


On the Performance of Mildly Greedy Players in k -Coloring Games

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Abstract

We study the performance of mildly greedy players in k -coloring games, a relevant subclass of anti-coordination games. A mildly greedy player is a selfish agent who is willing to deviate from a certain strategy profile only if her payoff improves by a factor of more than ϵ , for some given $\epsilon \geq 0$. In presence of mildly greedy players, stability is captured by the concept of $(1 + \epsilon)$ -approximate Nash equilibrium. In this paper, we first show that, for any k -coloring game, the $(1 + \epsilon)$ -approximate price of anarchy, i.e., the price of anarchy of $(1 + \epsilon)$ -approximate pure Nash equilibria, is at least $\frac{k-1}{(k-1)\epsilon+k}$, and that this bound is tight for any $\epsilon \geq 0$. Then, we evaluate the approximation ratio of the solutions achieved after a $(1 + \epsilon)$ -approximate one-round walk starting from any initial strategy profile, where a $(1 + \epsilon)$ -approximate one-round walk is a sequence of $(1 + \epsilon)$ -approximate best-responses, one for each player. We provide a lower bound of $\min \left\{ \frac{k-2}{k}, \frac{k-1}{(k-1)\epsilon+k} \right\}$ on this ratio, for any $\epsilon \geq 0$ and $k \geq 5$; for the cases of $k = 3$ and $k = 4$, we give finer bounds depending on ϵ . Our work generalizes the results known for cut games, the special case of k -coloring games restricted to $k = 2$.

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1 Introduction

In this paper, we study the following k -coloring game. We are given an undirected weighted graph $G = (V, E)$ with a weight function $c : E \mapsto \mathbb{R}^+$ that assigns to each edge $\{i, j\} \in E$ a non-negative real number. Vertices represent autonomous, selfish agents² while edges capture mutual idiosyncrasies or conflicts. In addition, a set of $k \geq 2$ colors, denoting the options or strategies available to all agents, is given. Each agent has to select a color with the goal of maximizing her *utility* (or *payoff*), defined as the sum of the weights of the edges connecting her vertex to agents who have chosen a different color. Intuitively, agents benefit from distinguishing themselves from their neighbors in the graph.

This game forms one of the basic payoff structures in Game Theory and models well a variety of real-world scenarios, specifically all those situations in which agents need to mutually anti-coordinate their strategies in order to maximize their profit or minimize their damage. Examples include companies deciding which products to develop to avoid direct competition, airlines selecting flight paths to reduce congestion, antennas selecting frequencies to reduce interference, coalition formation processes [21, 29]. For a more detailed application, consider engineers working at a software company, each aiming to acquire technical skills that would boost their individual career prospects. Ideally, to stand out, they would prefer a rare but valuable skill rather than one that is common among colleagues. We can model this as a k -coloring game on a graph $G = (V, E)$, where each vertex $v_i \in V$ represents an engineer and an edge $(v_i, v_j) \in E$ indicates employees i and j collaborate or compete within a team. The k available colors correspond to distinct skill sets (e.g., C++, Python, Rust, Go, Java). By choosing a color (skill) that differs from their neighbors', engineers strategically enhance their uniqueness in the collaboration network.

We are interested in evaluating the performance of self-emerging solutions in k -coloring games with respect to the *utilitarian social welfare*, which sums up the utilities of all players. The utilitarian social welfare is the standard *social function* used in the literature to measure the quality of a strategy profile, and, in the context of k -coloring games, coincides, up to a multiplicative factor of 2, with the total weight of the edges whose endpoints have different colors. To compare the quality of outcomes, emerging from the strategic behavior of selfish players, against the ones that could potentially be enforced by a dictatorial authority, Koutsoupias and Papadimitriou [28] introduced the well-known concept of *Price of Anarchy* (*PoA*). The PoA measures the *worst-case* ratio between the social value of a *Nash equilibrium* (in short NE), i.e., a strategy profile in which no player can improve her payoff by unilaterally changing her strategy, and the social value of a *social optimum*, that is a strategy profile optimizing the social function. As subsequent results have shown that even determining the existence of a NE in several games of interest is a computationally demanding task [1, 22, 29], a large amount of work has been devoted to analyzing how well less demanding solution concepts perform.

A first approach is to relax selfishness by allowing the presence of *mildly greedy players*, i.e., players who are willing to accept any strategy profile in which no substantial improvement can be obtained through a unilateral deviation. This naturally leads to the notion of approximate NE [6, 12, 14, 19]. Formally, given a value $\epsilon \geq 0$, a $(1 + \epsilon)$ -*approximate NE* is a strategy profile in which no player can improve her payoff by a multiplicative factor of more than $1 + \epsilon$ by unilaterally deviating to a different strategy. The $(1 + \epsilon)$ -*approximate Price of Anarchy* ($(1 + \epsilon)$ -*PoA*) is the generalization of the PoA to $(1 + \epsilon)$ -approximate NEa. Clearly, a NE is a $(1 + \epsilon)$ -approximate NE with $\epsilon = 0$ and the PoA is the $(1 + \epsilon)$ -PoA with $\epsilon = 0$.

² Throughout the paper, we use the terms agent and player interchangeably.

Other less demanding solution concepts can be derived from *best-response dynamics*, which are iterative processes in which, starting from a given strategy profile, players are processed sequentially and, at each step, one of them is allowed to change her current strategy by *best-responding* to the strategies played by the others. Clearly, when best-responses can be computed in polynomial time, a best-response dynamic with a polynomial number of steps can be efficiently computed. Indeed, solutions generated by sequences of best-responses of bounded length can exhibit appealing properties, and hence have been widely investigated. In particular, special cases of these dynamics are *covering walks*, *k-covering walks*, *one-round walks*, *k-round walks* and *random one-round walks* [30]. Note that one-round walks constitute the simplest of these dynamics and thus have been the target of several recent works [8, 9, 20, 30]. It is also worth noting that best-response dynamics can also accommodate mildly greedy players, and this translates into the corresponding definition of $(1 + \epsilon)$ -*approximate best-response dynamics*, that is, dynamics in which each player changes her strategy only when this improves her utility by a multiplicative factor of at least $1 + \epsilon$. This concept has attracted considerable interest, as studies have demonstrated that (approximate) best-response dynamics play a key role in identifying both (approximate) NEa and high-quality solutions in a wide range of (non-cooperative) games [30]. For example, in the classical 2-coloring game, a one-round walk may yield a solution whose social welfare is arbitrarily far from the optimum. Instead, if one admits mildly greedy players, it can be shown that the approximation ratio of the solutions achieved after a $(1 + \epsilon)$ -approximate one-round walk starting from any initial strategy profile is equal to $1/(2 + \epsilon)$ for any $\epsilon \geq 1$ and equal to $\frac{2\epsilon}{(\epsilon+1)(\epsilon+2)}$ for any $0 \leq \epsilon < 1$ [7, 20].

In this paper, we focus on the performance of mildly greedy players in k -coloring games. Using the primal-dual method of Bilò [4], we are able to lower bound the $(1 + \epsilon)$ -PoA of any instance of the k -coloring game by $\frac{k-1}{(k-1)\epsilon+k}$. Moreover, we construct specific instances of the game for which such a bound is tight for any possible value of k and ϵ . Next, we shift our attention to *approximate one-round walks*, i.e., approximate best-response dynamics in which each player plays exactly once. In particular, we study the approximation ratio of the solutions achieved after a $(1 + \epsilon)$ -approximate one-round walk starting from any initial strategy profile. This notion can be seen as an adaptation of the $(1 + \epsilon)$ -PoA to $(1 + \epsilon)$ -approximate one-round walks and is defined as the worst-case ratio between the social value of a strategy profile realized at the end of the walk and the social optimum. For this ratio, we provide a lower bound of $\min \left\{ \frac{k-2}{k}, \frac{k-1}{(k-1)\epsilon+k} \right\}$, for any $\epsilon \geq 0$ and $k \geq 5$, while for $k = 3$ and $k = 4$, we give finer bounds depending on ϵ . Our results generalize those known for *cut games* – the special case of 2-coloring games – for which the $(1 + \epsilon)$ -PoA and the approximation ratio of the solutions achieved after a $(1 + \epsilon)$ -approximate one-round walk starting from any initial strategy profile have been shown to be equal to $\frac{1}{2+\epsilon}$ and $\min \left\{ \frac{1}{2+\epsilon}, \frac{2\epsilon}{(1+\epsilon)(2+\epsilon)} \right\}$, respectively [7].

