European Journal of Operational Research 332 (2026) 1032–1047

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/eor

Interfaces with Other Disciplines

Network-based optimal control of pollution growth

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ARTICLE INFO

Keywords:
Optimal control problems
Value function
Graphs and networks
Pollution control
Transboundary pollution

ABSTRACT

This paper studies a model for the optimal control of pollution diffusion over time and space by a centralized economic agent. The controls are the investments in two types of production: a less polluting ("green") technology and a more polluting ("brown") one. The goal is to maximize an intertemporal utility function which takes into account the cost of pollution. The main novelty is the fact that the spatial component has a network structure. Moreover, in such a time-space setting, we analyze the trade-off between the use of green and brown technologies: this is also a novelty in such a setting. Extending methods from previous works, we can explicitly solve the problem in the case of strictly convex or linear pollution costs.

1. Introduction

Pollution control is a significant issue that arises in various research areas. In the field of economic and operational research, the problem has been studied from both the regulator and industry perspectives. In this context, a variety of questions have been addressed, ranging from the most fundamental to the more obscure. As a first example, pollution control can be made by the institutions using environmental regulation or direct control (e.g. to face environmental emergencies), as studied in the paper (Bawa, 1975). There, the author compares the two, determining the proper mix between them.

Moreover, the transboundary nature of most pollutants (both air and water pollutants), which goes more in the direction of the regulator perspective, has been extensively studied in the operational research literature, in both applied and theoretical papers.

In Kerl et al. (2015), the authors evaluate fluctuating pollutant formation from source emissions integrated within an electricity production model. In Xu and Masui (2009), Zhao et al. (2013), the authors address the issue of reducing local pollution (in the air and water, respectively) in China. In Stam et al. (1992) the authors propose a multicriteria software package to evaluate various scenarios and trade-offs in Europe.

Examples of more theoretical works include Bertinelli et al. (2014), Boucekkine et al. (2019, 2021, 2022), El Ouardighi et al. (2020), Ferrari and Koch (2019), de Frutos et al. (2022, 2021), de Frutos and Martín-Herrán (2019a,b, 2020), Leibowicz (2020).

Among the above more theoretical papers we first recall that in Bertinelli et al. (2014), El Ouardighi et al. (2020), de Frutos et al. (2022), de Frutos and Martín-Herrán (2020) models in a two-country setting are studied, while in Ferrari and Koch (2019) a strategic model of pollution control for a firm, representative of the productive sector of a country, is considered. While these three papers study some relevant policy implications, no spatio-temporal dynamics of pollution are investigated there.

Geographic heterogeneity in the context of pollution mitigation (which is a key theme in the literature of Operations Research) is considered, for example, in the work of Leibowicz (2020), which presents a theoretical spatial framework for sustainable urban land use and transportation planning that explicitly accounts for greenhouse gas-related damages. The model is a static one.

Going to dynamic models, to the best of our knowledge, the first works where a spatio-temporal evolution for pollution in a game setting is investigated are de Frutos et al. (2021), de Frutos and Martín-Herrán (2019a,b). More specifically, in the above three papers, the authors study a finite player model formulated in continuous time and space, in which the spatial and temporal evolution of pollution is modelled using a parabolic partial differential equation. The players behave strategically and maximise their welfare net of environmental damage

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[1] All the authors have equally contributed to the paper.

[2] Fausto Gozzi and Marta Leocata are supported by the Italian Ministry of University and Research (MIUR), in the framework of PRIN projects 2017FKHBA8 001 *(The Time-Space Evolution of Economic Activities: Mathematical Models and Empirical Applications)* and 20223PNJ8K *(Impact of the Human Activities on the Environment and Economic Decision Making in a Heterogeneous Setting: Mathematical Models and Policy Implications)*. Giulia Pucci is supported by the Swedish Research Council grant (2020–04697).

https://doi.org/10.1016/j.ejor.2026.03.012
Received 3 January 2026; Accepted 8 March 2026
Available online 14 March 2026

caused by the pollutant stock. In de Frutos et al. (2021), de Frutos and Martín-Herrán (2019a), players choose the level of emission; in de Frutos and Martín-Herrán (2019b), they choose the level of emission and investment in clean technology. Concerning the methodology, in de Frutos and Martín-Herrán (2019a,b) the authors approximate the problem by discretizing space and performing some numerical analysis, while in de Frutos et al. (2021) they find an explicit solution to the Hamilton-Jacobi-Bellman system and deduce from it a closed loop equilibrium.
On the other hand the papers (Boucekkine et al., 2019, 2021, 2022) employ different tools from infinite dimensional control/game theory and they provide a more general theoretical treatment of related spatiotemporal pollution problems. In particular, in Boucekkine et al. (2021), the authors study a central planner problem in which the negative externality of pollution is modelled as a continuous variable in time and space. In this context, the regulator chooses investments and pollution abatement controls, taking into account that production generates pollution and that pollution is transboundary. Notably, the authors derive closed-form solutions to the infinite-dimensional optimal control problem under consideration. This is achieved through a mathematical reformulation of the problem. The approach is not based on dynamic programming or the maximum principle, but on the functional reformulation of the objective function. Some of the ideas of such papers is used here to obtain closed-form solutions.
Our work departs from the above contributions introducing two significant innovations: *modelling the space as a network structure and introducing a less polluting (though more expensive) production method*.
Concerning the first novelty, the decision to model the space as a network of interconnected locations rather than as a continuous space is motivated by the fact that the network model is more flexible than the corresponding continuous-time model. This allows us to consider various network structures and provides a better basis for estimation, given that the datasets are clearly discrete. Currently, studies on networks are popular and have been widely applied to the study of economic effects of heterogeneous interactions between different entities. See, for example, Ballester et al. (2006), Calvia et al. (2024), Elliott and Golub (2013). See also Nerantzis et al. (2020) for applications in water distribution systems. Regarding the application of networks to pollution control problems, we observe that de Frutos and Martín-Herrán (2019a,b) also treat the discrete space case. However, in these papers, the discrete space is the result of discretising the PDE, rather than being a network with pollution flowing over arcs. In this respect, we must also mention the recent paper (Xue & Wang, 2024), which presents a multi-region differential game is used to investigate the interaction and co-evolution of multiple pollution stocks, with pollution moving across locations through a network. Our work differs from the aforementioned work in terms of both the mathematical approach and the modelling motivations. Indeed, inspired by Boucekkine et al. (2021), we adopt a different solution method that is not based on HJB equations. This enables us to study three types of geographic discrepancy: discrepancy in productivity, discrepancy in the self-cleaning capacity of nature, and discrepancy in network structure.
The second novelty is that the agent also has the possibility, at some cost, to shift part of the traditional production to less polluting processes. Including two investment options reflects the distinct environmental impacts of pollution-intensive production and low-emission alternatives: coal power, synthetic fertilisers and gasoline vehicles contribute heavily to emissions, whereas renewables, regenerative farming and electric vehicles reduce pollution and environmental damage. However, despite the obvious environmental benefits, these low-pollution processes incur higher costs and generally have lower productivity due to challenges such as energy fluctuations and storage issues. Throughout this work, we will use the terms 'non-renewable' or 'brown' to refer to traditional investments, and 'renewable' or 'green' to refer to the new, lower-emission alternative. When selecting an investment strategy, the agent must balance environmental benefits with economic trade-offs. This paper's model allows us to examine such trade-offs and their influence on policies, which seems to be a novelty in such a dynamic context. More precisely, we can solve the model in cases where both 'green' and 'brown' technologies coexist, and where only 'brown' technology is available. This provides a kind of negative benchmark, and enables us to quantify the benefits of introducing green production technologies (see Remark 3.9).
From the mathematical point of view, the novelties in our work require some nontrivial changes in the techniques employed. For the sake of clarity we then compare our approach with the ones of the closer papers (Boucekkine et al., 2019, 2021, 2022):

- Since the space is now modelled by a network, the state equation becomes a system of ODEs. On one side, this is, in principle, easier than the case of continuous space, as the dimension of the state variable here is finite. On the other hand, this allows us to deal with more general diffusion operators $\mathcal{L}$, which need to be treated differently (see on this the recent paper Calvia et al. (2024), where the network structure is used in a different context).
- The explicit solutions are found by exploiting the linearity of the state equation and of the pollution costs to reduce the problem to a parametric static problem (see Theorem 3.1). However, the introduction of the less polluting production process R makes it more difficult, with respect to previous papers (see Boucekkine et al., 2021, 2022), to find explicit solutions. In particular (see Section 3.2) we can treat the case of generic strictly convex cost f, characterizing the solutions also in this case, which is new with respect to the literature.

The main results of the paper are: Theorem 3.1 on the reduction to a parametric static problem; Theorem 3.2 on the solution of the case when a uniformly convex cost function on renewable is considered; Theorem 3.3 on the solution of the case when a linear cost function on renewable is considered; Theorem 3.4 on the asymptotic behaviour of pollution in the particular case of time-independent coefficients.
The content of the paper is as follows: Section 2 presents the general model and examines its well-posedness. In Section 3, the optimal control is studied and explicit solutions are found for different cost functions. Moreover, in the same section, a comparison with the case where only one type of investment, the brown one, is admitted is presented. Finally, in Section 4, we consider a quadratic cost for renewable technologies and focus on the analysis of some numerical simulations.

2. A model for pollution on networks

In this section, we provide an overview of the model. Our focus is on a central planning challenge within a spatially organised economy. Within this economy, a single commodity serves multiple roles: it is consumed, utilised as input in both renewable and non-renewable production (invested), and produced at various locations. Additionally, it is important to note that this commodity is not subject to trade between different locations; however, pollution does cross geographical boundaries.
In our model, the space variable is described as a network of interconnected geographic locations. When we refer to a network, we are describing a graph with weights, where the nodes correspond to these locations (e.g., cities, regions, etc.), the edges represent the connections between select locations, and the weights signify the importance of each connection which can vary with time. This network structure enhances the realism of pollution transportation within the model and aligns with the inherent network-like nature of pollutant data, which often exhibits a similar network pattern.
We model the network of $n \geq 2$ geographically distributed locations by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set of vertices that corresponds to locations and $\mathcal{E}$ is a set of edges connecting vertices. The graph is

simple, weighted, and finite. We identify $\mathcal{V}$ with the set $\{1,\dots,n\}$ of sites, where pollution is accumulated, capital input is invested and output is produced, consumed and locally re-invested and $\mathcal{E}$ as a subset of $\{(i,j)\in\{1,\dots,n\}^2$ s.t $i\neq j\}$. We say that two vertices $i,j\in\mathcal{V}$ are connected, and we write $i\sim j$ if there exists an edge connecting them, i.e., $i\sim j \iff (i,j)\in\mathcal{E}$.

We denote by $W=(w_{ij}(t))_{i,j\in\mathcal{V}}$ the matrix of the graph, with $w_{i,j}(t)$ representing the intensity of the geographical connection from node j to node i. The transboundary nature of pollution is represented by the action of a linear operator $L(\cdot):\mathbb{R}_+\to\mathbb{R}^{n\times n}$ on the nodes of the graph. This captures the idea that pollution may enter or exit any location as a result of interaction between them, and it is defined as $L(t)=(\ell_{i,j}(t))_{i,j\in\mathcal{V}}$ with

$$\ell_{i,j}(t)=\begin{cases} w_{i,j}(t), & i\neq j\\ -\sum_{k=1,k\neq i}^{n} w_{k,i}(t), & i=j.\end{cases}$$

At time t and at any location $i\in\mathcal{V}$, there is a single individual consuming $C_i(t)$, investing $I_i(t)$ in non-renewable production and $R_i(t)$ in renewable production. The production coming from the investments is denoted by $Y_i(t)$ and is given by

$$Y_i(t)=a_i^I(t)I_i(t)+a_i^R(t)R_i(t). \tag{1}$$

Remark 2.1. *$a_i^I(t)$ and $a_i^R(t)$ represent the productivity or technological levels for non-renewable and renewable investments, at location i in time t. They can represent possible technological spillovers across sites, disparities in technological advancement across space , and similar dynamics.*

