Xi, \quad F(P_1^*) = \mathbf{B} + \mathbf{C} + \Xi, \quad F(P_2^*) = \mathbf{A} + \mathbf{C} + \Xi,$$

where

$$\begin{aligned}\mathbf{A} &= \frac{(\alpha_i \cdot (a_i^I - 1)^{-1})^{-\frac{1-\gamma}{\gamma}}}{1-\gamma} - \alpha_i (a_i^I - 1)^{-1} (\alpha_i (a_i^I - 1)^{-1})^{-\frac{1}{\gamma}}, \\ \mathbf{B} &= -\alpha \left((a_i^I - 1)^{-1} B^* - \varphi(B^*)^\theta \right), \\ \mathbf{C} &= \alpha_i \left(-(a_i^I - 1)(a_i^R - 1)^{-1} R_i + \varepsilon_i R_i \right) - \lambda_i R_i,\end{aligned}$$

where $B_i^* = \left(\frac{\alpha}{(A_i^I - 1)\alpha_i \varphi_i^\theta} \right)^{\frac{1}{\theta-1}}$. Observe that, $\mathbf{B} > 0$. Moreover for $\lambda_i = -\alpha_i \varepsilon_i + \alpha_i \frac{(a_i^R - 1)}{(a_i^I - 1)}$, $C = 0$ and when $\gamma \in (0, 1)$ $\mathbf{A} > 0$. In conclusion, when $\gamma \in (0, 1)$, $F(P_0^*) > F(P_1^*)$ and $F(P_0^*) > F(P_2^*)$ and the maximum is attained on P_0^* . If $\gamma > 1$, $F(P_1^*) > F(P_0^*) > F(P_2^*)$ and the maximum is attained on P_1^* .

If $\gamma > 1$, we observe that in proximity of the null consumption the utility diverges to $-\infty$. Thus, we evaluate the function F in the two remaining lines of the boundary

$$F(P_0^*) = \mathbf{A} + \mathbf{B} + \mathbf{C}, \quad F(P_2^*) = \mathbf{A} + \mathbf{C},$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are the same quantities defined above. Observe that for $\gamma > 1$ $\mathbf{A} < 0$. Thus $F(P_0^*) > F(P_2^*)$ and the maximum is attained on P_0^* . \square

4.2 Long time behaviour of the optimal state trajectory

We consider now the special case when the coefficients are time-independent.

Theorem 4.4. *Let Assumption 4.1 hold. Assume that the coefficients A^I, A^R and φ are independent, i.e. $A^I(t) \equiv A^I$, $A^R(t) \equiv A^R$ and $\varphi(t) \equiv \varphi$ and that $\delta_i \neq 0, \forall i \in \mathcal{V}$. Then*

$$\lim_{t \rightarrow \infty} \sum_{i=1}^n |P_i^*(t) - P_{i,\infty}^*|^2 = 0,$$

where P_∞^* is the unique solution to the matrix equation

$$\mathcal{L}P + N^* = 0.$$

Proof. Notice that in this case, the expressions of the optimal controls are time independent too. Let λ_0 be the spectral bound of \mathcal{L} . Since $\delta \neq 0$, the operator \mathcal{L} is strictly dissipative, hence $\lambda_0 < 0$. Let us write

$$\mathcal{L} = \mathcal{L}_0 - \lambda_0, \quad \text{where} \quad \mathcal{L}_0 := \mathcal{L} + \lambda_0,$$

and note that \mathcal{L}_0 is dissipate by definition, hence $e^{s\mathcal{L}_0}$ is a contraction. Then, we can rewrite:

$$\begin{aligned}P^*(t) &:= e^{t\mathcal{L}_0} e^{-\lambda_0 t} p + \int_0^t e^{-\lambda_0(t-s)} e^{(t-s)\mathcal{L}_0} N^* ds, \quad t \geq 0, \\ &= e^{t\mathcal{L}_0} e^{-\lambda_0 t} p + \int_0^t e^{-\lambda_0 s} e^{s\mathcal{L}_0} N^* ds, \quad t \geq 0,\end{aligned}$$

and take the limit above when $t \rightarrow \infty$. Since $e^{s\mathcal{L}_0}$ is a contraction, the first one on the right-hand side converges to 0, whereas the second one converges to

$$P_\infty^* := \int_0^\infty e^{-\lambda_0 s} e^{s\mathcal{L}_0} N^* ds, \quad t \geq 0.$$