2 Related Work

k -coloring games have been first investigated in [27, 29], where it is shown that a NE always exists and can be computed in polynomial time if the graph is unweighted. In weighted graphs, instead, a NE always exists, but the problem of computing one is PLS-complete even for $k = 2$ [34]. Note that 2-coloring games are exactly equivalent to max-cut games, where players partition vertices into sets to maximize the number of edges between sets. If the maximum degree of the graph is at most 3, then a NE can be computed in polynomial time for 2-coloring games [32]. A related investigation for the same class of games is presented in [3, 13],

where the authors give an algorithm that, for any $\epsilon > 0$, computes a $(3 + \epsilon)$ -approximate NE in time polynomial in $\frac{1}{\epsilon}$ and the size of the graph. A better algorithm, returning a 2.7371-approximate NE has been recently designed in [15]. All these results are obtained by suitably exploiting the potential function method. The performance of NEa in k -coloring games has been addressed in [23], while the authors of [18] consider NEa where players also have an extra profit depending on the chosen color. Finally, results on the existence of strong NEa, i.e., solutions resistant to coalitional deviations, for k -coloring games (also referred as *max- k -cut games*) have been given in [2, 16, 24–26].

The more general version of k -coloring games defined on directed graphs does not admit any potential function, and the problem of understanding whether they have a NE is NP-complete for any fixed $k \geq 2$, even in the unweighted case [29]. These games have been extensively studied in [17, 21], in which the authors provide (i) a randomized algorithm with polynomial expected running time that, given any constant $k \geq 2$, computes a constant-approximate NE when the minimum outgoing degree of the graph is at least as big as the sum of the logarithm of the maximum outgoing degree and that of the maximum ingoing degree; (ii) a deterministic polynomial time algorithm computing a $(1 + \epsilon)$ -approximate NE, for any $\epsilon > 0$, by using $O(\frac{\log(n)}{\epsilon})$ colors; (iii) an experimental evaluation of such algorithms, which are compared with an heuristic based on best-response dynamics.

Several methods have been proposed to characterize the quality of self-emerging solutions in non-cooperative systems [4, 31, 33]. Among them, the primal-dual framework in [4] has been successfully applied in a variety of situations. For example, the authors of [5] used this method to find a lower bounding instance for the sequential PoA of cut games. In [9, 11] the primal-dual method has been exploited to design efficient tax functions for weighted congestion games with polynomial latency functions. Many other successful applications of the method can be found in [4, 10] and references therein.

3 Preliminaries

In this section, we establish background information and terminology used throughout the paper. We mostly follow definitions and notation given in [7].

Given a positive integer r , we denote by $[r]$ the set $\{1, \dots, r\}$. Let $G = (V, E)$ be an undirected weighted graph, with $|V| = n$ and let c be a weight function $c : E \mapsto \mathbb{R}^+$ that assigns to each edge $\{i, j\} \in E$ a non-negative real number. Throughout the paper, we use the standard notation $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of a graph G , respectively, when they are not explicitly defined. Moreover, we assume that all vertices in $V(G)$ are numbered from 1 to n , so that $V(G) = [n]$. We shall drop the argument G from the notation when the reference graph G is clear from the context.

An instance of the k -coloring game $\mathcal{K}(G)$ can be uniquely associated to a graph G and a set of colors $K = [k]$ as follows. Each vertex $i \in V$ is a player whose set of strategies is exactly the set of k -colors K . A *strategy profile* $\sigma = (\sigma(1), \dots, \sigma(n))$, also called a *state*, is a vector such that, for each $i \in V$, $\sigma(i) \in K$ denotes the color chosen by player i . Given a strategy profile σ , the notation (σ_{-i}, τ) denotes the strategy profile in which player i picks $\tau \in K$ as strategy in place of σ_i , while the strategy of the other players remains unchanged. Denote with $\text{Adj}(i) := \{j \in V : \{i, j\} \in E\}$ the set of neighboring vertices of vertex i , for each $i \in V$, and with $c_{ij} := c(\{i, j\})$ the weight of edge $\{i, j\} \in E$. The utility of player i in strategy profile σ is

$$U_i(\sigma) = \sum_{j \in \text{Adj}(i): \sigma(i) \neq \sigma(j)} c_{ij}.$$

In other words, the utility of player i is equal to the sum of the weights of the edges $\{i, j\} \in E$ incident to i , for which, in σ , player j chooses a color different from the one selected by player i . As a measure of the quality of a strategy profile σ , half of the sum of the players' utilities is a commonly used metric. Formally, we consider the following social function

$$\text{COL}(\sigma) = \sum_{\{i,j\} \in E: \sigma(i) \neq \sigma(j)} c_{ij},$$

which sums the weights of the edges whose endpoints have different colors. It is easy to see that $\text{COL}(\sigma) = \frac{1}{2} \sum_{i \in V} U_i(\sigma)$ for each strategy profile σ , so that COL can be seen as an alternative, and more intuitive, definition for the utilitarian social welfare.

Fix a strategy profile σ and a value $\epsilon \geq 0$. A $(1 + \epsilon)$ -approximate improving deviation for player i in σ is any strategy $\tau \in K$ such that $U_i((\sigma_{-i}, \tau)) > (1 + \epsilon)U_i(\sigma)$. A $(1 + \epsilon)$ -approximate NE is a strategy profile σ such that, for each $i \in N$ and $\tau \in K$, we have $U_i((\sigma_{-i}, \tau_i)) \leq (1 + \epsilon)U_i(\sigma)$, that is, no player has a $(1 + \epsilon)$ -approximate improving deviation in σ . A $(1 + \epsilon)$ -approximate best-response dynamics is a sequence of strategy profiles in which, at each step, a mildly greedy player is allowed to make the following type of choice: if she does not possess a $(1 + \epsilon)$ -approximate improving deviation, then she confirms her current choice; otherwise, she changes her strategy in favor of the strategy giving her the largest utility. A $(1 + \epsilon)$ -approximate one-round walk is a $(1 + \epsilon)$ -approximate best-response dynamics in which each player is allowed to change strategy, in turn, exactly once. Formally, a $(1 + \epsilon)$ -approximate one-round walk W is a sequence of $n + 1$ strategy profiles $W = (\sigma^{(0)}, \sigma^{(1)}, \dots, \sigma^{(n)})$ such that $\sigma^{(0)}$ and $\sigma^{(n)}$ are, respectively, the starting and final states of the walk and, for each $i \in V$, state $\sigma^{(i)}$ is obtained from $\sigma^{(i-1)}$ as a result of the choice of player i . We shall denote with $f(W)$ the final state of a $(1 + \epsilon)$ -approximate one-round walk W .

For a k -coloring game $\mathcal{K}(G)$, let $\text{NE}_\epsilon(\mathcal{K}(G))$ be its set of $(1 + \epsilon)$ -approximate NEa and let σ^* be the strategy profile maximizing the social function COL. Then, the $(1 + \epsilon)$ -approximate PoA of $\mathcal{K}(G)$ is defined as:

$$\text{PoA}_\epsilon(\mathcal{K}(G)) = \min_{\sigma \in \text{NE}_\epsilon(\mathcal{K}(G))} \frac{\text{COL}(\sigma)}{\text{COL}(\sigma^*)},$$

which, by definition, belongs to the interval $[0, 1]$. This notion can be generalized to the entire class of k -coloring games, denoted as \mathcal{K} , by defining the $(1 + \epsilon)$ -approximate PoA of class \mathcal{K} as:

$$\text{PoA}_\epsilon(\mathcal{K}) = \inf_{\mathcal{K}(G) \in \mathcal{K}} \text{PoA}_\epsilon(\mathcal{K}(G)).$$

Let $\text{ORW}_\epsilon(\mathcal{K}(G))$ be the set of $(1 + \epsilon)$ -approximate one-round walks starting from any initial strategy profile of a k -coloring game $\mathcal{K}(G)$. The approximation ratio of the solutions achieved after a $(1 + \epsilon)$ -approximate one-round walk starting from any initial strategy profile of $\mathcal{K}(G)$ is defined as:

$$\text{Apx}_\epsilon(\mathcal{K}(G)) = \min_{W \in \text{ORW}_\epsilon(\mathcal{K}(G))} \frac{\text{COL}(f(W))}{\text{COL}(\sigma^*)}.$$

As above, this notion can be generalized to the entire class \mathcal{K} as:

$$\text{Apx}_\epsilon(\mathcal{K}) = \inf_{\mathcal{K}(G) \in \mathcal{K}} \text{Apx}_\epsilon(\mathcal{K}(G)).$$

4 Primal-Dual Scheme for the Approximate PoA

In this section, we provide the primal and dual formulations of the k -coloring game following the approach of Bilò et al. [4]. Note in fact that, since such game is a weighted congestion game, the primal-dual framework of [4] can be naturally applied to it. As in any primal-dual method, the key challenge is to find the correct formulation that better fits the method. Although the same framework has been applied to the cut-game (i.e., the case with $k = 2$) [7], the generalization to the k -coloring game with $k \geq 3$ requires a fundamentally different model. In fact, it is no longer sufficient to know that two adjacent agents select different colors; the decision variables must also record the choice made by each agent. This results in a formulation whose dual counterpart poses more challenges due to the presence of an higher number of primal constraints, defined on any combination of vertices and colors, as discussed in what follows.