Inspired by the work (Boucekkine et al., 2021), we assume, for simplification, that capital inputs do not accumulate over time, nor are they exchanged across space. At any location, the output is produced, consumed, and locally invested (no trade across locations), implying the following resource constraints:

$$C_i(t)+I_i(t)+R_i(t)=Y_i(t). \tag{2}$$

Which, together with (1), yields to

$$C_i(t)=(a_i^I(t)-1)I_i(t)+(a_i^R(t)-1)R_i(t)$$

We consider the following control problem with an infinite time horizon on $\mathcal{V}$. At each time $t\in\mathbb{R}_+$ and location $i\in\mathcal{V}$, the planner chooses the control variables: i.e. the investment in traditional or brown production $I_i(t)$ and the investment in green production $R_i(t)$. They contribute to the dynamics of the pollution stock $P_i(t)$. Specifically, on each node $i\in\mathcal{V}$, the pollution's dynamics evolve according to the following ODE:

$$\begin{cases}\frac{d}{dt}P_i(t)=\sum_{j=1}^n w_{ij}(t)P_j(t)-\sum_{j=1}^n w_{ji}(t)P_i(t)\\ \qquad -\delta_i(t)P_i(t)+I_i(t)+\varepsilon_i(t)R_i(t)\\ P_i(0)=p_i\in\mathbb{R}_+.\end{cases} \tag{3}$$

Here, the function $\delta_i(t)$ accounts for the effect of nature's self-cleaning mechanisms at node $i\in\mathcal{V}$, which helps mitigate pollution over time. The function $\varepsilon_i(t)$ represents the pollution intensity associated with the renewable investment at location i. Considering values of $\varepsilon_i(t)$ strictly smaller than 1 is reasonable, as green investments are meant to have a smaller impact on pollution growth compared to brown ones. Finally, $p_i\in\mathbb{R}_+$, represents the initial pollution value for each $i\in\mathcal{V}$.
Consider a social planner, who aims at controlling investment levels $(I_i,R_i)_{i=1,\dots,n}$ to maximize the following social welfare function

$$J((p_i,(I_i,R_i))_{i=1,\dots,n}): = \int_0^{+\infty} e^{-\rho t}\left(\sum_{i=1}^n\left(\frac{C_i(t)^{1-\gamma}}{1-\gamma}-\omega_iP_i(t)-f_i(R_i(t))\right)\right)dt. \tag{4}$$

Here, ω_i measures local environmental awareness for each location, ρ is a given discount factor, γ denotes the intertemporal substitution in consumption. The function $f_i:\mathbb{R}_+\to\mathbb{R}_+\ \forall i\in\mathcal{V}$ represents the maintenance and operational costs related to renewable investments. This social welfare function constitutes the social benefit of a community, resulting in a trade-off between different interests, namely technological production and sensitivity to environmental problems. Specifically, the investments $I_i(t)$ and $R_i(t)$ increase utility through consumption but also lead to higher pollution, which incurs cost and decreases utility.

Remark 2.2. *Although we recognize that nonrenewable investments also incur costs, in the context of transitioning toward a greener environment, we have chosen to focus exclusively on the running costs coming from renewable energy, highlighting the critical need to shift towards sustainable solutions to address environmental issues.*

We use vector notation to describe the pollution stock, $P(t)$ and the consumption level $C(t)$

$$P(t):=(P_1(t),\dots,P_n(t)),\quad C(t):=(C_1(t),\dots,C_n(t)),$$

and the investments

$$I(t)=(I_1(t),\dots,I_n(t)),\quad R(t)=(R_1(t),\dots,R_n(t)).$$

By defining the operator $\mathcal{L}(t):=(L(t)-\delta(t))$, and the net emissions function $N(\cdot):\mathbb{R}_+\to\mathbb{R}_+^n$

$$N(t):=I(t)+\varepsilon(t)R(t),$$

Eq. (3) can be rewritten as

$$\begin{cases}\frac{d}{dt}P(t)=\mathcal{L}(t)P(t)+N(t)\\ P(0)=p\in\mathbb{R}_+^n,\end{cases} \tag{5}$$

where $\delta(t)=\mathrm{diag}(\delta_1(t),\dots,\delta_n(t))$, $\varepsilon(t)=\mathrm{diag}(\varepsilon_1(t),\dots,\varepsilon_n(t))$. By using Eq. (1) and (2), we can explicitly rewrite the functional in terms of the investments

$$J(p,(I,R)): = \int_0^{+\infty} e^{-\rho t}\left(\left\langle\frac{((A^I(t)-\mathbb{1})I(t)+(A^R(t)-\mathbb{1})R(t))^{1-\gamma}}{1-\gamma},\mathbf{1}\right\rangle - \langle\omega,P(t)\rangle-\langle f(R(t)),\mathbf{1}\rangle\right)dt, \tag{6}$$

with $\mathbf{1}$ representing the vector of ones in $\mathbb{R}^n$, $A^I(t)=\mathrm{diag}(a_1^I(t),\dots,a_n^I(t))$, $A^R(t)=\mathrm{diag}(a_1^R(t),\dots,a_n^R(t))$, $\omega=(\omega_1,\dots,\omega_n)$ and $f(R(t))=(f_1(R_1(t)),\dots,f_n(R_n(t)))$. The set of admissible controls $\mathcal{A}(p)$ is defined as

$$\mathcal{A}(p):=\left\{(I,R)\in PC(\mathbb{R}_+;\mathbb{R}_+^n\times\mathbb{R}_+^n):\ \int_0^\infty e^{-\rho t}\|N(t)\|dt<\infty,\ \int_0^\infty e^{-\rho t}\|f(R(t))\|dt<\infty\right\}.$$

Here $PC(\mathbb{R}_+;\mathbb{R}_+^n\times\mathbb{R}_+^n)$, denotes the set of piecewise continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+^n\times\mathbb{R}_+^n$.
We call **(P)** the following state constraint optimal control problem:

$$\text{Maximize}\quad J(p,(I,R))\quad \text{over}\quad (I,R)\in\mathcal{A}(p). \tag{P}$$

2.1. On the state equation and the well-posedness of the objective function

Assumption 1.

(i) For any $i,j\in\mathcal{V}$ and for any $t\in\mathbb{R}_+$, the function $t\mapsto w_{ij}(t)$ is continuous and satisfies $w_{ij}(t)\geq 0$. There are no self-loops i.e. $w_{ii}(t)=0\ \forall t>0$.
(ii) For any $i\in\mathcal{V}$, the functions $\delta_i, a_i^I, a_i^R, \varepsilon_i, f_i$, are piecewise continuous. They satisfy

$$0\leq\varepsilon_i(t)<1\qquad a_i^I(t),a_i^R(t)\geq 1,$$

and moreover $\delta_i>0$ is bounded.
(iii) $\rho>0$, $\gamma\in(0,1)\cup(1,\infty)$, $\omega_i>0\ \forall i\in\mathcal{V}$,
(v) The linear operator $\mathcal{L}(t)$ is continuously differentiable and there exists a positive constant β such that the time derivative of $\mathcal{L}(t)$ satisfies $\|\dot{\mathcal{L}}(t)\|\leq\beta$.

Table 1
List of parameters involved in the model and their interpretation.

Parameter	Interpretation
$w_{i,j}(t)$	intensity of the geographical connection from node j to node i
$a_i^I(t)$	technological level for brown investment at the node i and time t
$a_i^R(t)$	technological level for green investment at the node i and time t
$\delta_i(t)$	decay factor at the node i and time t
$\varepsilon_i(t)$	pollution intensity associated with the green investment at the node i and time t
ω_i	local environmental awareness at the node i
ρ	given discount factor
γ	intertemporal substitution in consumption

Under Assumption 1 and by standard results for time-varying linear ODE (Rugh, 1996), we get that for any admissible control strategy, the unique, continuous solution to (5), is given by

$$P(t) = \Phi(t,0)p + \int_0^t \Phi(t,s)N(s)ds, \quad t \geq 0 \tag{7}$$

where the *state transition matrix* $\Phi(t,s)$ solves

$$\begin{cases} \frac{d}{dt}\Phi(t,s) = \mathcal{L}(t)\Phi(t,s), & t \geq s \\ \Phi(s,s) = I \end{cases}$$

Remark 2.3.

(ii) The state transition matrix $\Phi(t,s)$ is given by the Peano-Baker series and converges absolutely and uniformly for $t,s \in [-T,T]$ where $T>0$ is arbitrary (see Theorem 3.3 in Rugh, 1996).

(ii) By its definition, for each t, $\zeta = 0$ is an eigenvalue for $L(t)$ and the vector $(1,\dots,1)$ is an eigenvector associated with it. All other eigenvalues $\zeta(t)$ of $L(t)$ satisfy $2\min_i \ell_{ii}(t) \leq Re(\zeta(t)) < 0$. Given $\delta_i(t) > 0$, the eigenvalues of $\mathcal{L}(t)$ lie strictly within the left-half complex plane.

(iii) Assumption 1-(vi) by Theorem 8.7 in Rugh (1996), guarantees exponential stability for the state transition matrix $\Phi(t,s)$ in the time-varying case. Specifically, $\Phi(t,s)$ satisfies $\|\Phi(t,s)\| \leq 1$ for any $t \geq s$.

Remark 2.4 (Time invariant case). *For time-independent weight matrix $L(t) \equiv L$ and natural decay $\delta(t) \equiv \delta$, then $\mathcal{L}(t) \equiv \mathcal{L}$ and the transition matrix is $\Phi(t,s) = e^{\mathcal{L}(t-s)}$. In this case, Eq. (7) can be rewritten as*

$$P(t) = e^{\mathcal{L}t}p + \int_0^t e^{\mathcal{L}(t-s)}N(s)ds, \quad t \geq 0$$

where the matrix exponential is defined by the power series $e^{\mathcal{L}t} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathcal{L}^k t^k$. The operator

$$\zeta\mathbb{1} - \mathcal{L} : \mathbb{R}^n \to \mathbb{R}^n$$

is invertible with bounded inverse $(\zeta\mathbb{1} - \mathcal{L})^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and the resolvent formula (see Theorem 1.10 in Chapter II of Engel et al., 2000) holds for every $\zeta > 0$:

$$(\zeta\mathbb{1} - \mathcal{L})^{-1}h = \int_0^\infty e^{-(\zeta\mathbb{1}-\mathcal{L})t}h\,dt, \quad \forall h \in \mathbb{R}^n. \tag{8}$$

Finally, notice that the condition on the eigenvalues given by Remark 2.3(ii), guarantees exponential stability for the state transition matrix $\Phi(t,s)$ in the time-invariant case. Specifically, $\Phi(t,s)$ satisfies $\|\Phi(t,s)\| \leq 1$ for any $t \geq s$ (Tables 1, 2).

Proposition 2.1. *$J(p,(I,R))$ is well defined for all $p \in \mathbb{R}^n_+$ and $(I,R) \in \mathcal{A}(p)$, possibly equal to $+\infty$ or $-\infty$ (depending, respectively, on the occurrences $\gamma \in (0,1)$ and $\gamma \in (1,\infty)$).*

Proof. The proof is presented in Appendix A.1 □

3. Solution of the optimal control problem

3.1. Rewriting the objective function

The planner aims at solving the optimization problem

$$v(p) := \sup_{(I,R)\in\mathcal{A}(p)} J(p,(I,R)).$$

The function v denotes the value function of the optimization problem, and a triple (I^*, R^*) is said to be an optimal control for the problem starting at p if it satisfies $J(p;(I^*,R^*)) = v(p)$.

We now define a vector α (which can also be seen as a function of the nodes), which we use to rewrite the objective functional conveniently. Set

$$\alpha(s) := \int_s^\infty e^{-\rho(t-s)}\Phi^\intercal(t,s)\omega\,dt. \tag{9}$$

Remark 3.1. *Notice that $\Phi^\intercal(t,s)$, the transpose of the state transition matrix, does not always coincide with the state transition matrix associated with $\mathcal{L}^\intercal$. But this holds if $\mathcal{L}(t)$ commutes with itself at different times.*

Proposition 3.1. *For all $j \in \mathcal{V}$ and $s \geq 0$, $\alpha_j(s)$ is bounded by*

$$\frac{\min_{i\in\mathcal{V}}\omega_i}{\rho + \max_{j\in\mathcal{V}}\sup_{s\geq 0}\delta_j(s)} \leq \alpha_j(s) \leq \frac{\max_{i\in\mathcal{V}}\omega_i}{\rho + \min_{j\in\mathcal{V}}\inf_{s\geq 0}\delta_j(s)}.$$

Proof. The proof is presented in Appendix A.2 □

Remark 3.2. *Proposition 3.1, together with Assumptions 2.1(ii-iii), guarantees that for each $j \in \mathcal{V}$, α_j remains bounded above and bounded away from zero.*

Proposition 3.2. *We have, for all $p \in \mathbb{R}^n_+$ and $(I,R) \in \mathcal{A}(p)$,*

$$J(p,(I,R)) = -\langle\alpha(0),p\rangle + \int_0^{+\infty} e^{-\rho t} \Bigg[\left\langle \frac{((A^I(t)-1)I(t) + (A^R(t)-1)R(t))^{1-\gamma}}{1-\gamma}, \mathbf{1}\right\rangle - \langle\alpha(t), I(t)+\varepsilon(t)R(t)\rangle - \langle f(R(t)),\mathbf{1}\rangle\Bigg]dt. \tag{10}$$

Proof. The proof is presented in Appendix A.3 □

As a consequence of (10), we get the following useful result.