And we can conclude by expressing the limit P_∞^* as $P_\infty^* = (\lambda_0 - \mathcal{L}_0)^{-1} N^*$. I.e. P_∞^* is the solution to $(\lambda_0 - \mathcal{L}_0)P = N^*$ or, equivalently, to $\mathcal{L}P + N^* = 0$. \square

4.3 Some investigations on the quadratic cost function

In this section, we present the results of a series of quantitative exercises where the cost of renewable technology is quadratic, i.e.

$$f_i(R_i) = \lambda_i R_i^2,$$

and all the relevant parameters of the model are kept constant, i.e.

$$A^I(t) \equiv A^I, \quad A^R(t) \equiv A^R, \quad \varphi(t) \equiv \varphi. \quad (19)$$

Parameters values listed below are chosen according to the motivation described in [BFFG21]. Parameter a_i^I stays between [2.5, 6.6], $\rho = 0.03$, $\gamma = 0.5$ and δ stays in the range [0.3, 0.5]. However, since the latter is not the main object of investigation of the current work it is chosen as constant on each node, namely $\delta_i = 0.4$ for each $i \in \mathcal{V}$. Regarding the parameters concerning abatement, we choose $\varphi = 0.11$ and $\theta = 0.2$ and $w_i = 1$ for each $i \in \mathcal{V}$. Note that both parameters φ and θ must be chosen not too high, otherwise condition (4.1) are not satisfied and the positivity of Pollution is broken. Initial condition of pollution, p is chosen constant in space, $p_i = p_0 \forall i \in \mathcal{V}$ and just to fix the idea we choose $p_0 = 1$.

We now discuss the parameter's numerical value representing this work's main novelties: parameters related to the new green technology ($a_i^R, \varepsilon_i, \lambda_i$) and the network structure (n, L). We expect the level of the parameter a_i^R to be calibrated to have the GDP in a range higher than the range for a_i^I , for instance, [0.25, 0.5]. Thus a_i^R should stay in the range [2, 4]. Quite difficult is to choose a numerical value for ε_i and λ_i . Since the impact of the investment I is normalized (and so it is one), we choose $\varepsilon_i < 1$ and in particular $\varepsilon_i = 0.1$. Since we were not able to find a proper value for λ_i , we have experimented with different values in the range [0.01, 10]. Regarding the network structure, we consider $n = 20$ nodes, and our benchmark choice on the links between nodes is given by the matrix $L = (\ell_{ij})$,

$$\ell_{ij} = \begin{cases} \frac{1}{n}, & i \neq j \\ -\frac{n-1}{n}, & i = j. \end{cases}$$

We have also tested the impact of different diffusion matrices, L^1, L^2, L^3

$$\ell_{ij}^1 = \begin{cases} \frac{10}{n}, & i \neq j \\ \frac{10(n-1)}{n}, & i = j, \end{cases} \quad \ell_{ij}^2 = \begin{cases} 1, & j = i + 1 \text{ or } j = i - 1 \\ -2, & i = j, i \neq 1, n \\ -1, & i = j = 1 \text{ or } i = j = n. \end{cases}.$$

One of the advantages of introducing the network structure is the possibility of representing different geographical situations. We consider the case in which there exists one node, $i^* = 10$ where pollution

tends to remain, so all the connections from $i^* = 10$ to j are lower than the other connection. A matrix representing the described situation is given by

$$\ell_{ij}^3 = \begin{cases} 0.5\ell_{ij}^2, & j = 10 \\ \ell_{ij}^2, & \text{otherwise.} \end{cases}$$

A straightforward consequence of the static choice on parameters (19), is that the profile of optimal investments and abatement are also constant in time and in particular they coincide with their long-time distribution. The same argument holds for production, consumption, and emission but not for Pollution. Indeed in the plot presented in the section, we will just present the long-time distribution of pollution. Four figures are presented. In these figures, we illustrate some of the quantities between investments, abatement, consumption, production, long-time pollution, emission. We have also represented the relationship between production, long-time pollution. In the horizontal axis, the value of the production/income is represented. More precisely we have considered the vector of production, $A^I I^* + A^R R^*$, and then we listed the elements of the vector in a descendent order and then reordered accordingly also the elements of pollution.