Given a k -coloring game $\mathcal{K}(G)$, let us denote with σ a generic $(1 + \epsilon)$ -approximate NE of $\mathcal{K}(G)$, and with σ^* the social optimum. As per the general framework, we are going to consider σ and σ^* as fixed constants, while, for each $\{i, j\} \in E(G)$, the values c_{ij} defining the edge weights are variables that must suitably be chosen so that: (i) σ is indeed a $(1 + \epsilon)$ -approximate NE of $\mathcal{K}(G)$, and (ii) the value of the social optimum σ^* is equal to 1. These two constraints together allow to formulate the problem of computing the worst-case ratio $\frac{\text{COL}(\sigma)}{\text{COL}(\sigma^*)}$ as the minimization of the social value $\text{COL}(\sigma)$, since (ii) normalizes the value of the social optimum to 1.

Given a strategy profile σ , we introduce variables α_{is}^σ for indices $i \in V$ and $s \in K$ so that

$$\alpha_{is}^\sigma = \begin{cases} 1, & \text{if } \sigma(i) = s \\ 0, & \text{otherwise.} \end{cases}$$

That is, α_{is}^σ indicates whether player i picks color s in σ or not. Note that the case in which $k = 2$ is easier to model since one can exploit binary variables and their complements to describe if a pair of agents picks different colors.

The primal formulation, that we denote by $LP(\sigma, \sigma^*)$, reads:

$$\min \sum_{\{i,j\} \in E} c_{ij}(1 - \alpha_{j\sigma(i)}^\sigma) \quad (1)$$

$$\text{s.t. } (1 + \epsilon) \sum_{j \in \text{Adj}(i)} c_{ij}(1 - \alpha_{j\sigma(i)}^\sigma) - \sum_{j \in \text{Adj}(i)} c_{ij}(1 - \alpha_{js}^\sigma) \geq 0 \quad \forall i \in V, s \in K, s \neq \sigma(i) \quad (2)$$

$$\sum_{\{i,j\} \in E} c_{ij}(1 - \alpha_{j\sigma^*(i)}^{\sigma^*}) = 1 \quad (3)$$

$$c_{ij} \geq 0 \quad \forall \{i, j\} \in E \quad (4)$$

The objective function (1) encodes the cost of the social function under strategy σ , for which the cost of the $\{i, j\}$ is considered when the color assigned by σ to j is different from that assigned to i (this is encoded by the negation of $\alpha_{j\sigma(i)}^\sigma$). Constraints (2) ensure that no player i has an $(1 + \epsilon)$ -approximate improving deviation in σ (i.e. that σ is a $(1 + \epsilon)$ -approximate NE). Equality (3) normalizes to 1 the cost of the social optimum σ^* .

4.1 Dual Formulation

To obtain the dual formulation $DLP(\sigma, \sigma^*)$ of $LP(\sigma, \sigma^*)$, we assign to each constraint (2) a non-negative, dual variable y_{is} , with $i \in V$, $s \in K$, $s \neq \sigma(i)$, and a variable γ to the equality constraint (3). The dual problem follows:

$$\begin{aligned}
& \max \quad \gamma & (5) \\
\text{s.t.} \quad & (1 + \epsilon) \left[\sum_{s \in K \setminus \{\sigma(i)\}} (y_{is}(1 - \alpha_{j\sigma(i)}^\sigma)) + \sum_{s \in K \setminus \{\sigma(j)\}} (y_{js}(1 - \alpha_{i\sigma(j)}^\sigma)) \right] \\
& - \left[\sum_{s \in K \setminus \{\sigma(i)\}} (y_{is}(1 - \alpha_{js}^\sigma)) + \sum_{s \in K \setminus \{\sigma(j)\}} (y_{js}(1 - \alpha_{is}^\sigma)) \right] \\
& + \gamma(1 - \alpha_{j\sigma^*(i)}^\sigma) \leq 1 - \alpha_{j\sigma(i)}^\sigma \quad \forall \{i, j\} \in E & (6) \\
& y_{is} \geq 0 \quad \forall i \in V, s \in K, s \neq \sigma(i) & (7)
\end{aligned}$$

By the Weak Duality Theorem, any feasible solution of this dual problem provides a lower bound on the optimal solution of $LP(\sigma, \sigma^*)$, i.e. a lower bound on the ratio $\frac{\text{COL}(\sigma)}{\text{COL}(\sigma^*)}$. Note that, since we have not made assumptions about the game $\mathcal{K}(G)$, if the provided dual solution is independent on the particular choice of σ and σ^* , we obtain a lower bound on the ratio $\frac{\text{COL}(\sigma)}{\text{COL}(\sigma^*)}$ for any possible pair of profiles σ and σ^* in any possible k -coloring game $\mathcal{K}(G)$, that is, we provide a lower bound on the $(1 + \epsilon)$ -approximate PoA of k -coloring games.

4.2 A Lower Bound

In this section, we provide a feasible solution to $DLP(\sigma, \sigma^*)$, thus yielding a lower bound on the $(1 + \epsilon)$ -approximate PoA of k -coloring games.

► **Theorem 1.** *For any k -coloring game $\mathcal{K}(G)$ and value $\epsilon \geq 0$, $\text{PoA}_\epsilon(\mathcal{K}(G)) \geq \frac{k-1}{k(1+\epsilon)-\epsilon}$.*

Proof. It suffices to show that the solution in which $\gamma = \frac{k-1}{(k-1)\epsilon+k}$ and $y_{is} = \frac{1}{2[(k-1)\epsilon+k]}$, $\forall i \in V, s \in K$ is feasible for $DLP(\sigma, \sigma^*)$. Plugging such values in the dual constraints (6) leads, for each $\{i, j\} \in E$, to

$$\begin{aligned}
(1 + \epsilon) & \left[\frac{k-1}{2[(k-1)\epsilon+k]} - \sum_{s \in K \setminus \{\sigma(i)\}} \frac{\alpha_{j\sigma(i)}^\sigma}{2[(k-1)\epsilon+k]} + \frac{k-1}{2[(k-1)\epsilon+k]} - \sum_{s \in K \setminus \{\sigma(j)\}} \frac{\alpha_{i\sigma(j)}^\sigma}{2[(k-1)\epsilon+k]} \right] \\
& - \left[\frac{k-1}{2[(k-1)\epsilon+k]} - \sum_{s \in K \setminus \{\sigma(i)\}} \frac{\alpha_{js}^\sigma}{2[(k-1)\epsilon+k]} + \frac{k-1}{2[(k-1)\epsilon+k]} - \sum_{s \in K \setminus \{\sigma(j)\}} \frac{\alpha_{is}^\sigma}{2[(k-1)\epsilon+k]} \right] \\
& + \frac{k-1}{(k-1)\epsilon+k} \leq 1 - \alpha_{i\sigma(j)}^\sigma & (8)
\end{aligned}$$

where we considered $\alpha_{j\sigma^*(i)}^\sigma = 0$, since it also implies the validity of the case $\alpha_{j\sigma^*(i)}^\sigma = 1$. Note that $\alpha_{j\sigma(i)}^\sigma = 1 \implies (\alpha_{i\sigma(j)}^\sigma = 1 \wedge \alpha_{js}^\sigma = \alpha_{is}^\sigma = 0)$ and $\alpha_{j\sigma(i)}^\sigma = 0 \implies (\alpha_{i\sigma(j)}^\sigma = 0 \wedge \alpha_{js'}^\sigma = \alpha_{i\bar{s}}^\sigma = 1)$ for some $s', \bar{s} \in K$, $s' \neq \sigma(i)$ and $\bar{s} \neq \sigma(j)$. Consider $\alpha_{j\sigma(i)}^\sigma = 1$. Inequality (8) becomes

$$-\frac{k-1}{(k-1)\epsilon+k} + \frac{k-1}{(k-1)\epsilon+k} \leq 0$$

which is true.