Theorem 3.1. *Let $(I^*(t), R^*(t))$ be an admissible strategy, i.e. $(I^*(t),R^*(t)) \in \mathcal{A}(p)$. Assume moreover that, for a.e. $t \in \mathbb{R}_+$, and for each $i \in \mathcal{V}$., the triplet $(I_i^*(t), R_i^*(t))$ is a maximum point for the function $F_{it} : \mathbb{R}^2_+ \to \mathbb{R}$ where*

$$F_{it}(I_i,R_i) = \frac{((a_i^I(t)-1)I_i + (a_i^R(t)-1)R_i)^{1-\gamma}}{1-\gamma} - \alpha_i(t)(I_i + \varepsilon_i(t)R_i) - f_i(R_i), \tag{11}$$

then $(I^*(t),R^*(t))$ is optimal for the problem **(P)**.

Proof. The proof is presented in Appendix A.4 □

Remark 3.3. *From Theorem 3.1, it is clear that the optimal control I_i, R_i are interlaced, and they depend on the other nodes only through the parameter $\alpha_i(t)$, which depends on the matrix $\mathcal{L}$.*

Remark 3.4. *If the value function is finite, from the rewriting of the functional (4) presented in (10) and (A.2), we can deduce some monotonic relationships between the value function and the various parameters of the model, in particular*

- *the value function v is increasing with respect to technological productivities, a_i^I, a_i^R for each individual node i.*

- *the value function v is decreasing with respect to pollution intensity ε_i, with respect to pollution awareness ω_i (since α is a linear in ω) at each location i.*

By assuming quadratic or linear costs, $f_i(R_i) = \lambda_i R_i$ or $f_i(R_i) = \lambda_i R_i^2$ for some parameter $\lambda_i > 0$, we can deduce that the value function is decreasing also with respect to the cost parameter λ_i.

3.2. Explicit solution of the problem and optimal path

To ensure the existence of an optimal solution for **(P)**, we will make the additional assumption.

Assumption 2.

(i) $\forall\, t \in \mathbb{R}_+,\ \forall\, i \in \mathcal{V},\ a_i^I(t), a_i^R(t) > 1.$
(ii) There exist $C \geq 0, g \in \mathbb{R}$ such that $\forall t \in \mathbb{R}_+, \forall i \in \mathcal{V}$,

$$(a_i^I(t)-1)^{\frac{1-\gamma}{\gamma}} + (a_i^R(t)-1)^{\frac{1-\gamma}{\gamma}} + \frac{(a_i^R(t)-1)}{(a_i^I(t)-1)} \leq Ce^{gt},$$

(iii) $\rho > g$.

Theorem 3.2. *Let Assumption 2 hold. Consider a cost function f satisfying $f(0) = 0$, $f', f'' \geq \epsilon$ for some $\epsilon > 0$, and $\int_0^\infty e^{-\rho t}\|f(R)\|dt < \infty$. Then,*

- *If*

$$\varepsilon_i(t) \leq \left(\frac{a_i^R(t)-1}{a_i^I(t)-1}\right) - \frac{f_i'(0)}{\alpha_i(t)}, \tag{12}$$

$$f_i'\left(\frac{1}{a_i^R(t)-1}\left(\frac{a_i^I(t)-1}{\alpha_i(t)}\right)^{1/\gamma}\right) \geq \alpha_i(t)\left(\frac{a_i^R(t)-1}{a_i^I(t)-1} - \varepsilon_i(t)\right), \tag{13}$$

  the inner point

$$\begin{cases} R_{1,i} = \left[(f_i')^{-1}\left(\alpha_i(t)\left(\frac{a_i^R(t)-1}{a_i^I(t)-1} - \varepsilon_i(t)\right)\right)\right] \\ I_{1,i} = \left[\left(\frac{\alpha_i(t)}{a_i^I(t)-1}\right)^{-1/\gamma} - (a_i^R(t)-1)R_{1,i}\right]\frac{1}{(a_i^I(t)-1)}, \end{cases} \tag{14}$$

  belongs to $\mathcal{A}(p)$ and is optimal starting at each p. The optimal consumption flow is for each $i \in \mathcal{V}$

$$C_{1,i}(t) = \left(\frac{a_i^I(t)-1}{\alpha_i(t)}\right)^{\frac{1}{\gamma}}.$$

  The optimal emissions flow is for each $i \in \mathcal{V}$

$$N_{1,i}(t) := I_{1,i}(t) + \varepsilon R_{1,i}(t). \tag{15}$$

- *If condition (12) holds and also*

$$f_i'\left(\frac{1}{a_i^R(t)-1}\left(\frac{a_i^I(t)-1}{\alpha_i(t)}\right)^{1/\gamma}\right) < \alpha_i(t)\left(\frac{a_i^R(t)-1}{a_i^I(t)-1} - \varepsilon_i(t)\right), \tag{16}$$

  a boundary point with $I_{2,i} = 0$ is global maximum. In this case, an explicit formula for the optimal control is not available. However, controls are the unique solution of the non-linear system of equations,

$$\begin{cases} R_{2,i} \text{ s.t. } \left((a_i^R(t)-1)R_{2,i}\right)^{-\gamma}\cdot(a_i^R(t)-1) - \varepsilon_i(t)\alpha_i(t) = f_i'(R_{2,i}) \\ I_{2,i} = 0, \end{cases} \tag{17}$$

  belongs to $\mathcal{A}(p)$ and is optimal starting at each p. The optimal consumption flow is, for each $i \in \mathcal{V}$,

$$C_{2,i}(t) = \left(\frac{a_i^R(t)-1}{\varepsilon_i(t)\alpha_i(t) + f_i'(R_{2,i})}\right)^{\frac{1}{\gamma}}$$

  The optimal emissions flow is $N_{2,i}(t) := \varepsilon_i(t)R_{2,i}(t)$ for each $i \in \mathcal{V}$.

- *If*

$$\varepsilon_i(t) > \left(\frac{a_i^R(t)-1}{a_i^I(t)-1}\right) - \frac{f_i'(0)}{\alpha_i(t)} \tag{18}$$

  the boundary point

$$\begin{cases} I_{3,i} = \left[\left(\frac{\alpha_i(t)}{a_i^I(t)-1}\right)^{-1/\gamma}\right]\frac{1}{(a_i^I(t)-1)} \\ R_{3,i} = 0, \end{cases} \tag{19}$$

  belongs to $\mathcal{A}(p)$ and is optimal starting at each p. The optimal consumption flow is for each $i \in \mathcal{V}$

$$C_{3,i}(t) = \left(\frac{a_i^I(t)-1}{\alpha_i(t)}\right)^{\frac{1}{\gamma}}.$$

  The optimal emissions flow is $N_{3,i}(t) := I_{3,i}(t)$. for each $i \in \mathcal{V}$.

Proof. The proof is presented in Appendix A.5 □

Remark 3.5. *The requirement $a^I, a^R > 1$ in Assumption 2 ensures that neither form of production is a priori excluded from the optimization problem and that all subsequent assumptions and conditions are well defined for any value of γ. Moreover, Assumption 2 (ii-iii) guarantees that the optimal solutions lie within the set of admissible controls $\mathcal{A}(p)$.*

Remark 3.6. *The result stated in the theorem above admits a clear interpretation. Specifically, we observe that if the pollution rate associated with green investment is sufficiently high, it is optimal to invest exclusively in brown energy, implying $R \equiv 0$. Conversely, when the pollution rate of renewable investment is sufficiently low, the optimal strategy depends on the trade-off between the marginal cost of renewable investment and the effect of accumulated pollution on the relative attractiveness of brown versus renewable investment, as captured by conditions (13) and (16). Specifically, if the marginal cost of renewables grows quickly enough, i.e. (13) holds, then it is optimal to supplement green investment with brown one. On the other hand, if the inequality is reversed, i.e. (16) holds, the marginal cost of green investment grows more slowly, and investing exclusively in green production alone is optimal.*

Theorem 3.3. *If Assumption 2 holds. Consider a linear cost function $f_i(r) = \lambda_i r$, $\lambda_i > 0$.*

- *If*

$$\varepsilon_i(t) < \left(\frac{a_i^R(t)-1}{a_i^I(t)-1}\right) - \frac{\lambda_i}{\alpha_i(t)}, \tag{20}$$

  the boundary point

$$\begin{cases} I_{1,i} = 0 \\ R_{1,i} = (a_i^R(t)-1)^{\frac{1-\gamma}{\gamma}}\left(\lambda_i + \varepsilon_i(t)\alpha_i(t)\right)^{-\frac{1}{\gamma}}, \end{cases} \tag{21}$$

  belongs to $\mathcal{A}(p)$ and is optimal starting at each p. The optimal consumption flow is for each $i \in \mathcal{V}$

$$C_{1,i}(t) = \left(\frac{a_i^R(t)-1}{\lambda_i + \varepsilon_i(t)\alpha_i(t)}\right)^{\frac{1}{\gamma}}.$$

  The optimal emissions flow is $N_{1,i}(t) := \varepsilon_i(t)R_{1,i}(t)$. for each $i \in \mathcal{V}$.

- *If*

$$\varepsilon_i(t) > \left(\frac{a_i^R(t)-1}{a_i^I(t)-1}\right) - \frac{\lambda_i}{\alpha_i(t)}, \tag{22}$$

  the boundary point

$$\begin{cases} I_{2,i} = (a_i^I(t)-1)^{\frac{1-\gamma}{\gamma}}\alpha_i(t)^{-\frac{1}{\gamma}} \\ R_{2,i} = 0, \end{cases} \tag{23}$$

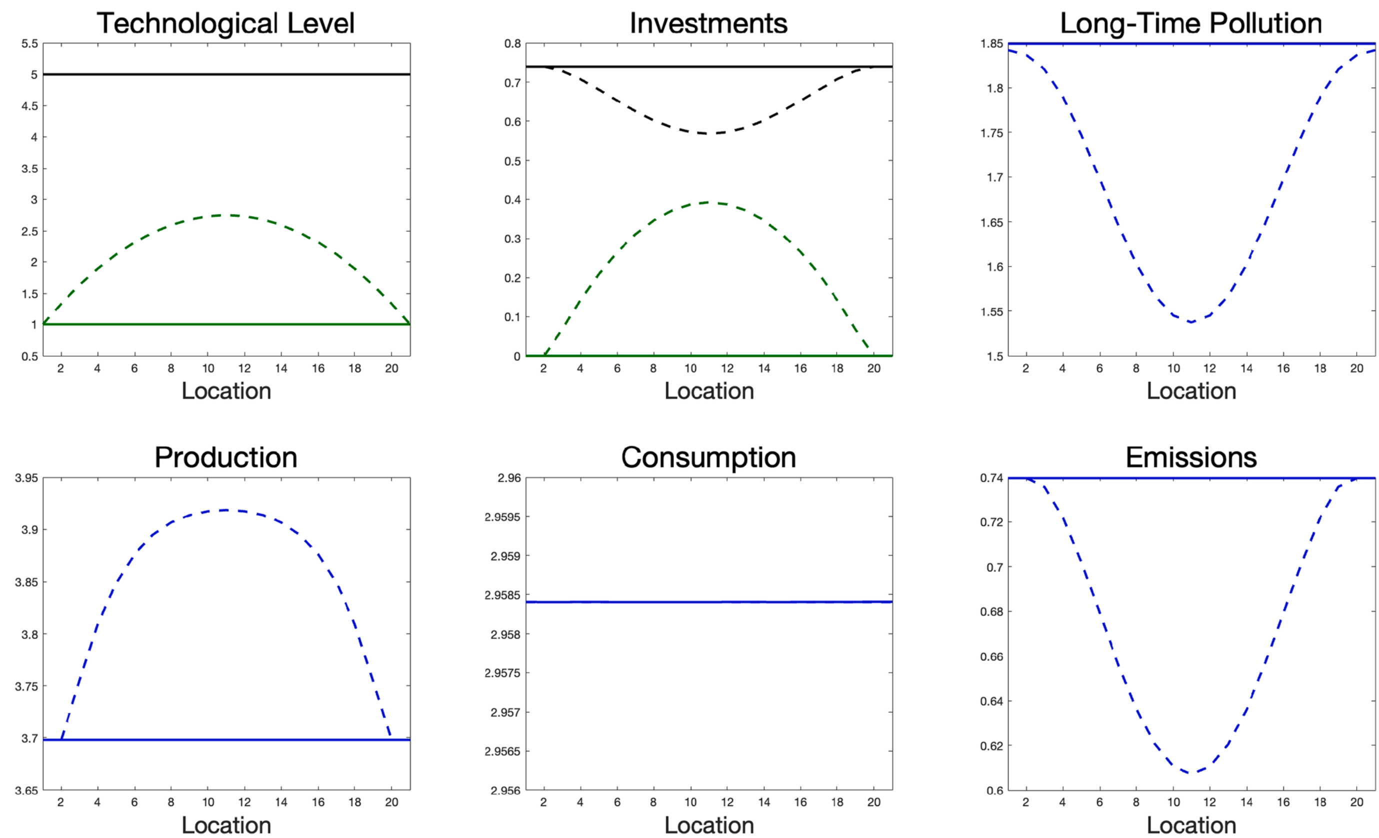

Fig. 1. Impact of renewable investments on optimal investment choices, long-term pollution levels, production, consumption and emissions across nodes. The parameter values specific to this figure are: $L = L^1$, $\delta_i = 0.4$, $a_i^I = 5$. In the straight-line scenario $a_i^R = 1$, in the dashed-lines scenario $a_i^R = 2.75$ at the core and $a_i^R = 1$ at the periphery, $\forall\, i \in \mathcal{V}$.

belongs to $\mathcal{A}(p)$ and is optimal starting at each p. The optimal consumption flow is for each $i \in \mathcal{V}$

$$C_{2,i}(t) = \left(\frac{a_i^I(t) - 1}{\alpha_i(t)}\right)^{\frac{1}{\gamma}}.$$

The optimal emissions flow is $N_{2,i}(t) := I_{2,i}(t)$. for each $i \in \mathcal{V}$.