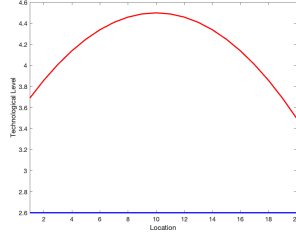
In Figure 1 we show the impact of different types of connections in the optimals. We solved the optimization problem numerically by considering different matrices: L, L^1, L^2, L^3 . We observe that the only relevant difference is observed at the level of pollution, while all other variables (abatement, consumption, emission, investment, production) are not changed. In Figure 1f we can observe that the larger the connections, the greater the spread of pollution. In fact, for L^2 the curve presents a bell-shaped curve, for L the curve is flatter than the latter but still maintains a bell-shaped profile. For L_1 the curve is even flatter. For L^3 , an inversion in the shape of the pollution is observed at position 10. This is perfectly consistent with the fact that we have imposed that the pollution at node 10 tends to settle. The reason why the variation in the diffusion matrix does not strongly affect all other variables lies in the fact that the dependence of all these variables on the diffusion matrix passes only through α , defined as (11), and this variable is invariant to the tested variations in the diffusion matrix.

Figure 2 shows the emergence of spatial discrepancy in input productivity. In Figure 2e and Figure 2a we observe that the social planner will invest less in brown technology and disinvest more where the technology level is higher. A different behavior is observed in the investment in green technology, in fact, the social planner will invest more where the technology level is higher. The opportunity given by this other source of energy allows the social planner to produce more where there is correspondence at a high technological level. In this way, the typical Stokey-like picture is captured (pollution goes down with production across location). This is the main difference with the case where only brown investments are allowed, where the Stokey-like picture is not captured in the case of heterogeneity of the technological parameter (see Section 4 in [BFFG21]).

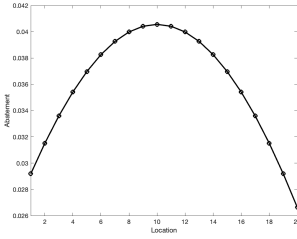
In Figure 3 we present a further investigation of the spatial distribution of investment, keeping the same spatial structure at the technology level, see Figure 2a. To understand this, we perform an exploration by changing the value of the cost parameter. We assume that the cost parameter is constant in space, $\lambda_i = \lambda \forall i \in \mathcal{V}$ and test the situation for $\lambda = 0.01, 1.5, 4$. We observe that if it is optimal to invest in both types of technologies ($\lambda = 1.5$ or $\lambda = 5$), the spatial distribution of green investment is always in agreement with the distribution of technology level, while the spatial distribution of brown investment has a discrepancy with the spatial distribution of technology level. If it is optimal to invest in only one type of energy (see the $\lambda = 0.01$ case), the spatial distribution of investment presents a spatial discrepancy in input productivity. This investigation confirms the

scenario presented in Figure 2, namely, that potential technology differentiation generates a Stokey-like picture. Indeed, in Figure 3e we observe that high output corresponds to lower pollution.

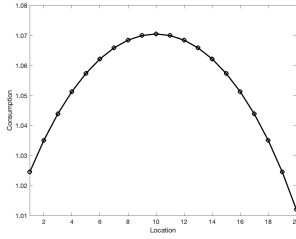
Figure 4 provides an initial analysis of the impact of new technology on consumption. We consider the impact of the presence of green investment on consumption. To handle different levels of significance of green investment, different cost parameters are considered. As done previously, the cost parameter is assumed constant in space, $\lambda_i = \lambda \forall i \in \mathcal{V}$ and we tested the situation for $\lambda = 0.01, 1.5, 4$. We calculated numerically the optimal consumption for these different cases (see green lines in Figure 4a, Figure 4b and Figure 4c). Furthermore, the optimal consumption is obtained numerically when $a_i^R \leq 1$ for all $i \in \mathcal{V}$ (see the black lines in Figure 4a, Figure 4b and Figure 4c). We thus observe that while it is optimal to invest in both types of energy, the consumption remains that of the case with only one energy source, in fact in Figure 4b and Figure 4c the green line and the black line overlap. When it is optimal to invest only in green energy, consumption increases, see Figure 4a. This fact suggests the fairly natural idea that welfare would benefit from having affordable green technology and is also consistent with the fact that the value function is decreasing with respect to the parameter λ .