On the other hand, when $\alpha_{j\sigma(i)}^\sigma = \alpha_{i\sigma(j)}^\sigma = 0$, by the definition of α_{is}^σ , there exists a color \bar{s} that the strategy profile σ assigns to vertex i , with $\bar{s} \neq \sigma(j)$. The same holds for some color $s' \neq \sigma(i)$ that σ assigns to j . Feeding Inequality (8) with such values, one obtains

$$(1 + \epsilon) \left(\frac{k-1}{(k-1)\epsilon+k} \right) - \frac{k-2}{(k-1)\epsilon+k} + \frac{k-1}{(k-1)\epsilon+k} \leq 1$$

which is equivalent to

$$\frac{(k-1)\epsilon + k}{(k-1)\epsilon + k} \leq 1$$

which again always hold. ◀

4.3 A Matching Upper Bound

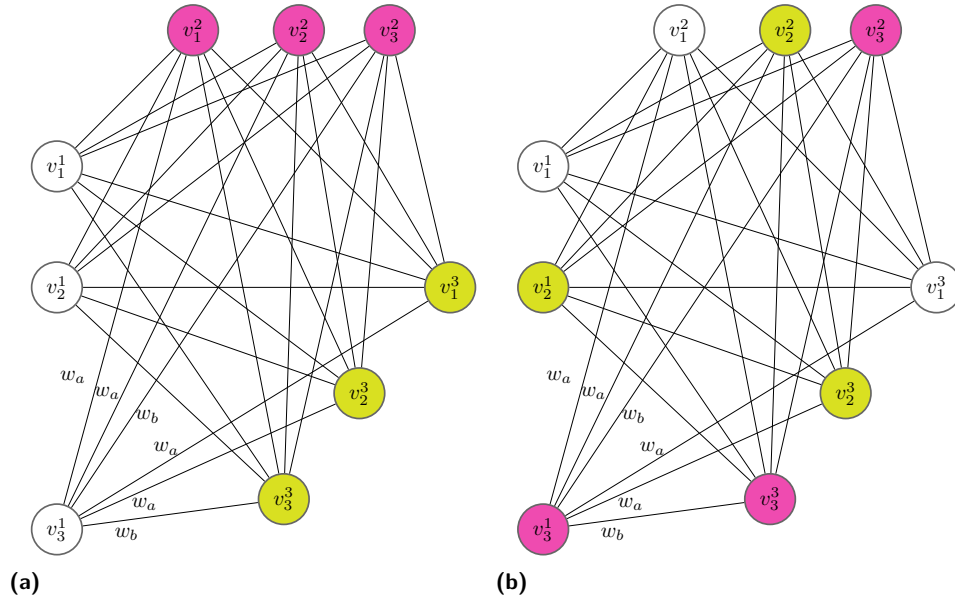
In this section, we provide a family of instances of the k -coloring game for which the lower bound $\frac{k-1}{(k-1)\epsilon+k}$ given by Theorem 1 is tight for any possible values of k and ϵ .

► **Theorem 2.** For any $k \geq 2$, there exists an instance of the k -coloring game $\mathcal{K}(G)$ in which $PoA_\epsilon(\mathcal{K}(G)) = \frac{k-1}{(k-1)\epsilon+k}$, for any $\epsilon \geq 0$.

Proof. Consider a k -partite complete graph, i.e. a graph $G = (\cup_{i=1}^k V_i, E)$ in which, for each $i \in [k]$, each vertex in V_i is connected to all the vertices in V_j , for each $j \in [k] \setminus \{i\}$. In particular, we consider k -partite complete graphs in which $|V_i| = k$ for each $i \in [k]$. We are going to assign weights to the edges of the graph so that the social optimum value is equal to 1, and there exists a $(1 + \epsilon)$ -approximate NE whose social value is $\frac{k-1}{(k-1)\epsilon+k}$.

Let v_i^s be the i -th vertex in the s -th vertex subset V_s . Assign the weight $w_a := \frac{2}{k^2(k-1)((k-1)\epsilon+k)}$ to the edges (type (a) edges) connecting vertex v_i^s to vertex $v_j^{s'}$ for $i, j \in [k]$, $i \neq j$ and $s, s' \in [k]$, $s \neq s'$, and the weight $w_b := \frac{2((1+\epsilon)(k-1)-(k-2))}{k^2(k-1)((k-1)\epsilon+k)}$ to the edges (type (b) edges) of the form $\{v_i^s, v_i^{s'}\}$ for $i \in [k]$ and $s, s' \in [k]$, $s \neq s'$.

The social optimum is obtained by assigning the same color i to all the vertices of a same vertex subset V_i , for $i \in [k]$, as shown in Figure (1a).



■ **Figure 1** Panel (a): the 3-partite complete graph in which each vertex in the same vertex subset picks the same color. Panel (b): the 3-partite complete graph in which each vertex v_i^s for a fixed $i \in [3]$, in each vertex subset $s, s \in [3]$, picks the same color i . For the sake of readability, edge weights are shown only for the relevant subset of edges.

By a combinatorial argument, one can see that the social optimum is equal to 1. Indeed, we have a total of $\frac{k^2}{2}(k-1)^2$ type (a) edges plus a total of $\frac{k^2}{2}(k-1)$ type (b) edges. Multiplying such quantities by the respective weights one gets:

$$\frac{k^2}{2}(k-1)^2 \left[\frac{2}{k^2(k-1)((k-1)\epsilon+k)} \right] + \frac{k^2}{2}(k-1) \left[\frac{2((1+\epsilon)(k-1)-(k-2))}{k^2(k-1)((k-1)\epsilon+k)} \right]$$

which simplifies to $\frac{(k-1)}{(k-1)\epsilon+k} + \frac{(1+\epsilon)(k-1)-(k-2)}{(k-1)\epsilon+k}$. Finally, rearranging the terms, we obtain: $\frac{(k-1)\epsilon+k}{(k-1)\epsilon+k} = 1$.

Now, we need to build a $(1+\epsilon)$ -approximate NE attaining the lower bound of $\frac{k-1}{(k-1)\epsilon+k}$. Consider the strategy profile in which color i is assigned to each vertex v_i^s for $i \in [k]$ and $s \in [k]$, as in Figure (1b). Observe that the profit of a generic vertex v_i^s , for $i \in [k]$ and $s \in [k]$, is equal to $(k-1)^2$ times the weight w_a of type (a) edges. Summing this quantity over all vertices of the graph and then dividing by 2, one gets

$$\frac{k^2}{2}(k-1)^2 \left[\frac{2}{k^2(k-1)((k-1)\epsilon+k)} \right],$$

which is equivalent to $\frac{k-1}{(k-1)\epsilon+k}$. Therefore, it remains to show that any alternative strategy for vertex v_i^s guarantees a profit that is not larger than $(1+\epsilon)$ times the profit yielded by the current choice of i . As we argued, the payoff of each vertex is equal to $(k-1)^2$ times w_a , while, by the set of weights defined on the k -partite complete graph, the profit obtainable by changing to *any* other color $j \in [k] \setminus \{i\}$ is equal to $(k-1)(k-2)$ times w_a plus $(k-1)$ times the weight w_b of type (b) edges. One can show that

$$(1+\epsilon)(k-1)^2 \left[\frac{2}{k^2(k-1)((k-1)\epsilon+k)} \right] = (k-1) \left[\frac{2(k-2)}{k^2(k-1)((k-1)\epsilon+k)} + \frac{2((1+\epsilon)(k-1)-(k-2))}{k^2(k-1)((k-1)\epsilon+k)} \right]$$

holds. Indeed, it follows that

$$\frac{2(1+\epsilon)(k-1)}{k^2((k-1)\epsilon+k)} = \frac{2(k-2)}{k^2((k-1)\epsilon+k)} + \frac{2((1+\epsilon)(k-1)-(k-2))}{k^2((k-1)\epsilon+k)}$$

which becomes

$$\frac{2(1+\epsilon)(k-1)}{k^2((k-1)\epsilon+k)} = \frac{2(1+\epsilon)(k-1)}{k^2((k-1)\epsilon+k)}$$

which is always true. ◀

5 Primal-Dual Scheme for the Approximate One-round Walk

In this section, we give the primal formulation for the $(1+\epsilon)$ -approximate one-round walk of the k -coloring game. In particular, let W be a $(1+\epsilon)$ -approximate one-round walk for a fixed game $\mathcal{K}(G)$, with $\epsilon \geq 0$. Denote by σ^S and σ^F the initial and final states of W , respectively, and by \mathcal{CH} the set of players who change their strategy during the walk, that is, $\mathcal{CH} = \{i \in V : \sigma_i^F \neq \sigma_i^S\}$. Using the same approach as in the previous section, we obtain the following linear program $LP(\sigma^S, \sigma^F, \sigma^*)$:

$$\begin{aligned}
 & \min \sum_{\{i,j\} \in E} c_{ij}(1 - \alpha_{j\sigma^F(i)}^{\sigma^F}) \\
 \text{s.t. } & (1 + \epsilon) \sum_{j \in \text{Adj}(i): j < i} c_{ij}(1 - \alpha_{j\sigma^S(i)}^{\sigma^F}) + (1 + \epsilon) \sum_{j \in \text{Adj}(i): j > i} c_{ij}(1 - \alpha_{j\sigma^S(i)}^{\sigma^S}) \\
 & - \sum_{j \in \text{Adj}(i): j < i} c_{ij}(1 - \alpha_{jl}^{\sigma^F}) \\
 & - \sum_{j \in \text{Adj}(i): j > i} c_{ij}(1 - \alpha_{jl}^{\sigma^S}) \geq 0 \quad \forall i \in V \setminus \mathcal{CH}, l \in K, l \neq \sigma^S(i) \tag{9}
 \end{aligned}$$