- *If*

$$\varepsilon_i(t) = \left(\frac{a_i^R(t) - 1}{a_i^I(t) - 1}\right) - \frac{\lambda_i}{\alpha_i(t)}, \tag{24}$$

the inner point

$$\begin{cases} (a_i^I(t) - 1)I_{3,i} + (a_i^R(t) - 1)R_{3,i} = (a_i^I(t) - 1)^{\frac{1}{\gamma}}\alpha_i(t)^{-\frac{1}{\gamma}} \\ 0 < R_{3,i} < (a_i^R(t) - 1)^{-1}\left((a_i^I(t) - 1)^{\frac{1}{\gamma}}\alpha_i(t)^{-\frac{1}{\gamma}}\right), \end{cases} \tag{25}$$

belongs to $\mathcal{A}(p)$ and is optimal starting at each p. The optimal consumption flow is, for each $i \in \mathcal{V}$,

$$C_{3,i}(t) = \left(\frac{a_i^I(t) - 1}{\alpha_i(t)}\right)^{\frac{1}{\gamma}}.$$

The optimal emissions flow is $N_{3,i}(t) := I_{3,i}(t) + \varepsilon_i(t)R_{3,i}(t)$. for each $i \in \mathcal{V}$.

Proof. The proof is presented in Appendix A.6 □

Remark 3.7. *Since the case of linear costs cannot be regarded as a special case of Theorem 3.3, we analyze it separately. In doing so, we uncover a result that appears natural when linear costs are considered. We essentially distinguish two relevant cases. First, if the effective pollution rate is relatively low, specifically if condition (20) holds, it is optimal to invest solely in green production, with no investment in browns. Conversely, if it is relatively high, as in condition (22), it is optimal to invest exclusively in brown production, with zero green investment. The third case, corresponding to the equality in condition (24), defines an inner point where both investments coexist; however, this occurs on a set of null measure and is therefore negligible in practical terms.*

3.3. Long-time behaviour of the optimal state trajectory

We now consider the special case when the coefficients are time-independent. We denote with $(I^*(t), R^*(t))$ optimal consumptions stated in the Theorem (3.2) or in Theorem (3.3).

Theorem 3.4. *Let Assumption 2 hold. Assume that the coefficients $a_i^I, a_i^R, \delta_i, \epsilon_i$ are time independent, $\forall i \in \mathcal{V}$. Then*

$$\lim_{t\to\infty} \sum_{i=1}^{n} |P_i^*(t) - P_{i,\infty}^*|^2 = 0,$$

where $P_i^*(t)$ is the unique solution of the equation

$$\frac{d}{dt}P(t) = \mathcal{L}(t)P(t) + N^*$$

with $N^* = I^* + \epsilon R^*$ and P_∞^* is the unique solution to the matrix equation $\mathcal{L}P + N^* = 0$.

Proof. The proof is presented in Appendix A.7 □

3.4. The model with $a_i^R(t) \equiv 1\ \forall i \in \mathcal{V}$.

We will now address the case in which on every node, the non-renewable productivity factor is strictly greater than one, while the renewable one equals one, namely $a_i^R(t) \equiv 1$ and $a_i^I(t) > 1\ \forall i \in \mathcal{V}$. In this

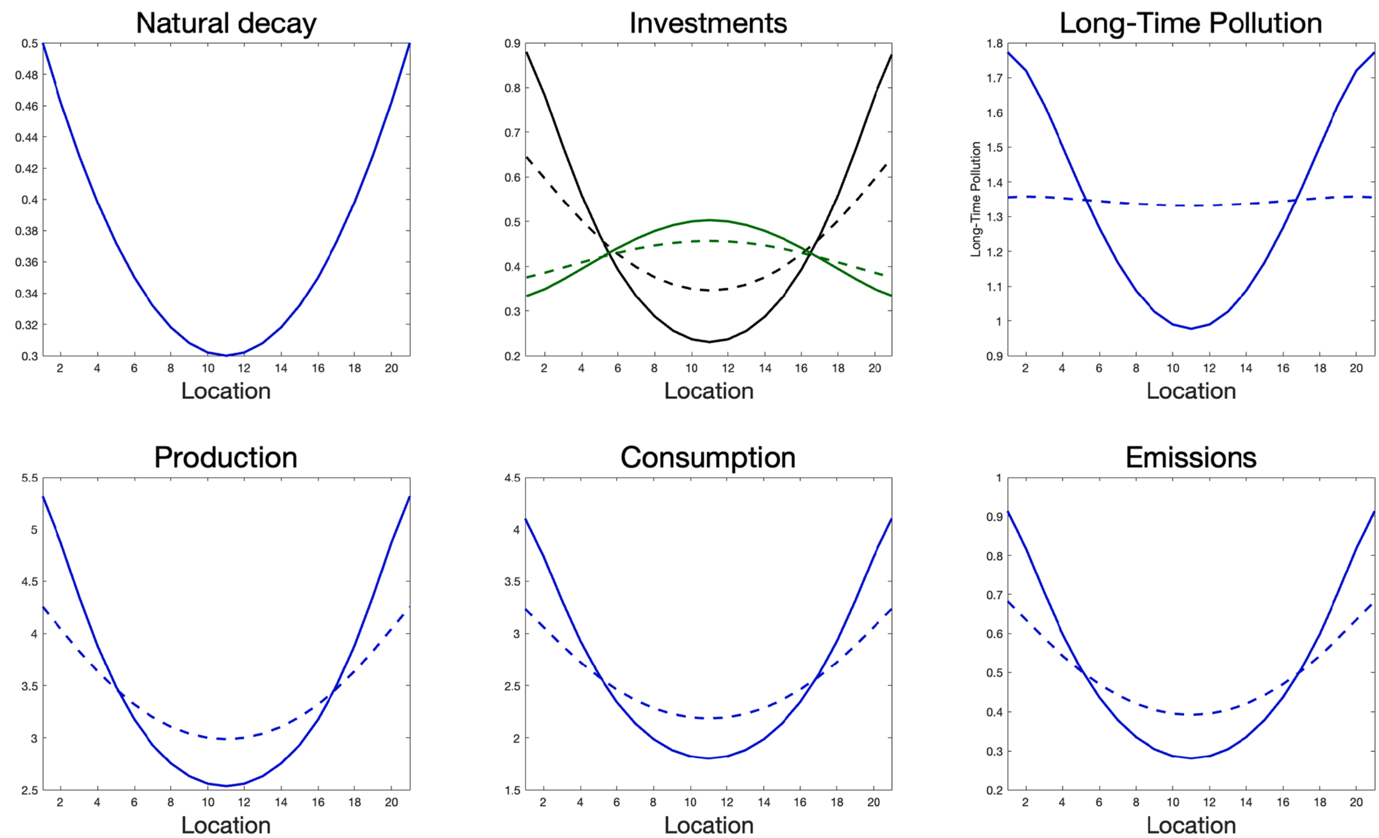

Fig. 2. Comparison of optimal investment choices, long-term pollution levels, production, consumption and emissions across nodes for different choices of the operator L. The parameter values specific to this figure are: $a_i^I = 5$, $a_i^R = 2.75$, $\forall\, i \in \mathcal{V}$. The parameter δ varies across nodes, namely $\delta_i = 0.3$ at the core and $\delta_i = 0.5$ at the periphery. The straight lines represent the case $L = L^1$ and the dashed lines the case $L = L^2$.

setting, any investment in renewable energy sources is economically unfeasible, and thus the entire production relies on a single (traditional) energy source. So, to simplify our setting, we will directly consider the only possible investment I. With this simplification, the production takes the form

$$Y_i(t) = a_i^I(t) I_i(t),$$

and the resource constraint implies that the consumption is simply given by

$$C_i(t) = (a_i^I(t) - 1) I_i(t)$$

The dynamics of P can be written as

$$\begin{cases} \frac{d}{dt} P(t) = (L - \delta) P(t) + I(t) \\ P(0) = p \in \mathbb{R}_+^n, \end{cases}$$

and the social welfare to be maximised

$$J(p, I) := \int_0^{+\infty} e^{-\rho t} \left(\sum_{i=1}^n \left(\frac{C_i(t)^{1-\gamma}}{1-\gamma} - \omega_i P_i(t) \right) \right) dt. \tag{26}$$

Finally, the set of admissible controls

$$\mathcal{A}(p) := \left\{ I \in PC(\mathbb{R}_+; \mathbb{R}_+^n) : \int_0^{\infty} e^{-\rho t} \left(\sum_{i=1}^n |I_i(t)|^2 \right)^{\frac{1}{2}} dt < \infty \right\}. \tag{27}$$

To guarantee the existence of a solution, we impose the following assumptions, in addition to those stated in Assumption 2.1.

Assumption 3.

(i) There exist $C \geq 0$, $g \in \mathbb{R}$ such that

$$(a_i^I(t) - 1)^{\frac{1-\gamma}{\gamma}} \leq C e^{gt}, \qquad \forall t \in \mathbb{R}_+, \quad \forall i \in \mathcal{V},$$

(ii) $\rho > g$.

Remark 3.8. *Assumptions 3 $(i)-(ii)$ guarantee that the value function is finite. Given this, to solve the problem, we use the alternative form (10) of the objective functional. In such form, we take the control which, for every t maximizes the integrand in (10). This is a candidate optimal control.*

Theorem 3.5. *The optimal investment I_i, given by*

$$I_i(t) = \alpha_i(t)^{-\frac{1}{\gamma}} \left(a_i^I(t) - 1 \right)^{\frac{1-\gamma}{\gamma}},$$

belongs to $\mathcal{A}(p)$ in (27) and is optimal for (26) starting at each p. The optimal consumption flow is for each $i \in \mathcal{V}$

$$C_i(t) = \left(\frac{a_i^I(t) - 1}{\alpha_i(t)} \right)^{\frac{1}{\gamma}}.$$

The optimal emissions flow is $N_i(t) := I_i(t)$. for each $i \in \mathcal{V}$.

Proof. The proof is presented in Appendix A.8 □

Notice that in our model, investment depends on the transboundary nature of pollution. The regulator must consider not only the local technological factors but also the potential impact of making investments in a specific location on the neighbouring areas in terms of pollution. It is worth noting that local investments may not necessarily increase with local productivity, denoted as $A^I(t)$. In some cases, higher local productivity might result in lower investments ($I(t)$), which leads to reduced local emissions, albeit at the cost of a slight reduction in production.