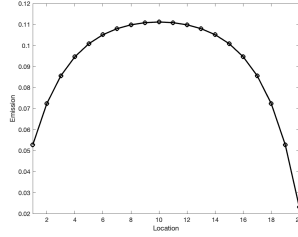
(a) Technological Level compared to the location



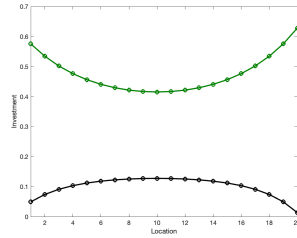
(b) Abatement compared to the location.



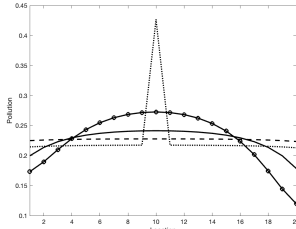
(c) Consumption compared to location



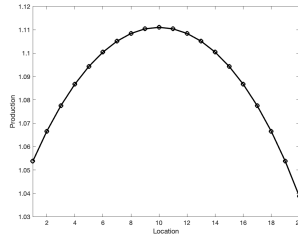
(d) Emission compared to location



(e) Investment I (in black) and R (in green) compared to location

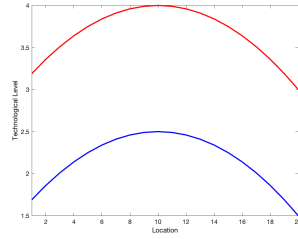


(f) Pollution compared to location

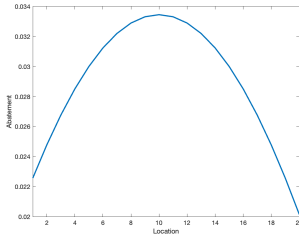


(g) Production compared to location

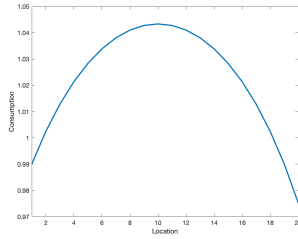
Figure 1: Numerical representation of the situation when Technological Productivity is heterogeneous in space and different diffusion matrices are considered. In Figure 1a A^I is plotted in red, and A^R is plotted in blue. All other parameters are constant in time and space. Continuous lines (—) are related to the diffusion L , dashed lines (--) are related to the diffusion L^1 , dotted lines (\cdots) are related to the diffusion L^2 and continuous lines with marker ($-o$) are related to the diffusion L^4 . The values of other parameters are described at the beginning of subsection 4.3, while the value of $\lambda_i = 1 \forall i \in \mathcal{V}$.



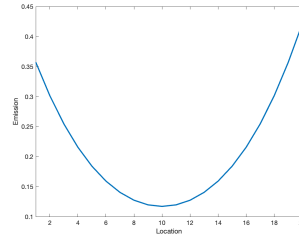
(a) Technological Level compared to the location



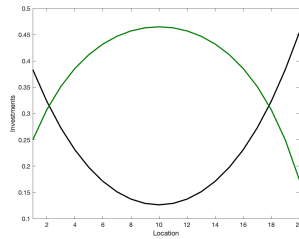
(b) Abatement compared to the location



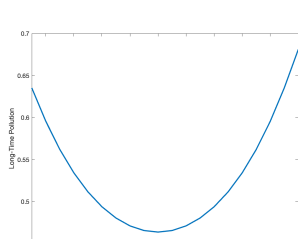
(c) Consumption compared to the location