$$\begin{aligned}
 & -(1 + \epsilon) \sum_{j \in \text{Adj}(i): j < i} c_{ij}(1 - \alpha_{j\sigma^S(i)}^{\sigma^F}) - (1 + \epsilon) \sum_{j \in \text{Adj}(i): j > i} c_{ij}(1 - \alpha_{j\sigma^S(i)}^{\sigma^S}) \\
 & + \sum_{j \in \text{Adj}(i): j < i} c_{ij}(1 - \alpha_{j\sigma^F(i)}^{\sigma^F}) + \sum_{j \in \text{Adj}(i): j > i} c_{ij}(1 - \alpha_{j\sigma^F(i)}^{\sigma^S}) \geq 0 \quad \forall i \in \mathcal{CH} \tag{10}
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{j \in \text{Adj}(i): j < i} c_{ij}(1 - \alpha_{j\sigma^F(i)}^{\sigma^F}) + \sum_{j \in \text{Adj}(i): j > i} c_{ij}(1 - \alpha_{j\sigma^F(i)}^{\sigma^S}) \\
 & - \sum_{j \in \text{Adj}(i): j < i} c_{ij}(1 - \alpha_{jl}^{\sigma^F}) \\
 & - \sum_{j \in \text{Adj}(i): j > i} c_{ij}(1 - \alpha_{jl}^{\sigma^S}) \geq 0 \quad \forall i \in \mathcal{CH}, l \in K, l \neq \sigma^F(i) \tag{11}
 \end{aligned}$$

$$\sum_{\{i,j\} \in E} c_{ij}(1 - \alpha_{j\sigma^*(i)}^{\sigma^*}) = 1 \tag{12}$$

$$c_{ij} \geq 0 \quad \forall \{i, j\} \in E \tag{13}$$

Inequalities (9) state that for each player $i \in V$ not changing her strategy, and any color $l \in K$, $l \neq \sigma^S(i)$, the utility she is gaining at the time she is processed by the walk is at least $1 + \epsilon$ times the one she would get by choosing color l . Inequalities (10) ensure that each player $i \in \mathcal{CH}$ has an $(1 + \epsilon)$ -approximate improving deviation in $\sigma_S(i)$ (although the definition of improving deviations requires the inequality to be strict, this simplification is without loss of generality). Moreover, differently from the cut game (i.e., k -coloring games with $k = 2$), an agent may have several different such deviations. To model the dynamics in which each player is changing strategy to the one that is the most promising at the time it is processed, inequalities (11) are required.

5.1 Dual Formulation

Observe that, by providing a feasible solution to a corresponding dual program, one can obtain a lower bound to the approximation ratio of the solutions achieved after a $(1 + \epsilon)$ -approximate one-round walk starting from any strategy profile. The resulting dual formulation $DLP(\sigma^S, \sigma^F, \sigma^*)$ of linear program $LP(\sigma^S, \sigma^F, \sigma^*)$, defined above, is as follows:

$$\max \gamma \tag{14}$$

$$\begin{aligned}
 \text{s.t. } & (1 + \epsilon) \left[\sum_{l \in K \setminus \{\sigma^S(j)\}} (y_{jl}(1 - \alpha_{i\sigma^S(j)}^{\sigma^F})) + \sum_{l \in K \setminus \{\sigma^S(i)\}} (y_{il}(1 - \alpha_{j\sigma^S(i)}^{\sigma^S})) \right] \\
 & - \left[\sum_{l \in K \setminus \{\sigma^S(j)\}} (y_{jl}(1 - \alpha_{il}^{\sigma^F})) + \sum_{l \in K \setminus \{\sigma^S(i)\}} (y_{il}(1 - \alpha_{jl}^{\sigma^S})) \right] \\
 & + \gamma(1 - \alpha_{j\sigma^*(i)}^{\sigma^*}) \leq 1 - \alpha_{j\sigma^F(i)}^{\sigma^F} \quad \forall \{i, j\} \in E, i, j \notin \mathcal{CH}, i < j \tag{15}
 \end{aligned}$$

$$\begin{aligned}
& - (1 + \epsilon) \left[x_j(1 - \alpha_{i\sigma^F(j)}^{\sigma^F}) + x_i(1 - \alpha_{j\sigma^S(i)}^{\sigma^S}) \right] + x_j(1 - \alpha_{i\sigma^F(j)}^{\sigma^F}) + x_i(1 - \alpha_{j\sigma^F(i)}^{\sigma^S}) \\
& + \gamma(1 - \alpha_{j\sigma^*(i)}^{\sigma^*}) + \sum_{l \in K \setminus \{\sigma^F(j)\}} z_{jl}(1 - \alpha_{i\sigma^F(j)}^{\sigma^F}) + \sum_{l \in K \setminus \{\sigma^F(i)\}} z_{il}(1 - \alpha_{j\sigma^F(i)}^{\sigma^S}) \\
& - \sum_{l \in K \setminus \{\sigma^F(i)\}} z_{il}(1 - \alpha_{jl}^{\sigma^S}) - \sum_{l \in K \setminus \{\sigma^F(j)\}} z_{jl}(1 - \alpha_{il}^{\sigma^F}) \\
& \leq 1 - \alpha_{j\sigma^F(i)}^{\sigma^F} \quad \forall \{i, j\} \in E, i, j \in \mathcal{CH}, i < j \tag{16}
\end{aligned}$$

$$\begin{aligned}
& (1 + \epsilon) \left[\sum_{l \in K \setminus \{\sigma^S(j)\}} y_{jl}(1 - \alpha_{i\sigma^S(j)}^{\sigma^F}) - x_i(1 - \alpha_{j\sigma^S(i)}^{\sigma^S}) \right] - \sum_{l \in K \setminus \{\sigma^S(j)\}} y_{jl}(1 - \alpha_{il}^{\sigma^F}) \\
& + x_i(1 - \alpha_{j\sigma^F(i)}^{\sigma^S}) + \gamma(1 - \alpha_{j\sigma^*(i)}^{\sigma^*}) + \sum_{l \in K \setminus \{\sigma^F(i)\}} z_{il}(1 - \alpha_{j\sigma^F(i)}^{\sigma^S}) \\
& - \sum_{l \in K \setminus \{\sigma^F(i)\}} z_{il}(1 - \alpha_{jl}^{\sigma^S}) \leq 1 - \alpha_{j\sigma^F(i)}^{\sigma^F} \quad \forall \{i, j\} \in E, i \in \mathcal{CH}, j \notin \mathcal{CH}, i < j \tag{17}
\end{aligned}$$

$$\begin{aligned}
& (1 + \epsilon) \left[\sum_{l \in K \setminus \{\sigma^S(i)\}} y_{il}(1 - \alpha_{j\sigma^S(i)}^{\sigma^S}) - x_j(1 - \alpha_{i\sigma^F(j)}^{\sigma^F}) \right] + x_j(1 - \alpha_{i\sigma^F(j)}^{\sigma^F}) \\
& - \sum_{l \in K \setminus \{\sigma^S(i)\}} y_{il}(1 - \alpha_{jl}^{\sigma^S}) + \gamma(1 - \alpha_{j\sigma^*(i)}^{\sigma^*}) + \sum_{l \in K \setminus \{\sigma^F(j)\}} z_{jl}(1 - \alpha_{i\sigma^F(j)}^{\sigma^F}) \\
& - \sum_{l \in K \setminus \{\sigma^F(j)\}} z_{jl}(1 - \alpha_{il}^{\sigma^F}) \leq 1 - \alpha_{j\sigma^F(i)}^{\sigma^F} \quad \forall \{i, j\} \in E, i \notin \mathcal{CH}, j \in \mathcal{CH}, i < j \tag{18}
\end{aligned}$$

$$y_{is} \geq 0 \quad \forall i \in V, s \in K, s \neq \sigma(i) \tag{19}$$

5.2 Analysis of the Dual Formulation

In this subsection, we derive a relaxation of the dual constraints by assuming that all dual variables of the same type are assigned the same value. That is, we impose $x_i := x$ for each $i \in V$ and, for each $i \in V$ and $l \in K$, $y_{il} := y$ and $z_{il} := z$. For a fixed pair of indices $i, j \in V$, with $i < j$, let \mathbb{I}^F be the indicator function that equals 1 if and only if i and j belong to different sets in the solution returned at the end of the walk, that is, $\sigma^F(i) \neq \sigma^F(j)$. Similarly, let \mathbb{I}^S be the indicator function which equals 1 if and only if i and j belong to different sets in the starting solution of the walk, i.e., $\sigma^S(i) \neq \sigma^S(j)$, let \mathbb{I}^* be the indicator function which equals 1 if and only if i and j belong to different sets in the social optimum, i.e., $\sigma^*(i) \neq \sigma^*(j)$ and let \mathbb{I}^{ij} be the indicator function which equals 1 if and only if $\sigma^F(i) \neq \sigma^S(j)$. With the above assumptions, we obtain the following equivalences:

- $1 - \alpha_{j\sigma^*(i)}^{\sigma^*} = \mathbb{I}^*$;
- $1 - \alpha_{j\sigma^F(i)}^{\sigma^F} = 1 - \alpha_{i\sigma^F(j)}^{\sigma^F} = \mathbb{I}^F$;
- $1 - \alpha_{i\sigma^S(j)}^{\sigma^S} = 1 - \alpha_{j\sigma^F(i)}^{\sigma^S} = \mathbb{I}^{ij}$;
- $1 - \alpha_{j\sigma^S(i)}^{\sigma^S} = 1 - \alpha_{i\sigma^S(j)}^{\sigma^S} = \mathbb{I}^S$.