Remark 3.9 (Comparison between pollutions)**.** *We investigate the impact of introducing a new source of energy into the economic system. In particular,*

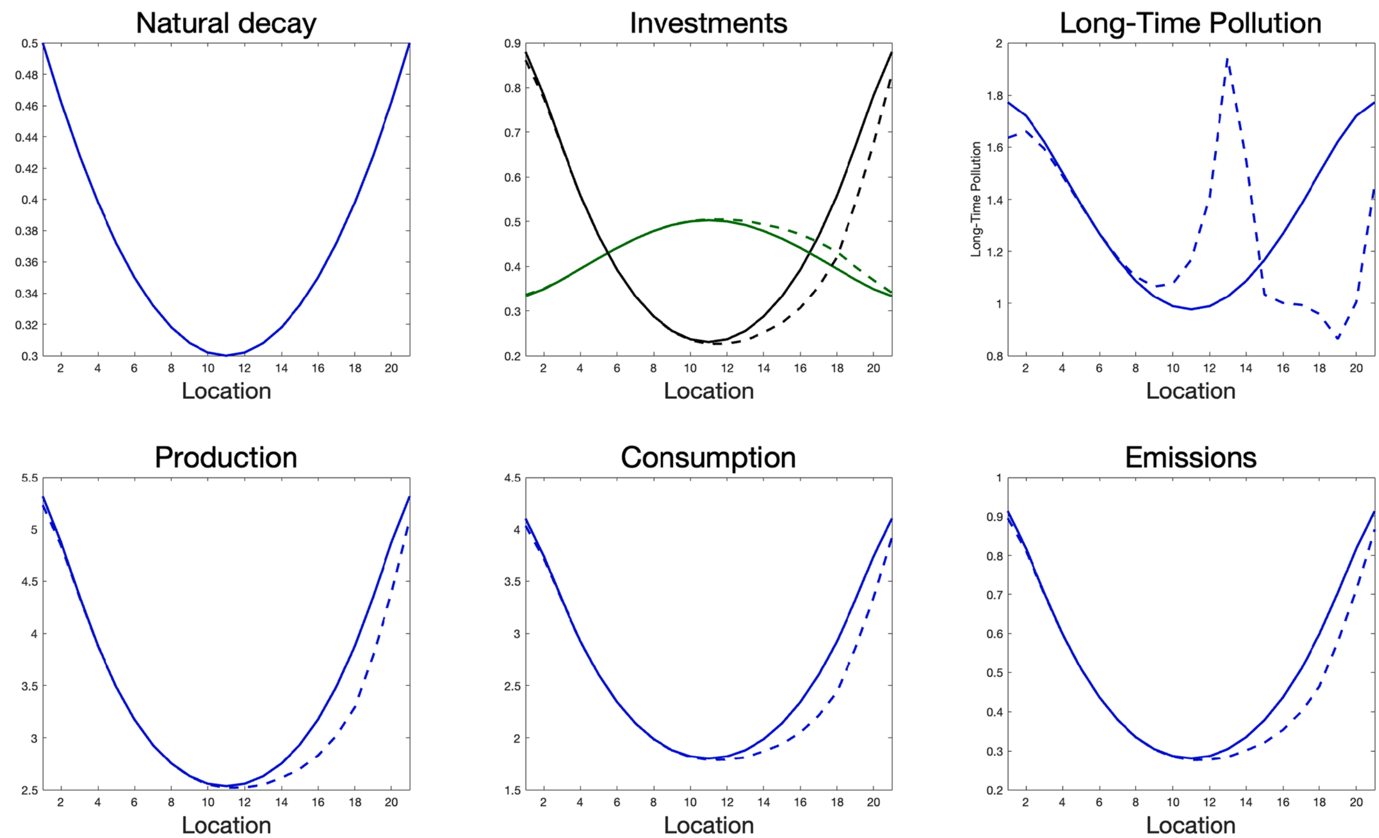

Fig. 3. Comparison of optimal investment choices, long-term pollution levels, production, consumption and emissions across nodes for different choices of the operator L. The parameter values specific to this figure are: $a_i^I = 5$, $a_i^R = 2.75$, $\text{wind} = 0.4$, $\forall\, i \in \mathcal{V}$. The parameter δ varies across nodes, namely $\delta_i = 0.3$ at the core and $\delta_i = 0.5$ at the periphery. The straight lines represent the case $L = L^1$ and the dashed lines the case $L = L^3$.

we analyze how the adoption of a cleaner energy source affects pollution levels. Our study compares the optimal emission flow in two scenarios:

1. *When only a single energy source is available, as described in Theorem 3.5.*
2. *When two energy sources are considered, as described in Eq. (15) in Theorem (3.2).*

Since we assume that f is strictly increasing, it follows that

$$N_{1,i} = N_i - (f_i')^{-1}\left(\alpha_i(t)\left(\frac{a_i^R(t)-1}{a_i^I(t)-1} - \varepsilon_i(t)\right)\right)\left(\frac{a_i^R(t)-1}{a_i^I(t)-1} - \varepsilon_i(t)\right) < N_i.$$

This result implies that, in each location, the optimal emission flow when two energy sources are available is lower than the optimal emission flow when only a single, more polluting energy source is used.

4. Some numerical investigations when the renewable cost function is quadratic

In this section, we present the results of a series of quantitative exercises where the cost of renewable technology is quadratic, i.e. $f_i(R_i) = \lambda_i R_i^2$, and all the relevant parameters of the model are kept constant. A straightforward consequence of the static choice of parameters is that the profile of optimal investments is also constant in time, and in particular, they coincide with their long-time distribution. The same argument holds for production, consumption and emission, but not for pollution. Indeed, in the plots presented in the section, we will just present its long-time distribution.

The large number of parameters in the model gives us great flexibility in terms of potential scenarios to study. We have therefore selected some interesting ones focusing on how optimal investment strategies are affected by the use (or non-use) of renewable investments and how these choices change according to heterogeneities on the nodes. In particular, our analysis will focus on the following points.

- The role of Renewable Investment: We aim to analyze how optimal investment choices change by including renewable investment in the model.
- Geographic or Economic Heterogeneity: By introducing parameter heterogeneity among nodes, we study how geographic and economic differences influence investment decisions. We explore the effects of spatial factors (e.g. modelling rural or urban settings) by varying natural decay across nodes and modifying the diffusion matrix L to represent different types of pollution flow. As for the economic heterogeneities, we consider different values of productivity factors (for both green and brown production).
- Combined effects of Heterogeneity: We consider scenarios that combine geographic and economic disparities. We will study how the results of the previous analysis are enhanced or mitigated when combined. This allows us to investigate, for example, how economically but not geographically advantaged network nodes, such as densely populated urban centres, respond differently in terms of optimal investments compared to less productive but geographically advantaged areas.

4.1. Calibration of the parameters

We start by focusing on the characterization and calibration of the parameters based on existing literature. Given the similarity of the two models, we will refer mainly to Boucekkine et al. (2021) to assign numerical values.

For the network structure, we consider $n = 21$ nodes arranged on a circular network. We model the flow of pollution on the network through

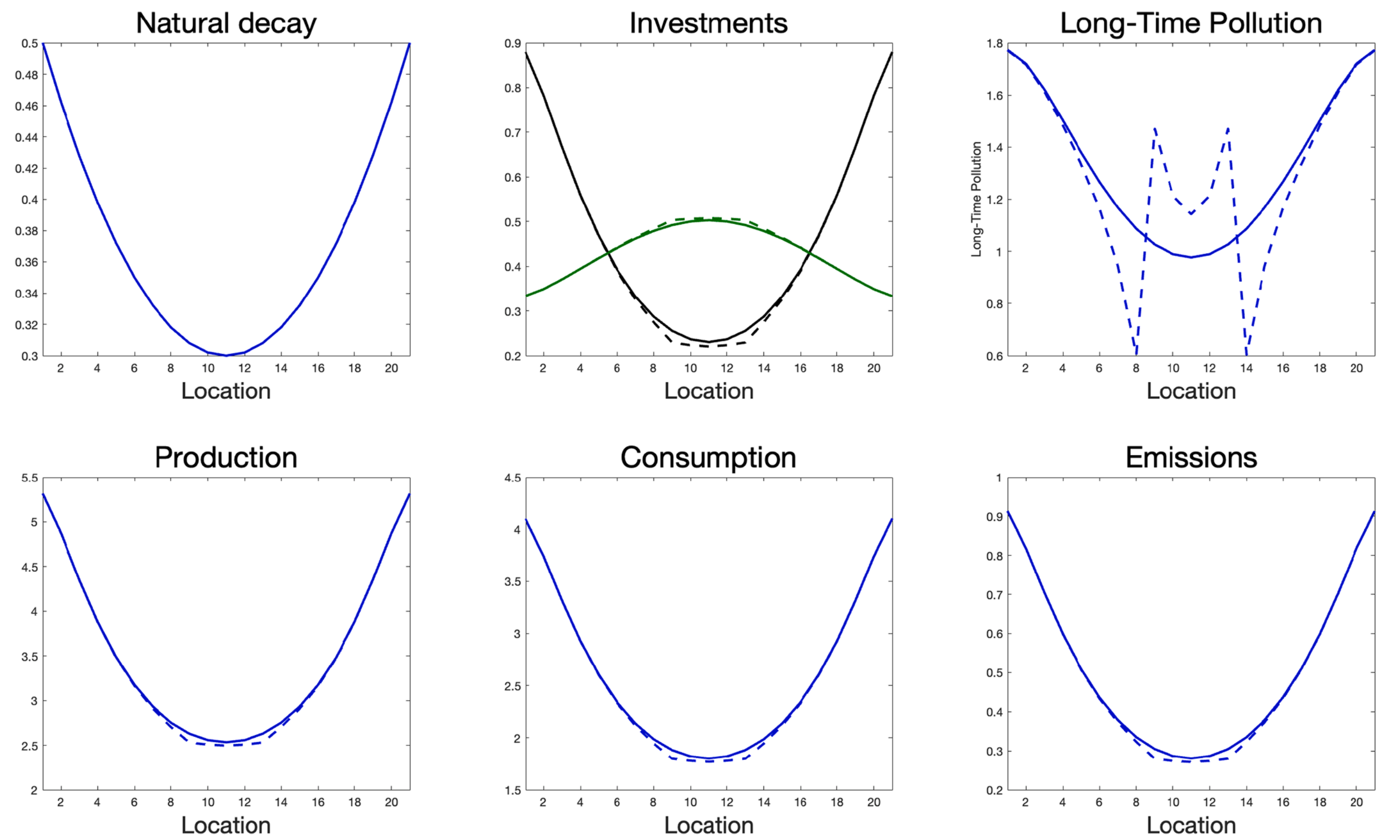

Fig. 4. Comparison of optimal investment choices, long-term pollution levels, production, consumption and emissions across nodes for different choices of the operator L. The model parameters are: $a_i^I = 5$, $a_i^R = 2.75$, $\forall\, i \in \mathcal{V}$. The parameter δ varies across nodes, namely $\delta_i = 0.3$ at the core and $\delta_i = 0.5$ at the periphery. The straight lines represent the case $L = L^1$ and the dashed lines the case $L = L^4$.

the matrix L. The elements of $L = (\ell_{ij})$ control how the pollution is dispersed from one node to the other, reflecting geographic factors such as proximity and connectivity between locations. Each entry $\ell_{ij}, i \neq j$ reflects how easily the pollution generated at node j can spread to node i, and the diagonal term $\ell_{jj} = -\sum_{i\neq j} \ell_{ij}$ keeps track of all the pollution leaving node j. Notice that the matrix is not required to be symmetrical, this allows us to consider asymmetric pollution flows between regions, capturing phenomena such as the influence of directional factors, such as prevailing winds or currents, which may make pollution disperse more strongly in one direction than another with no need for additional advection terms.

We first consider a *nearest-neighbor Laplacian* with periodic boundary conditions, where the matrix $L^1 = (\ell^1_{ij})$ is given by:

$$\ell^1_{ij} = \begin{cases} \frac{1}{2}, & j = i+1 \text{ or } j = i-1 \\ -1, & i = j \\ \frac{1}{2}, & i = 1,\ j = n \text{ or } i = n,\ j = 1 \end{cases}$$

Here, pollution spreads from a node to its immediate neighbour, meaning the diffusion is relatively local and slow.

Afterwards, we also test a *distance based laplacian* on all nodes where the matrix $L^2 = (\ell^2_{ij})$ is given by

$$\ell^2_{ij} = \begin{cases} \frac{1}{\min(|i-j|, n-|i-j|)} & \text{if } i \neq j. \\ -\sum_{k\neq i} \frac{1}{\min(|i-k|, n-|i-k|)} & \text{if } i = j, \end{cases}$$

This representation defines each element ℓ^1_{ij} as the reciprocal of the circular distance between nodes i and j. The idea is that the pollution on each node can spread to all the others with a weight that diminishes with distance. Compared to the matrix L^1, we have global diffusion, even though the intensity reduces as the distance grows.