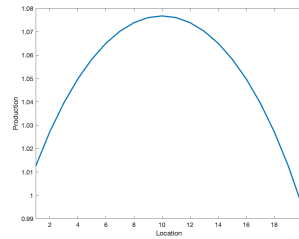
(d) Emission compared to the location



(e) Investment I (in black) and R (in green) compared to location

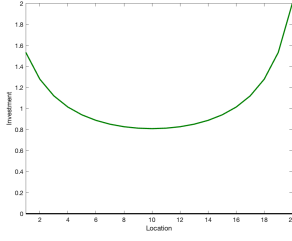


(f) Pollution compared to location

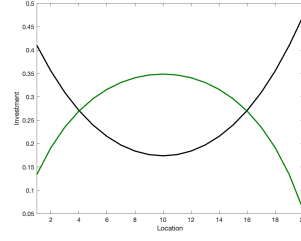


(g) Production compared to location

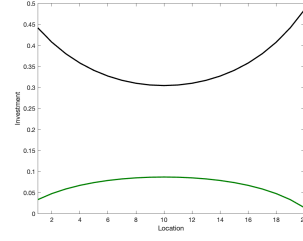
Figure 2: Numerical representation of the situation when Technological Productivity is heterogeneous in space. In figure 2a A^I is represented in red and A^R is represented in blue. The values of other parameters are described above, while the value of $\lambda_i = 1 \forall i \in \mathcal{V}$.



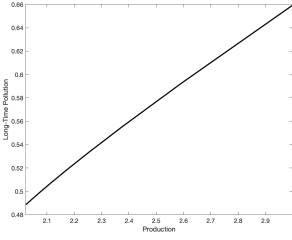
(a) Investment I (in black) and R (in green) compared to location for $\lambda_i = 0.01 \forall i \in \mathcal{V}$



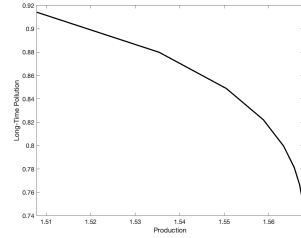
(b) Investment I (in black) and R (in green) compared to location for $\lambda_i = 1.5 \forall i \in \mathcal{V}$



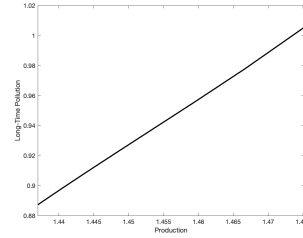
(c) Investment I (in black) and R (in green) compared to location for $\lambda_i = 4 \forall i \in \mathcal{V}$



(d) Long-Time Pollution compared to Production for $\lambda_i = 0.01 \forall i \in \mathcal{V}$

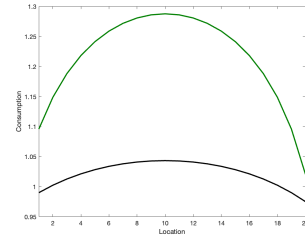


(e) Long-Time pollution compared to Production for $\lambda_i = 1.5 \forall i \in \mathcal{V}$

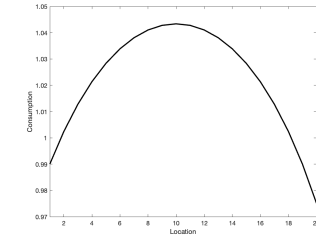


(f) Long-Time pollution compared to Production for $\lambda_i = 4 \forall i \in \mathcal{V}$

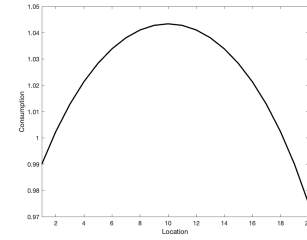
Figure 3: Numerical investigation of the impact of the cost parameter λ_i on optimal investments and on the relation between Long-Time Pollution and Production. The Technological Productivity is the one described in Figure 2a and the values of other parameters are described at the beginning of the subsection 4.3



(a) Consumption compared to location for $\lambda_i = 0.05 \forall i \in \mathcal{V}$



(b) Consumption compared to location for $\lambda_i = 1 \forall i \in \mathcal{V}$



(c) Consumption compared to location for $\lambda_i = 5 \forall i \in \mathcal{V}$

Figure 4: Numerical investigation of the impact of the cost parameter λ_i on Consumption. In green it is represented the consumption for different level of λ_i , In black it is plotted the consumption when the green technology is not taken into account ($a_i^R < 1$). The Technological Productivity is the one described in Figure 2a and the values of other parameters are described at the beginning of the subsection 4.3