Moreover, the summations occurring in the dual constraints simplify as follows:

- $\sum_{l \in K \setminus \{\sigma^F(j)\}} (1 - \alpha_{i\sigma^F(j)}^{\sigma^F}) = (k - 1)\mathbb{I}^F$;
- $\sum_{l \in K \setminus \{\sigma^F(i)\}} (1 - \alpha_{j\sigma^S(i)}^{\sigma^S}) = (k - 1)\mathbb{I}^S$;
- $\sum_{l \in K \setminus \{\sigma^F(j)\}} (1 - \alpha_{il}^{\sigma^F}) = k - 1 - \mathbb{I}^F$;

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$$\begin{aligned}
& \text{---} \quad \sum_{l \in K \setminus \{\sigma^S(i)\}} (1 - \alpha_{jl}^{\sigma^S}) = k - 1 - \mathbb{I}^S; \\
& \text{---} \quad \sum_{l \in K \setminus \{\sigma^F(i)\}} (1 - \alpha_{jl}^{\sigma^F}) = (k - 1)\mathbb{I}^{ij}; \\
& \text{---} \quad \sum_{l \in K \setminus \{\sigma^F(i)\}} (1 - \alpha_{jl}^{\sigma^S}) = k - 1 - \mathbb{I}^{ij}.
\end{aligned}$$

Substituting and rearranging, the four dual constraints become the following ones:

1. $y((1 + \epsilon)(k - 1)(\mathbb{I}^{ij} + \mathbb{I}^S) - 2k + 2 + \mathbb{I}^{ij} + \mathbb{I}^S) + \gamma\mathbb{I}^* \leq \mathbb{I}^F$ if $i, j \notin \mathcal{CH}$;
2. $x(\mathbb{I}^F - (1 + \epsilon)\mathbb{I}^S - \epsilon\mathbb{I}^{ij}) + z(k(\mathbb{I}^F + \mathbb{I}^{ij}) - 2k + 2) + \gamma\mathbb{I}^* \leq \mathbb{I}^F$ if $i, j \in \mathcal{CH}$;
3. $x(\mathbb{I}^{ij} - (1 + \epsilon)\mathbb{I}^S) + y((k - 1)((1 + \epsilon)\mathbb{I}^{ij} - 1) + \mathbb{I}^{ij}) + z(k(\mathbb{I}^{ij} - 1) + 1) + \gamma\mathbb{I}^* \leq \mathbb{I}^F$ if $i \in \mathcal{CH}, j \notin \mathcal{CH}$;
4. $x(\mathbb{I}^F - (1 + \epsilon)\mathbb{I}^{ij}) + y((k - 1)((1 + \epsilon)\mathbb{I}^S - 1) + \mathbb{I}^S) + z(k(\mathbb{I}^F - 1) + 1) + \gamma\mathbb{I}^* \leq \mathbb{I}^F$ if $i \notin \mathcal{CH}, j \in \mathcal{CH}$.

First, observe that, when $\mathbb{I}^* = 1$, the four constraints above become harder to satisfy with respect to the case in which $\mathbb{I}^* = 0$. So, in what follows, we shall always assume that $\mathbb{I}^* = 1$. For each of these constraints, we shall derive specialized inequalities with respect to the indicators \mathbb{I}^F , \mathbb{I}^S and \mathbb{I}^{ij} , depending on which combinations of their values are possible under the assumptions holding for each constraint. For constraint (1), since $i, j \notin \mathcal{CH}$, we have that $\mathbb{I}^F = \mathbb{I}^S$ must hold. Considering the two possible cases for the common value, we derive inequalities:

$$\begin{aligned}
\text{I1:} \quad & \gamma \leq 2(k - 1)y; \\
\text{I2:} \quad & \gamma \leq 1 - 2((k - 1)\epsilon + 1)y.
\end{aligned}$$

For constraint (2), since $i, j \in \mathcal{CH}$, we have that \mathbb{I}^F and \mathbb{I}^S may assume all possible combinations of values (in particular, we observe that the two cases in which $\mathbb{I}^F = 1 - \mathbb{I}^S$ require $k \geq 3$). However, it is easy to see that, for the pairs $(\mathbb{I}^S, \mathbb{I}^F)$ assuming the values $(0, 0)$, $(0, 1)$ or $(1, 0)$, $\mathbb{I}^{ij} = 1$ must hold. For the leftover case of $\mathbb{I}^S = \mathbb{I}^F = 1$, it can be $\mathbb{I}^{ij} \in \{0, 1\}$. We stress that, the case of $\mathbb{I}^{ij} = 0$, requires $k \geq 3$. Considering all possible cases, we derive inequalities:

$$\begin{aligned}
\text{I3:} \quad & \gamma \leq \epsilon x + (k - 2)z; \\
\text{I4:} \quad & \gamma \leq 1 + (\epsilon - 1)x - 2z; \\
\text{I5:} \quad & \gamma \leq (1 + 2\epsilon)x + (k - 2)z; \\
\text{I6:} \quad & \gamma \leq 1 + \epsilon x + (k - 2)z; \\
\text{I7:} \quad & \gamma \leq 1 + 2\epsilon x - 2z.
\end{aligned}$$

For constraint (3), since $i \in \mathcal{CH}$ and $j \notin \mathcal{CH}$, we have that $\mathbb{I}^F = \mathbb{I}^S = 0$ cannot be possible (we observe that the two case in which $\mathbb{I}^F = \mathbb{I}^S = 1$ requires $k \geq 3$). Moreover, it is easy to see that: for pairs $(\mathbb{I}^S, \mathbb{I}^F)$ assuming the values $(0, 1)$ or $(1, 1)$ $\mathbb{I}^{ij} = 1$ must hold; for the pair assuming the values $(1, 0)$, $\mathbb{I}^{ij} = 0$ must hold. Considering all possible cases, we derive inequalities:

$$\begin{aligned}
\text{I8:} \quad & \gamma \leq 1 - x - ((k - 1)\epsilon + 1)y - z; \\
\text{I9:} \quad & \gamma \leq (1 + \epsilon)x + (k - 1)y + (k - 1)z; \\
\text{I10:} \quad & \gamma \leq 1 + \epsilon x - ((k - 1)\epsilon + 1)y - z.
\end{aligned}$$

For constraint (4), since $i \notin \mathcal{CH}$ and $j \in \mathcal{CH}$, by symmetry on constraint (3), we have that $\mathbb{I}^F = \mathbb{I}^S = 0$ cannot be possible (note again that the two case in which $\mathbb{I}^F = \mathbb{I}^S = 1$ requires

$k \geq 3$). Moreover, it is easy to see that: for pairs $(\mathbb{I}^S, \mathbb{I}^F)$ assuming the values $(1, 0)$ or $(1, 1)$, $\mathbb{I}^{ij} = 1$ must hold; for the pair assuming the values $(0, 1)$, $\mathbb{I}^{ij} = 0$ must hold. Considering all possible cases, we derive inequalities:

$$\begin{aligned} \text{I11: } & \gamma \leq 1 - x + (k-1)y - z; \\ \text{I12: } & \gamma \leq (1 + \epsilon)x - ((k-1)\epsilon + 1)y + z(k-1); \\ \text{I13: } & \gamma \leq 1 + \epsilon x - ((k-1)\epsilon + 1)y - z. \end{aligned}$$