To incorporate asymmetries in the spread matrix, we add distortion in the diffusion of pollution, by the use of a "wind" with strength, e.g. $wind = 0.4$, acting only on a specific set of nodes $\tilde{\mathcal{J}} \subseteq \{1 \dots, n\}$. This adjustment increases the dispersal rate in the direction of the wind and lowers it. We define the $L^3 = (\ell^3_{ij})$ such that the off-diagonal elements are given by:

$$\ell^3_{ij} = \begin{cases} \frac{1}{2} \pm wind, & j = i \pm 1, i \in \tilde{\mathcal{J}} \\ \frac{1}{2}, & j = i \pm 1, i \notin \tilde{\mathcal{J}}. \end{cases}$$

The diagonal elements are set to ensure that the sum of each column is zero, maintaining the balance between pollution entries and exits from each node. Boundary conditions are applied as needed but are not explicitly stated here.

Another advantage of allowing asymmetries in the L matrix is that it can represent geographic areas in which pollution has difficulty entering or exit. For example, we can define two set of nodes $\tilde{\mathcal{I}}, \tilde{\mathcal{J}} \subseteq \{1 \dots, n\}$ and construct $L^4 = (\ell^4_{ij})$ such that

$$\ell^4_{ij} = \begin{cases} \zeta \ell_{i,j}, & j \in \tilde{\mathcal{J}}, i \in \tilde{\mathcal{I}}, \\ \ell_{ij}, & \text{otherwise}. \end{cases}$$

We will set the $\ell_{i,j}$, $i \neq j$ as in L^1, with the diagonal values $\ell_{i,i,}$ such that the sum of each column is zero. Depending on the value of $\zeta \geq 0$, pollution flows are restricted or enhanced.

For the calibration of the other parameters, we will let δ_i live in a range from 30% to 50% per year. We will make this parameter vary across the nodes to reflect geographical disparities between urban and rural areas, as well as other factors that may create geographically advantaged or disadvantaged regions.

In Boucekkine et al. (2021), the productivity factor a^I is calibrated to achieve an investment-to-gross domestic product ratio within $15-40\%$,

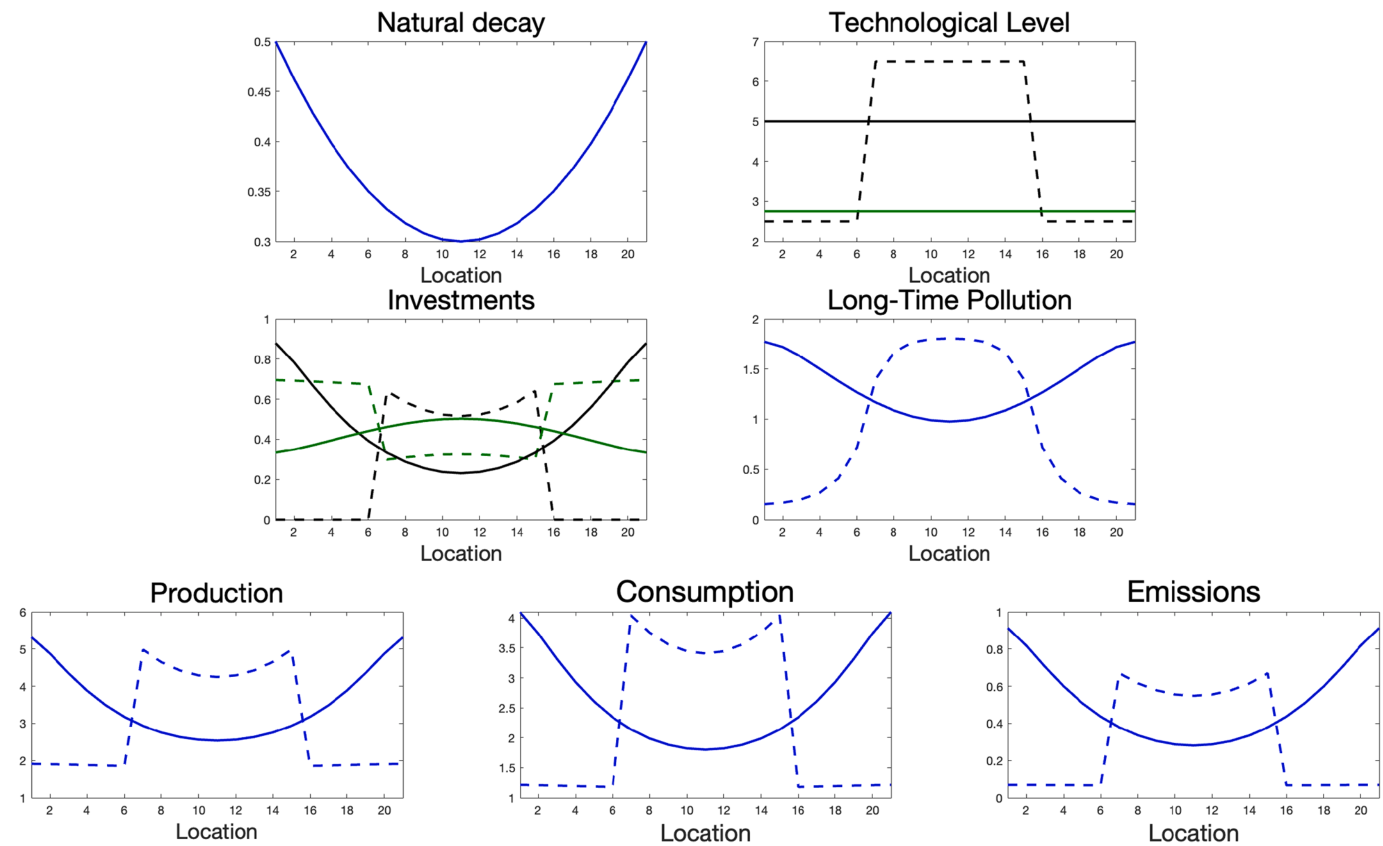

Fig. 5. Comparison of optimal investment choices, long-term pollution levels, production, consumption and emissions across nodes. The parameter values specific to this figure are: $a_i^R = 2.75$, $\forall\, i \in \mathcal{V}$. The parameter δ varies across nodes, namely $\delta_i = 0.3$ at the core and $\delta_i = 0.5$ at the periphery. The straight lines represent the case in which $a_i^I = 5$, $\forall\, i \in \mathcal{V}$, and the dashed lines the case in which $a_i^I = 6.5$ on the nodes from 7 to 15 and $a_i^I = 2.5$ on the other nodes.

which aligns with typical economic ranges. Here, because of the presence of two forms of investment contributing to the GDP, such reasoning is not as straightforward. Furthermore, the productivity of renewable investments is affected by several factors, including the kind of process taken into consideration, the geographic location, the type of energy employed and so on. Here we set a^I as in Boucekkine et al. (2021), thus in the range $[2.5, 6.5]$ and $a^R \in [1, 2.75]$, whereby setting $a^R = 1$ we will study the case in which no sustainable investment is taken into account. The environmental awareness ω, representing the agent's awareness of pollution levels and their impact on health, environment and overall quality of life, is set to a baseline value of $\omega = 1$ and can be interpreted as either a social factor or an implicit tax on pollution.
The discount factor is taken equal to 3%. Initial condition of pollution, p is chosen constant in space, $p_i = 1\ \forall i \in \mathcal{V}$. Finally, we fix the inverse of the inter-temporal elasticity of substitution, which measures the regulator's aversion to inequality in consumption, to be equal to $\gamma = 0.5$. This choice implies a relatively low aversion to consumption inequality and reflects a social planner who is willing to tolerate some fluctuations in consumption to prioritize other objectives, such as optimizing investments and reducing pollution.
It is quite challenging to choose numerical values for ε_i and λ_i as, once again, empirical data on renewable impact per unit investment varies significantly across investment types, geographical regions, employed energy and so on. For ε, which measures the impact of renewable production on emissions, we choose $\varepsilon_i = 0.1$, to reflect that renewable production contributes less to pollution compared to traditional sources, whose impact is normalized, and so it is 1. For λ_i, which represents the costs of (carrying, maintaining, installing, etc.) renewable investments, we set a value of $\lambda_i = 1$.
The following table summarizes the parameter values that are kept constant across all simulations, while the particular choices for the remaining parameters are provided in the captions of the corresponding figures.

4.2. *Numerical experiments*

In the following figures, we illustrate how optimal investments, long-term pollution levels, production, consumption and emissions vary by location. Non-renewable investments are always shown in black, renewable investments in green, and pollution reduction investments in red. When necessary, we also include plots of the model parameters that vary between nodes, such as natural decay and technological productivity.
We begin our analysis by investigating how the inclusion of renewable investment affects the optimal investment choices and, consequently, all the other quantities within the model. Fig. 1 illustrates two scenarios: the straight lines represent the case where renewable productivity a_i^I is set equal to 1 across all nodes, which corresponds to a scenario with no renewable investments. In contrast, the dashed lines represent the case where renewable productivity exceeds 1 on some nodes, leading to positive renewable investments. The particular choice of parameters is summarized in the caption. In the first subplot, the non-renewable productivity is shown in black, while the renewable productivity is represented in green. The straight line corresponds to $a_i^R = 1$ on all nodes, while the dashed line addresses the case where $a_i^R \geq 1$ on the nodes, with a peak of $a_i^R = 2.75$ at the central node and a gradual decrease towards the outer nodes. In each of the subplots, we report the behaviour of some of the model variables on the nodes. We observe that the optimal renewable investment R follows the spatial distribution of a^R, showing a peak at the central node. When renewable investments are introduced, non-renewable investment I decreases, with a peak of −25% at the central node. Emissions and long-term pollution levels decrease in regions

Table 2
Fixed parameter values for all simulations.

Parameter	Value
n	21
ω_i	1
ε_i	0.1
λ_i	1
ρ	0.03
γ	0.5

where renewable productivity exceeds 1. Specifically, long-term pollution is reduced by up to 20% at the central node. Production increases as a productivity source is added to the model, while the optimal consumption level remains unchanged.

Since all parameters are time-independent, by Remark 2.4, α is given by

$$\alpha = \int_0^\infty e^{-(\rho\mathbb{1}-\mathcal{L}^\intercal)t}\omega dt, \tag{28}$$

i.e. it is the unique solution to the linear system $(\rho\mathbb{1}-\mathcal{L}^\intercal)\alpha = \omega$. Moreover, for each $i \in \mathcal{V}$, the bounds established in Proposition 3.1 take the form:

$$\min_{j\in\mathcal{V}} \frac{\omega_j}{\rho+\delta_j} \le \alpha_i \le \max_{j\in\mathcal{V}} \frac{\omega_j}{\rho+\delta_j}$$

Notice that the $i-$th component of α reflects the cumulative effect of environmental awareness ω at location i, adjusted for how pollution spreads over time. Specifically, it reflects the combined effects of pollution dispersion across locations (through L), of natural decay δ and time discounting ρ. From the bound, it follows that, if ω_i and δ_i are constant across all nodes $i \in \mathcal{V}$, then α remains constant as well. Consequently, the choice of network structure governing the pollution dispersion in space does not affect the disutility. However, allowing natural decay and/or pollution awareness to vary across nodes means that different network structures will result in different disutility values at the nodes, and consequently, this will impact the optimal strategy.
In Fig. 2, we compare two different choices for the matrix L, as introduced in Section 4.1. The straight lines represent the case $L = L^1$, while the dashed lines $L = L^2$. We keep all model parameters constant across nodes, except for the natural decay δ, which has a minimum at the central node and maximums at the extremes, as shown in the first subplot. We observe that non-renewable investment follows the shape of the natural decay: it is lower in regions where δ is low and vice versa. In contrast, renewable investment exhibits the opposite behaviour, being higher in areas where self-cleaning capacity is weaker. Long-term pollution, emissions, production and consumption also follow the shape of δ. We note that all the quantities are affected by the choice of L: when using the distance-based Laplacian L^2, differences across nodes are reduced, leading to smoother overall distributions. Specifically, the long-term pollution appears to be almost homogeneous across locations. While L^1 spreads pollution only to the adjacent nodes, L^2 allows pollution to spread broadly across the network, resulting in mitigated sharp variations and a more balanced distribution for investments, production and all the other economic factors.

In Fig. 3, we further investigate geographic heterogeneities by comparing the effect of $L = L^1$ and $L = L^3$, respectively, in straight and dashed lines. As presented in Section 4.1, in L^3 we impose a "wind" effect that influences only a subset of the nodes, namely $\mathcal{J} = \{14, \dots, 19\}$. This accentuates dispersion towards higher-numbered nodes while reducing dispersion toward lower-numbered nodes. As a result, we observe a bottleneck effect at node 13, where pollution accumulates. Additionally, in the wind-affected regions, the behaviour of all economic quantities is pronounced compared to the case $L = L^1$: renewable investment is higher, while nonrenewable investment, production, consumption, and emissions are all lower.