References

- [Bar19] Francesco Bartaloni. Infinite horizon optimal control problems with non-compact control space. existence results and dynamic programming. 2019.
- [BFFG19] Raouf Boucekkine, Giorgio Fabbri, Salvatore Federico, and Fausto Gozzi. Growth and agglomeration in the heterogeneous space: a generalized ak approach. *Journal of Economic Geography*, 19(6):1287–1318, 2019.
- [BFFG21] Raouf Boucekkine, Giorgio Fabbri, Salvatore Federico, and Fausto Gozzi. From firm to global-level pollution control: The case of transboundary pollution. *European journal of operational research*, 290(1):331–345, 2021.
- [BFFG22] Raouf Boucekkine, Giorgio Fabbri, Salvatore Federico, and Fausto Gozzi. A dynamic theory of spatial externalities. *Games and Economic Behavior*, 132:133–165, 2022.
- [Bul01] Stanley R Bull. Renewable energy today and tomorrow. *Proceedings of the IEEE*, 89(8):1216–1226, 2001.
- [CGL⁺24] Alessandro Calvia, Fausto Gozzi, Marta Leocata, Georgios I. Papayiannis, Anastasios Xepapadeas, and Athanasios N. Yannacopoulos. An optimal control problem with state constraints in a spatio-temporal economic growth model on networks. *Journal of Mathematical Economics*, 113:102991, 2024.
- [dFGLPMH22] Javier de Frutos, Víctor Gatón, Paula M López-Pérez, and Guiomar Martín-Herrán. Investment in cleaner technologies in a transboundary pollution dynamic game: A numerical investigation. *Dynamic Games and Applications*, 12(3):813–843, 2022.
- [dFLPMH21] Javier de Frutos, Paula M López-Pérez, and Guiomar Martín-Herrán. Equilibrium strategies in a multiregional transboundary pollution differential game with spatially distributed controls. *Automatica*, 125:109411, 2021.
- [dFMH19a] Javier de Frutos and Guiomar Martín-Herrán. Spatial effects and strategic behavior in a multiregional transboundary pollution dynamic game. *Journal of Environmental Economics and Management*, 97:182–207, 2019.
- [dFMH19b] Javier de Frutos and Guiomar Martín-Herrán. Spatial vs. non-spatial transboundary pollution control in a class of cooperative and non-cooperative dynamic games. *European Journal of Operational Research*, 276(1):379–394, 2019.
- [dFMH20] Javier de Frutos and Guiomar Martín-Herrán. Non-linear incentive equilibrium strategies for a transboundary pollution differential game. *Games in Management Science: Essays in Honor of Georges Zaccour*, pages 187–204, 2020.
- [ENB00] Klaus-Jochen Engel, Rainer Nagel, and Simon Brendle. *One-parameter semigroups for linear evolution equations*, volume 194. Springer, 2000.
- [FGP08] Giuseppe Freni, Fausto Gozzi, and Cristina Pignotti. Optimal strategies in linear multisector models: Value function and optimality conditions. *Journal of Mathematical Economics*, 44(1):55–86, 2008.

- [FGS06] Giuseppe Freni, Fausto Gozzi, and Neri Salvadori. Existence of optimal strategies in linear multisector models. *Economic Theory*, 29:25–48, 2006.
- [FR11] Lorenzo Farina and Sergio Rinaldi. *Positive linear systems: theory and applications*. John Wiley & Sons, 2011.
- [JMHZ10] Steffen Jørgensen, Guiomar Martín-Herrán, and Georges Zaccour. Dynamic games in the economics and management of pollution. *Environmental Modeling & Assessment*, 15:433–467, 2010.
- [LY95] Xunjing Li and Jiongmin Yong. Control problems in infinite dimensions. *Optimal Control Theory for Infinite Dimensional Systems*, pages 1–23, 1995.
- [Trö24] Fredi Tröltzsch. *Optimal control of partial differential equations: theory, methods and applications*, volume 112. American Mathematical Society, 2024.
- [XW24] Linzhao Xue and Xianjia Wang. The impact of pollution transmission networks in a transboundary pollution game. *Journal of Cleaner Production*, 451:142010, 2024.