Given a set of inequalities \mathcal{S} , we say that an inequality $s \in \mathcal{S}$ is *dominated* by an inequality $s' \in \mathcal{S} \setminus \{s\}$ if each solution of s' is also a solution for s . So, all dominated inequalities can be removed from \mathcal{S} without altering its set of solutions. It is easy to see that inequalities I5, I6 and I9 are all dominated by I3; inequality I7 is dominated by I4 and inequalities I10, I11 and I13 are dominated by I8. So, after removing the dominated inequalities, we obtain that, for a pair of indices $i, j \in V$, with $i < j$, a feasible dual solution such that all variables of the same type assume the same value must satisfy the following set of six inequalities:

$$\begin{aligned} \overline{\text{I1}}: & \gamma \leq 2(k-1)y; \\ \overline{\text{I2}}: & \gamma \leq 1 - 2((k-1)\epsilon + 1)y; \\ \overline{\text{I3}}: & \gamma \leq \epsilon x + z(k-2); \\ \overline{\text{I4}}: & \gamma \leq 1 + (\epsilon - 1)x - 2z; \\ \overline{\text{I5}}: & \gamma \leq 1 - x - ((k-1)\epsilon + 1)y - z; \\ \overline{\text{I6}}: & \gamma \leq (1 + \epsilon)x - ((k-1)\epsilon + 1)y + z(k-1). \end{aligned}$$

5.3 Performance of the Approximate One-round Walk

To prove our results for $(1 + \epsilon)$ -approximate one-round walks, we give the following lemmas in which we provide different feasible dual solutions holding under different assumptions.

► **Lemma 3.** *If $\epsilon \leq \frac{k}{(k-1)(k-2)}$, then the solution such that $\gamma = \frac{k-2}{k}$, $x = 0$, $y = \frac{k-2}{2k(k-1)}$ and $z = \frac{1}{k}$ is feasible for the dual.*

Proof. We need to prove that all six inequalities $\overline{\text{IR}}$, with $R \in [6]$, are satisfied. Inequalities $\overline{\text{I1}}$, $\overline{\text{I3}}$ and $\overline{\text{I4}}$ trivially hold true. Substituting and rearranging in both $\overline{\text{I2}}$, $\overline{\text{I5}}$, and $\overline{\text{I6}}$, we get $\epsilon \leq \frac{k}{(k-1)(k-2)}$, which holds true under our assumption. ◀

► **Lemma 4.** *If $\epsilon \leq \frac{k}{(k-1)(k-2)}$, let $\beta := \epsilon^2 k(k-1) + \epsilon(k^2 + 2k - 2) + k$. Then, the solution such that $\gamma = \frac{2\epsilon(k-1)^2}{\beta}$, $x = \frac{k - \epsilon(k^2 - 3k + 2)}{\beta}$, $y = \frac{\epsilon(k-1)}{\beta}$ and $z = \frac{\epsilon((k-1)\epsilon + 2k - 1)}{\beta}$ is feasible for the dual.*

Proof. First, we observe that, under the assumption that $\epsilon \leq \frac{k}{(k-1)(k-2)}$, we have $x \geq 0$. We need to prove that all six inequalities $\overline{\text{IR}}$, with $R \in [6]$, are satisfied. Inequalities $\overline{\text{I1}}$, $\overline{\text{I3}}$, $\overline{\text{I4}}$, $\overline{\text{I5}}$ and $\overline{\text{I6}}$ trivially hold true. Substituting and rearranging in $\overline{\text{I2}}$, we get inequality $k \geq (k-1)(k-2)\epsilon^2 + (k^2 - 4k - 2)\epsilon$. Writing down $k^2 - 4k - 2$ as $(k-1)(k-2) - k$ and rearranging, we get inequality $\epsilon \leq \frac{k}{(k-1)(k-2)}$, which holds true under our assumption. ◀

► **Lemma 5.** *If $\epsilon \geq \frac{k}{(k-1)(k-2)}$, then the solution such that $\gamma = \frac{k-1}{(k-1)\epsilon + k}$, $x = 0$, $y = \frac{1}{2((k-1)\epsilon + k)}$ and $z = \frac{(k-1)\epsilon + 1}{2((k-1)\epsilon + k)}$ is feasible for the dual.*

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Proof. We need to prove that all six inequalities $\overline{\text{IR}}$, with $\text{R} \in [6]$, are satisfied. Inequalities $\overline{\text{I1}}$, $\overline{\text{I2}}$, $\overline{\text{I4}}$ and $\overline{\text{I5}}$ trivially hold true. Substituting and rearranging in $\overline{\text{I3}}$, we get $\epsilon \geq \frac{k}{(k-1)(k-2)}$, which holds true under our assumption. Finally, substituting and rearranging in $\overline{\text{I6}}$, we get $\epsilon \geq \frac{1}{k-2}$, which is implied by the assumption $\epsilon \geq \frac{k}{(k-1)(k-2)}$. ◀

Observe that, for $\epsilon \leq \frac{k}{(k-1)(k-2)}$, we can choose between two different solutions. Clearly, depending on the value of k , we shall select the one yielding the more significant lower bound, that is, the one having the largest value for variable γ . Imposing $\frac{2\epsilon(k-1)^2}{k(k-1)\epsilon^2 + (k^2 + 2k - 2)\epsilon + k} > \frac{k-2}{k}$ after rearranging, it turns out that the solution given in Lemma 4 outperforms the one from Lemma 3 if and only if $(k\epsilon - k + 2)((k-1)(k-2)\epsilon - k) < 0$. This is equivalent to $\epsilon \in \left(\min \left\{ \frac{k-2}{k}, \frac{k}{(k-1)(k-2)} \right\}, \max \left\{ \frac{k-2}{k}, \frac{k}{(k-1)(k-2)} \right\} \right)$. Now, observe that, for $k \geq 5$, $\max \left\{ \frac{k-2}{k}, \frac{k}{(k-1)(k-2)} \right\} = \frac{k-2}{k}$, which implies that a necessary condition for the solution given in Lemma 4 to outperform the one from Lemma 3 is to have $\epsilon \geq \frac{k}{(k-1)(k-2)}$, but the solution from Lemma 4 is not feasible under this premise. Thus, we conclude that, for $k \geq 5$, the solution from Lemma 3 is always preferable. For $k = 3, 4$, instead, we have that the solution from Lemma 3 is preferable as long as $\epsilon \leq \frac{k-2}{k}$.

Summarizing, for $k \geq 4$, we obtain the following lower bounds.

► **Theorem 6.** *For any $\epsilon \geq 0$, the approximation ratio of $(1 + \epsilon)$ -approximate one-round walks starting from any initial strategy profile in 4-coloring games is at least*

$$\begin{cases} \frac{1}{2} & \text{if } \epsilon \leq \frac{1}{2}, \\ \frac{9\epsilon}{6\epsilon^2 + 11\epsilon + 2} & \text{if } \frac{1}{2} \leq \epsilon \leq \frac{2}{3}, \\ \frac{3}{3\epsilon + 4} & \text{if } \epsilon \geq \frac{2}{3}. \end{cases}$$

► **Theorem 7.** *For any $\epsilon \geq 0$ and $k \geq 5$, the approximation ratio of $(1 + \epsilon)$ -approximate one-round walks starting from any initial strategy profile in k -coloring games is at least*

$$\begin{cases} \frac{k-2}{k} & \text{if } \epsilon \leq \frac{k}{(k-1)(k-2)}, \\ \frac{k-1}{(k-1)\epsilon + k} & \text{if } \epsilon \geq \frac{k}{(k-1)(k-2)}. \end{cases}$$

For the case of $k = 3$, a more intricate situation occurs, as we can prove the following additional feasible solutions.