In Fig. 4, we compare the choices $L = L^1$ and $L = L^4$. For L^4 we set $\zeta = 0$, $\tilde{\mathcal{I}} = \{8, 14\}$, $\tilde{\mathcal{J}} = \{9, 13\}$, this results in a matrix where $\ell_{8,9} = \ell_{14,13} = 0$, i.e. we create a zone between nodes 9 and 13 where pollution cannot exits. Compared to the case $L = L^1$, this results in higher pollution levels at nodes 9 and 13 due to accumulation, and lower levels at 8 and 14. However, the changes in pollution are localized to the affected area, leading to only slight adjustments in other quantities without causing substantial re-optimization of investments or production.

We conclude this analysis by introducing an additional form of heterogeneity across the nodes: spatial discrepancy in input productivity. To capture this, we vary the values of a^I and a^R across the nodes, allowing for different levels of productivity in each location. In Fig. 5, we compare the case of geographical heterogeneity already addressed in Fig. 2 with the case where non-renewable productivity is unevenly distributed across locations, concentrating in the central nodes to create an economically advantaged zone. It is interesting to notice that the geographic heterogeneity alone (straight lines) leads to a natural sorting where renewable investment dominates in geographically disadvantaged zones and non-renewable flourishes where pollution dissipates faster. Adding economic heterogeneity (dashed lines) completely overrides this effect: non-renewable productivity is now higher in the core, and consequently, it becomes more economically attractive to invest there, despite the higher pollution.

However, in the simulations presented, we observe that the model parameters do not have the same impact on the optimal strategies. In particular, model parameters such as δ_i, a_i^I, and a_i^R fundamentally determine the shape of the optimal strategies, whereas the parameter α_i has a second-level impact on the shape of the optimal strategies. This is a consequence of the fact that pollution appears linearly in the objective function.

5. Conclusions and directions of further research

In this paper, we introduce a spatio-temporal model where time is treated as a continuous variable while space is represented discretely through a network structure. Within this framework, we analyse both green and brown investments under the assumption of transboundary pollution.
By making certain simplifications regarding the dependence of pollution on the objective function, specifically, assuming a linear relationship between pollution levels and the social planner decisions, we derive explicit analytical controls and pollution trajectories. Beyond these mathematical insights, we explore optimal policy implications in the presence of both geographic and economic heterogeneity. In particular, we examine how optimal investment strategies evolve when a new renewable investment option is introduced. We also investigate disparities in pollution decay rates, variations in the diffusion matrix (to account for different pollution flow dynamics), and differences in input productivity, along with their combined effects.
However, several aspects remain unexplored in the current model and deserve further study.
Observe that, since the regulator is internalizing pollution flow and also that production is generating pollution, we can say that the free riding is prevented. It could be interesting to study the strategic perspective, (see Boucekkine et al. (2022), de Frutos and Martín-Herrán (2019a,b) for some investigations in this directions and also Augeraud-Véron et al. (2025), La Torre et al. (2021) for the global and local dimension of policies).
To enhance realism, a future direction of the work could be to add the possibility that investment at any given moment influences future production. This would mean adding another state variable, the capital, whose state equation describes its accumulation. This would add complexity, in particular if we allow the regulator to manage production exchanges between network nodes. This is part of our current projects.
Another promising direction involves establishing a rigorous mathematical connection between continuous-space models and network-based

models (see de Frutos and Martín-Herrán (2019b) for work in similar direction). Such theoretical results could serve as a foundation for approximating continuous spatial dynamics using discrete schemes or vice versa.

CRediT authorship contribution statement

Fausto Gozzi: Writing – original draft, Writing – review & editing; **Marta Leocata:** Writing – original draft, Writing – review & editing; **Giulia Pucci:** Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Fausto Gozzi reports was provided by LUISS University. Fausto Gozzi reports a relationship with LUISS University that includes: employment. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proofs

A.1. Proof of Proposition 2.1

Proof. The term $\frac{((A^I(t)-1)I(t)+(A^R(t)-1))^{1-\gamma}}{1-\gamma}$ in (6) is always either positive (if $\gamma \in (0,1)$) or negative (if $\gamma > 1$). Since by the definition of $\mathcal{A}(p)$, the map $t \to e^{-\rho t}\|f(R_t)\|$ is integrable, it suffices to show that $\int_0^t e^{-\rho t}\langle \omega, P(t)\rangle dt$ is well defined and finite. We have

$$\int_0^\infty e^{-\rho t}\langle \omega, P(t)\rangle dt = \int_0^\infty e^{-\rho t}\langle \omega, \Phi(t,0)p + \int_0^t \Phi(t,s)N(s)ds\rangle dt.$$

Now since $\omega, p \in \mathbb{R}^n$ and $\|\Phi(t,s)\| \le 1$, the first integral term is finite. Moreover for $T > 0$ we get, by Fubini-Tonelli's theorem:

$$\begin{aligned}\int_0^T \left(\int_0^t e^{-\rho t}\langle \omega, \Phi(t,s)N(s)\rangle ds\right)dt &= \int_0^T \left(\int_0^t e^{-\rho s}\langle \omega, e^{-\rho(t-s)}\Phi(t,s)N(s)\rangle ds\right)dt \\ &= \int_0^T e^{-\rho s}\langle \omega, N(s)\int_s^T e^{-\rho(t-s)}\Phi(t,s)dt\rangle ds.\end{aligned}$$

By sending T to $+\infty$ and recalling that $\|\Phi(t,s)\| \le 1$, the integral $\int_s^\infty e^{-\rho(t-s)}\Phi(t,s)dt$ is bounded. Since $\int_0^\infty e^{-\rho s}\|N(s)\|ds < \infty$, the entire expression is finite. □

A.2. Proof of Proposition 3.1

Proof. Define the column sum of the state transition matrix for fixed j, $c_j(t) = \sum_{i=1}^n \Phi_{i,j}(t,s)$, where $\Phi_{i,j}(t,s)$ denotes the (i,j)th entry of $\Phi(t,s)$. By differentiating with respect to t:

$$\begin{aligned}\frac{d}{dt}c_j(t) &= \sum_{i=1}^n \frac{d}{dt}\Phi_{i,j}(t,s) = \sum_{i=1}^n\sum_{l=1}^n \mathcal{L}_{i,l}\Phi_{l,j}(t,s) = \sum_{l=1}^n\sum_{i=1}^n \mathcal{L}_{i,l}\Phi_{l,j}(t,s) \\ &= \sum_{l=1}^n \Phi_{l,j}(t,s)\sum_{i=1}^n (L_{i,l}(t) - \delta_l(t)\mathbb{1}_{\{l=i\}}) = -\sum_{i=1}^n \Phi_{l,j}(t,s)\delta_l(t),\end{aligned}$$

where we used that the sum over columns of L is zero. By calling $\delta_{\min} = \min_{j\in\mathcal{V}}\inf_{s\ge0}\delta_j(s)$ and $\delta_{\max} = \max_{j\in\mathcal{V}}\sup_{s\ge0}\delta_j(s)$ it follows that

$$-\delta_{\max}c_j(t) \le \frac{d}{dt}c_j(t) \le -\delta_{\min}c_j(t)$$

with initial condition $c_j(s) = \sum_{i=1}^n \Phi_{i,j}(s,s) = \sum_{i=1}^n I_{i,j} = 1$. By the comparison principle for ODEs, this implies that $e^{-\delta_{\max}(t-s)} \le c_j(t) \le e^{-\delta_{\min}(t-s)}$.

Now consider the j–th element of the α vector:

$$\begin{aligned}\alpha_j(s) &= \int_s^\infty e^{-\rho(t-s)}[\Phi^\intercal(t,s)\omega]_j dt = \int_s^\infty e^{-\rho(t-s)}\sum_{i=1}^n \Phi_{i,j}(t,s)\omega_i dt. \\ &\le \max_{i\in\mathcal{V}}\omega_i\int_s^\infty e^{-\rho(t-s)}\sum_{i=1}^n \Phi_{i,j}(t,s) = \max_{i\in\mathcal{V}}\omega_i\int_s^\infty e^{-\rho(t-s)}c_j(t)dt \\ &\le \max_{i\in\mathcal{V}}\omega_i\int_s^\infty e^{-(\rho+\delta_{\min})(t-s)}dt = \frac{\max_{i\in\mathcal{V}}\omega_i}{\rho+\delta_{\min}}.\end{aligned}$$

The other bound comes symmetrically. □

A.3. Proof of Proposition 3.2

Proof. Using (7), we can rewrite the second term of the functional (6) in a more convenient way

$$\begin{aligned}\int_0^\infty e^{-\rho t}\langle w, P(t)\rangle dt &= \int_0^\infty e^{-\rho t}\langle \omega, \Phi(t,0)p + \int_0^t \Phi(t,s)N(s)ds\rangle dt \\ &= \langle \omega, \int_0^\infty e^{-\rho t}\Phi(t,0)pdt\rangle + \int_0^\infty e^{-\rho t}\langle \omega, \int_0^t \Phi(t,s)N(s)ds\rangle dt.\end{aligned} \tag{A.1}$$

Note that the first term of the right-hand side is the only one which depends on the initial datum p. By (8), the first term can be rewritten as

$$\langle \omega, \int_0^\infty e^{-\rho t}\Phi(t,0)pdt\rangle = \langle \int_0^\infty e^{-\rho t}\Phi^\intercal(t,0)\omega dt, p\rangle = \langle \alpha(0), p\rangle.$$

The second term in (A.1) can be rewritten by exchanging the integrals as:

$$\begin{aligned}&\int_0^\infty e^{-\rho t}\langle \omega, \int_0^t \Phi(t,s)N(s)ds\rangle dt = \int_0^\infty \left(\int_0^t e^{-\rho t}\langle \omega, \Phi(t,s)N(s)\rangle ds\right)dt = \\ &= \int_0^\infty \left(\int_0^t e^{-\rho s}\langle \omega, e^{-\rho(t-s)}\Phi(t,s)N(s)\rangle ds\right)dt = \int_0^\infty e^{-\rho s}\left\langle \omega, \int_s^\infty e^{-\rho(t-s)}\Phi(t,s)N(s)dt\right\rangle ds \\ &= \int_0^\infty e^{-\rho s}\left\langle \int_s^\infty e^{-\rho(t-s)}\Phi^\intercal(t,s)\omega dt, N(s)\right\rangle ds = \int_0^\infty e^{-\rho s}\langle \alpha(s), N(s)\rangle ds\end{aligned}$$

□

A.4. Proof of Theorem 3.1

Proof. This is a straightforward consequence of Proposition 3.2. Indeed, according to the reformulation presented in (10) the problem is reduced to a static one because the integral in (10) can be optimized pointwise, fixed time $t \in \mathbb{R}$ and fixed $i \in \mathcal{V}$. The objective function can be rewritten as

$$J(p,(I,R)) = -\langle \alpha(0), p\rangle + \sum_{i=1}^{n} \int_0^{+\infty} e^{-\rho t} F_i(I_i(t), R_i(t))dt, \tag{A.2}$$

where F_i is defined in (11). If $(I_i^*(t), R_i^*(t))$ is a maximum of the function F_i that is integrated in time, then $(I^*(t), R^*(t))$ is a maximum for the control problem without any constraint on the state variable and the control. If moreover $(I^*(t), R^*(t))$ belong to $\mathcal{A}(p)$, then it is also optimal for **(P)**. $\square$

A.5. Proof of Theorem 3.2

Proof. As suggested by Theorem 3.1, we look for a control that is admissible and that is optimal for the function F_i, defined in (11). First, we observe that F_i is coercive and concave on $\mathbb{R}^2_+$.
Indeed, if $\gamma \in (0,1)$, the utility term grows sub-linearly and we can estimate

$$F_i(I_i, R_i) \le c_1 \frac{(I_i + R_i)^{1-\gamma}}{1-\gamma} - \alpha_i(t)(I_i - \varepsilon_i(t)R_i) - f_i(R_i),$$

while if $\gamma \in (1, +\infty)$

$$F_i(I_i, R_i) \le -\alpha_i(t)I_i - \varepsilon_i(t)R_i - f_i(R_i).$$

Since F_i is coercive on the set $\mathbb{R}^2_+$, F_i admits a global maximum on $\mathbb{R}^2_+$. It is standard to prove that the function F_i is concave on $\mathbb{R}^2_+$ and it is strictly concave in the inner of $\mathbb{R}^2_+$. To prove the latter, one can calculate the Hessian matrix

$$H_f = \begin{bmatrix} -A\left(a_i^I(t)-1\right)^2 & -A\left(a_i^I(t)-1\right)\left(a_i^R(t)-1\right) \\ -A\left(a_i^I(t)-1\right)\left(a_i^R(t)-1\right) & -A\left(a_i^R(t)-1\right)^2 - f''(R_i(t)) \end{bmatrix}$$

where $A = \gamma((a_i^I(t)-1)I_i + (a_i^R(t)-1)R_i)^{-\gamma-1}$, and by the signs of all principal minors can deduce the strict concavity of the function in the inner of $\mathbb{R}^2_+$.
We look for the maximum between points that are not differentiable and the points satisfying the Karush-Kuhn-Tucker condition.
First, we observe that the points where consumption is null cannot be maximum points.
Assume $\gamma < 1$, and choose a point with consumption null, namely $(0,0)$. Then, we observe that by increasing δ in the direction of I_i, the function F_i increases as well. Indeed

$$F_{i,t}(0,0) = 0, \quad F_{i,t}(\delta, 0) = \frac{\left((a_i^I(t)-1)\delta\right)^{1-\gamma}}{1-\gamma} - \alpha\delta > F_{i,t}(0,0)$$

for δ small enough. A similar argument holds for $\gamma > 1$.
If condition (12) holds, then the critical points of the Lagrangian function belong to the inner $\mathbb{R}^2_+$. Indeed, the system

$$\begin{cases} \left((a_i^I(t)-1)I_i + (a_i^R(t)-1)R_i\right)^{-\gamma}(a_i^I(t)-1) - \alpha_i(t) = 0 \\ \left((a_i^I(t)-1)I_i + (a_i^R(t)-1)R_i\right)^{-\gamma} \cdot (a_i^R(t)-1) - \varepsilon_i(t)\alpha_i(t) - f'(R_i) = 0 \end{cases}$$

admits solution $(I_i, R_i) = \left(\left[\left(\frac{\alpha_i(t)}{a_i^I(t)-1} \right)^{-1/\gamma} - (a_i^R(t)-1)R^* \right] \frac{1}{(a_i^I(t)-1)}, (f_i')^{-1}\left(\alpha_i(t) \left(\frac{a_i^R(t)-1}{a_i^I(t)-1} - \varepsilon_i(t) \right) \right) \right)$ where condition (12) guarantees the positivity of R_i, while condition (13) guarantees the positivity of I_i.
Now we need to prove that (R_i, I_i) belongs to the set of admissible controls. Specifically, the optimal net emission term can be explicitly written as

$$N_i(t) = \alpha_i(t)^{-\frac{1}{\gamma}}(a_i^I(t)-1)^{\frac{1-\gamma}{\gamma}} + \left[\varepsilon_i(t) - \frac{a_i^R(t)-1}{a_i^I(t)-1} \right] (f_i')^{-1}\left(\alpha_i(t) \left(\frac{a_i^R(t)-1}{a_i^I(t)-1} - \varepsilon_i(t) \right) \right).$$

Define $x(t) := \alpha_i(t)\left(\frac{a_i^R(t)-1}{a_i^I(t)-1} - \varepsilon_i(t) \right)$. Condition (12) gives $x(t) > 0$ and by the inverse function rule, since $f_i'' \ge \epsilon > 0$, $(f_i')^{-1}$ is Lipschitz continuous. Together with assumption 2 (ii), this allows us to bound $\|N(t)\| \le Ce^{gt}$, and by 2 (iii) we conclude.
If (18) holds, the inner of $\mathbb{R}^2_+$ does not contain any critical points. In particular, the solution of the lagrangian system

$$\begin{cases} \left((a_i^I(t)-1)I_i + (a_i^R(t)-1)R_i\right)^{-\gamma}(a_i^I(t)-1) - \alpha_i(t) = 0 \\ \left((a_i^I(t)-1)I_i + (a_i^R(t)-1)R_i\right)^{-\gamma} \cdot (a_i^R(t)-1) - \varepsilon_i(t)\alpha_i(t) - f'(R_i) = -\mu_i^R, \quad \mu_i^R \ge 0 \end{cases}$$

is $(I_i, R_i) = \left(\left[\left(\frac{\alpha_i(t)}{a_i^I(t)-1} \right)^{-1/\gamma} \right] \frac{1}{(a_i^I(t)-1)}, 0 \right)$ with $\mu_i^R = \alpha_i(t)\left(\varepsilon_i(t) - \frac{(a_i^R(t)-1)}{(a_i^I(t)-1)} \right) + f_i'(0)$. Condition (18) guarantees that the lagrangian multiplication μ_i^R is positive and Assumption 2 ensures that $(I_i, R_i) \in \mathcal{A}(p)$. To conclude, when condition (12) is verified and condition (13) is not, namely

$$\left(\frac{\alpha_i(t)}{a_i^I(t)-1} \right)^{-1/\gamma} > (a_i^R(t)-1)(f_i')^{-1}\left(\alpha_i(t) \left(\frac{a_i^R(t)-1}{a_i^I(t)-1} - \varepsilon_i(t) \right) \right)$$

the inner of $\mathbb{R}^2_+$ does not contain any critical points and the global maximum belongs to the boundary, and since condition (12) is assumed to hold, the maximum is solution of the lagrangian system

$$\begin{cases} \left((a_i^R(t)-1)R_i\right)^{-\gamma}(a_i^I(t)-1) - \alpha_i(t) = -\mu_i^I \\ \left((a_i^R(t)-1)R_i\right)^{-\gamma} \cdot (a_i^R(t)-1) - \varepsilon_i(t)\alpha_i(t) - f'(R_i) = 0 \end{cases}$$

However, in this case, the optimal controls are not explicitly derived. Indeed, they are solutions of the non-linear system

$$\begin{cases} R_i \ s.t. \ \left((a_i^R(t)-1)R_i\right)^{-\gamma}\cdot(a_i^R(t)-1)-\varepsilon_i(t)\alpha_i(t)=f'(R_i). \\ I_i=0. \end{cases}$$

Notice that, since the function $\Phi(x)=\left((a_i^R(t)-1)x\right)^{-\gamma}\cdot(a_i^R(t)-1)-\varepsilon_i(t)\alpha_i(t)-f'(x)$, is monotonous strictly decreasing and $\lim_{x\to 0^+}\Phi(x)=\infty$, $\lim_{x\to+\infty}\Phi(x)=-\infty$, we conclude that the $\Phi(x)$ as only one zero and therefore R_i^* is unique. The admissibility of the optimal point follows from Assumption 2 (ii–iii), together with the facts that $\alpha_i(t)$ is uniformly bounded, and $f_i'\geq\epsilon>0$. □

A.6. Proof of Theorem 3.3

Proof. By following the same argument proposed in Theorem 3.2, it is possible to prove that by choosing f_i as linear F_i is coercive on the set $\mathbb{R}_+^2$, and therefore F_i admits a global maximum on $\mathbb{R}_+^2$. We look for the maximum between points that are not differentiable and the points satisfying the Karush-Kuhn-Tucker condition. By adapting the argument presented in Theorem 3.2, one can exclude points where the function F_i is not differentiable. In particular, one can observe that points where consumption is null cannot be maximum points.
Therefore, the global maximum is a solution of the KKT system.
If condition (24) holds, the critical points of the Lagragian belongs to the inner of $\mathbb{R}_+^2$. In particular, the system

$$\begin{cases} \left((a_i^I(t)-1)I_i+(a_i^R(t)-I_i)R_i\right)^{-\gamma}(a_i^I(t)-1)-\alpha_i(t)=0 \\ \left((a_i^I(t)-1)I_i+(a_i^R(t)-I_i)R_i\right)^{-\gamma}\cdot(a_i^R(t)-1)-\varepsilon_i(t)\alpha_i(t)-\lambda_i=0 \end{cases}$$

admits as solutions the points of the line

$$(a_i^I(t)-1)I+(a_i^R(t)-1)R_i=\left(\alpha_i(t)\cdot(a_i^I(t)-1)^{-1}\right)^{-\frac{1}{\gamma}}.$$

If condition (12) holds, the lagrangian system

$$\begin{cases} \left((a_i^I(t)-1)I_i+(a_i^R(t)-I_i)R_i\right)^{-\gamma}(a_i^I(t)-1)-\alpha_i(t)=-\mu_i^I, \quad \mu_i^I\geq 0 \\ \left((a_i^I(t)-1)I_i+(a_i^R(t)-I_i)R_i\right)^{-\gamma}\cdot(a_i^R(t)-1)-\varepsilon_i(t)\alpha_i(t)-\lambda_i=0 \end{cases}$$

admits as solutions the point $(I_i,R_i)=\left(0,\frac{1}{(a_i^R(t)-1)}\left(\frac{\lambda_i+\alpha_i(t)\varepsilon_i(t)}{(a_i^R(t)-1)}\right)^{-\frac{1}{\gamma}}.\right)$ To conclude, if condition (18) holds, the lagrangian system

$$\begin{cases} \left((a_i^I(t)-1)I_i+(a_i^R(t)-I_i)R_i\right)^{-\gamma}(a_i^I(t)-1)-\alpha_i(t)=0 \\ \left((a_i^I(t)-1)I_i+(a_i^R(t)-I_i)R_i\right)^{-\gamma}\cdot(a_i^R(t)-1)-\varepsilon_i(t)\alpha_i(t)-\lambda_i=-\mu_i^R, \quad \mu_i^R\geq 0 \end{cases}$$

admits as solutions the point $(I_i,R_i)=\left(\frac{1}{(a_i^I(t)-1)}\left(\frac{\alpha_i(t)}{(a_i^I(t)-1)}\right)^{-\frac{1}{\gamma}},0\right)$ □

A.7. Proof of Theorem 3.4

Proof. Notice that in this case, the expressions of the optimal controls are time-independent, too. Let θ_0 be the spectral bound of $\mathcal{L}$. Since $\delta>0$, the operator $\mathcal{L}$ is strictly dissipative, hence $\theta_0<0$. Let us write $\mathcal{L}=\mathcal{L}_0-\theta_0$ where $\mathcal{L}_0:=\mathcal{L}+\theta_0$, and note that $\mathcal{L}_0$ is dissipate by definition, hence $e^{s\mathcal{L}_0}$ is a contraction. Then, we can rewrite:

$$P^*(t):=e^{t\mathcal{L}_0}e^{-\theta_0 t}p+\int_0^t e^{-\theta_0(t-s)}e^{(t-s)\mathcal{L}_0}N^*ds=e^{t\mathcal{L}_0}e^{-\theta_0 t}p+\int_0^t e^{-\theta_0 s}e^{s\mathcal{L}_0}N^*ds, \quad t\geq 0,$$

and take the limit above when $t\to\infty$. Since $e^{s\mathcal{L}_0}$ is a contraction, the first one on the right-hand side converges to 0, whereas the second one converges to $P_\infty^*:=\int_0^\infty e^{-\theta_0 s}e^{s\mathcal{L}_0}N^*ds$, $t\geq 0$. And we can conclude by expressing the limit P_∞^* as $P_\infty^*=(\theta_0-\mathcal{L}_0)^{-1}N^*$. I.e. P_∞^* is the solution to $(\theta_0-\mathcal{L}_0)P=N^*$ or, equivalently, to $\mathcal{L}P+N^*=0$. □

A.8. Proof of Theorem 3.5

Proof. As suggested by Theorem 3.1, we look for a control that is admissible and that is optimal for the function F, defined in (11), where the control R is zero, namely

$$F(I)=\left\langle\frac{((A^I(t)-1)I)^{1-\gamma}}{1-\gamma},\mathbf{1}\right\rangle-\langle\alpha(t),I\rangle=\sum_{i=1}^n\frac{((a_i^I(t)-1)I_i)^{1-\gamma}}{1-\gamma}-\alpha_i(t)I_i=\sum_{i=1}^n F_i(I_i).$$

First, we need to check that $I^*\in\mathcal{A}(p)$. We have

$$(A^I(t)-\mathbb{1}))I^*(t)=\left(\frac{A^I(t)-\mathbb{1}}{\alpha(t)}\right)^{\frac{1}{\gamma}}\geq 0, \quad \forall t\in\mathbb{R}_+.$$

Moreover, considering $N^*(t)$ as in (3.5) and Assumption 3, we get the existence of some constant $C_0>0$ such that $0\leq N_i^*(t)\leq C_0e^{gt}$, $\forall t\in\mathbb{R}_+$, $\forall i\in\mathcal{V}$.
We conclude that $I^*\in\mathcal{A}(p)$.
Concerning optimality, as stressed in Theorem 3.1, the integrals in (10) can be optimized pointwisely, indeed fix $t\in\mathbb{R}_+$, $i\in\mathcal{V}$.
By strict concavity of F_i with respect to $I_i(t)$, the unique maximum point can be found just by first-order optimality conditions: $((a_i^I(t)-1)I_i(t))^{-\gamma}(a_i^I(t)-1)-\alpha_i(t)=0$.

The claim on the optimal control then follows by solving the above system, and all the remaining claims immediately follow from straightforward computations. □

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