► **Lemma 8.** *If $\epsilon \leq \sqrt{2}$, let $\beta := 3\epsilon^2 + 6\epsilon + 2$. Then, the solution such that $\gamma = \frac{4\epsilon}{\beta}$, $x = \frac{2-\epsilon}{\beta}$, $y = \frac{\epsilon}{\beta}$ and $z = \frac{\epsilon(2+\epsilon)}{\beta}$ is feasible for the dual.*

Proof. First, we observe that, under the assumption that $\epsilon \leq \sqrt{2}$, we have $x \geq 0$. We need to prove that all six inequalities $\overline{\text{IR}}$, with $\text{R} \in [6]$, are satisfied. Inequalities $\overline{\text{I1}}$, $\overline{\text{I3}}$, $\overline{\text{I4}}$, $\overline{\text{I5}}$ and $\overline{\text{I6}}$ trivially hold true. Substituting and rearranging in $\overline{\text{I2}}$, we obtain inequality $\epsilon^2 - 2 \leq 0$ which holds true under our assumption. ◀

► **Lemma 9.** *If $\epsilon \geq \sqrt{2}$, then the solution such that $\gamma = \frac{2}{2\epsilon+3}$, $x = \frac{\epsilon-1}{2(2\epsilon+3)}$, $y = \frac{1}{2(2\epsilon+3)}$ and $z = \frac{2+\epsilon}{2(2\epsilon+3)}$ is feasible for the dual.*

Proof. First, we observe that, under the assumption that $\epsilon \geq \sqrt{2}$, we have $x \geq 0$. We need to prove that all six inequalities $\overline{\text{IR}}$, with $\text{R} \in [6]$, are satisfied. Inequalities $\overline{\text{I1}}$, $\overline{\text{I2}}$, $\overline{\text{I5}}$ and $\overline{\text{I6}}$ trivially hold true. Substituting and rearranging in $\overline{\text{I3}}$, we get $\epsilon^2 - 2 \geq 0$, which holds true under our assumption. Finally, substituting and rearranging in $\overline{\text{I4}}$, we get $\epsilon^2 - 1 \geq 0$, which, being weaker than the previous one, holds true under our assumption. ◀

As discussed above, for $\epsilon \leq 1/3$, the solution from Lemma 3 is preferable to that of Lemma 4. A simple calculation shows that the solution from Lemma 8 is preferable to that of Lemma 4, as long as $\epsilon \geq 1$. In addition, it is easy to show that, as long as $\epsilon \leq 3/2$, the solution of Lemma 9 is preferable to that of Lemma 4. In summary, one obtains the following result.

► **Theorem 10.** *For any $\epsilon \geq 0$, the approximation ratio of $(1 + \epsilon)$ -approximate one-round walks starting from any initial strategy profile in 3-coloring games is at least*

$$\left\{ \begin{array}{ll} \frac{1}{3} & \text{if } \epsilon \leq \frac{1}{3}, \\ \frac{8\epsilon}{6\epsilon^2 + 13\epsilon + 3} & \text{if } \frac{1}{3} \leq \epsilon \leq 1, \\ \frac{4\epsilon}{3\epsilon^2 + 6\epsilon + 2} & \text{if } 1 \leq \epsilon \leq \sqrt{2}, \\ \frac{2}{2\epsilon + 3} & \text{if } \epsilon \geq \sqrt{2}. \end{array} \right.$$

We observe that any upper bound for the $(1 + \epsilon)$ -approximate PoA directly extends to the approximation ratio of the solutions achieved after a $(1 + \epsilon)$ -approximate one-round walk. So, by the upper bound of $\frac{k-1}{k(1+\epsilon)-\epsilon}$ proved for the $(1 + \epsilon)$ -approximate PoA in Theorem 2, we derive that the lower bounds shown in Theorems 6, 7 and 10 are tight as long as $\epsilon \geq \frac{k}{(k-1)(k-2)}$ for $k \geq 4$ and $\epsilon \geq \sqrt{2}$ for $k = 3$. Determining matching upper bounds, for all other missing cases, remains a challenging open problem. With this respect, in the next section, we provide an answer for the basic case of $\epsilon = 0$.

5.4 A Tight Instance for $\epsilon = 0$

In this section, we design a family of instances of the k -coloring game with $k \geq 3$, for which the lower bounds given in Theorems 6, 7 and 10 are tight, when $\epsilon = 0$.

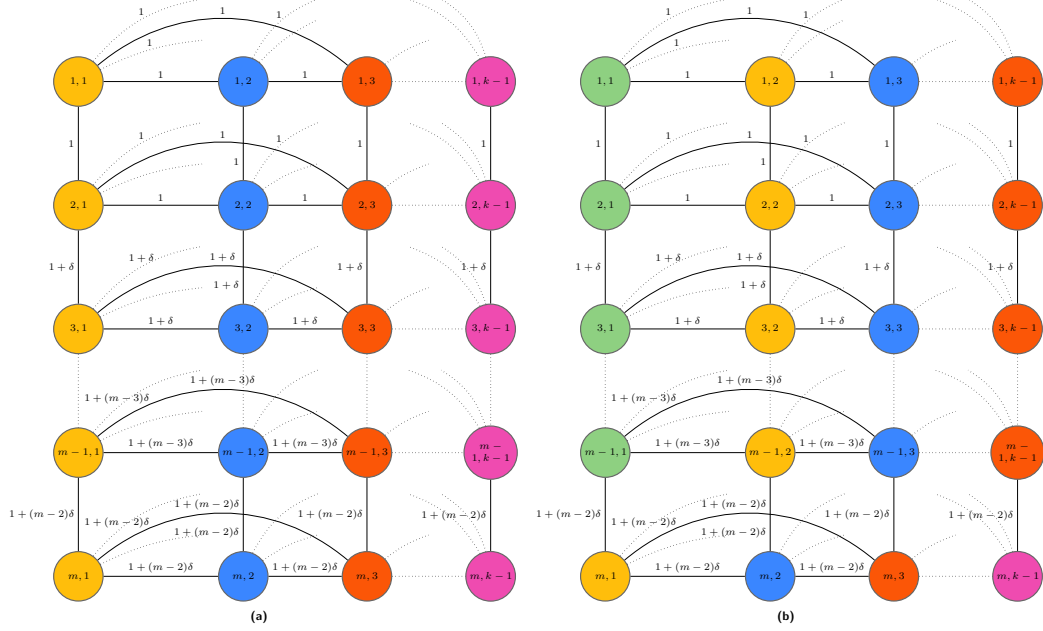
► **Theorem 11.** *For any $k \geq 3$ and $\mu > 0$, there exists an instance of the k -coloring game $\mathcal{K}(G)$ for which $\text{Ap}x_0(\mathcal{K}(G)) \leq \frac{k-2}{k} + \mu$.*

Proof. Consider the graph k -coloring game $\mathcal{K}(G_m)$ induced by the parametric graph G_m , $m > 1$. $G_m = (V, E)$ is an $m \times (k - 1)$ grid graph in which the vertices of each row i form a $(k - 1)$ -clique, denoted by K_{k-1}^i . Thus, each vertex (i, j) is adjacent to (i, j') , $j' \in [k - 1] \setminus \{j\}$, and to the vertices $(i - 1, j)$ and $(i + 1, j)$, i.e., the vertices in the previous and successive rows of row i in the same column j . All edges in clique K_{k-1}^1 have weight equal to 1. All edges in clique K_{k-1}^i and those connecting the vertices of the $i - 1$ -st row with the vertices in the i -th row have weight $1 + (i - 2)\delta$, for $i = 2, \dots, m$, where $\delta > 0$ is a positive constant. It is easy to see that this graph can be colored by using $k - 1$ colors so that the endpoints of each edge receive different colors. We thus have

$$\text{COL}(\sigma^*) = \frac{(k-1)(k-2)}{2} + \frac{(k-1)(k-2)}{2} \sum_{i=0}^{m-2} (1 + i\delta) + (k-1) \sum_{i=0}^{m-2} (1 + i\delta).$$

On the other hand, consider a one-round walk starting from the strategy profile in which the vertices in column j pick color j . Figure 2a shows this initial strategy profile on the parametric graph G_m . The walk starts from vertex $(1, 1)$, i.e., the one in the top-left corner of the grid, which will change strategy by picking color k , that is not chosen by any of its neighbors. The walk then proceeds on the vertex to the right of the starting one, namely vertex $(1, 2)$, which will change strategy by picking the color that vertex $(1, 1)$ initially had, which is not picked by any of its neighbors. Each following vertex $(1, j)$, for $j = 3, \dots, k - 1$, will pick the color that the vertex on its left initially had. The walk continues in such a

way up to row $m - 1$. The only vertices that will not change their strategy are those in the last row, namely vertices (m, j) , $j \in [k - 1]$. The final strategy profile σ^F is depicted in Figure 2b.



■ **Figure 2** Panel (a): the grid graph with the initial strategy profile in which the vertices of the same column pick the same color. Panel (b): the grid graph after the one-round walk described in the proof of Theorem 11. In both panels, some edge weights are missing for the sake of readability.

The social function computed on σ^F reads

$$\text{COL}(\sigma^F) = \frac{(k-1)(k-2)}{2} + \frac{(k-1)(k-2)}{2} \sum_{i=0}^{m-2} (1+i\delta) + (k-1)(1+(m-2)\delta).$$

It follows that

$$\text{Apx}_0(\mathcal{K}(G)) \leq \frac{\frac{(k-1)(k-2)}{2} + \frac{(k-1)(k-2)}{2} \sum_{i=0}^{m-2} (1+i\delta) + (k-1)(1+(m-2)\delta)}{\frac{(k-1)(k-2)}{2} + \frac{(k-1)(k-2)}{2} \sum_{i=0}^{m-2} (1+i\delta) + (k-1) \sum_{i=0}^{m-2} (1+i\delta)}.$$

By choosing δ small enough, e.g., $\delta \leq \frac{1}{m^3}$ and then letting $m \rightarrow \infty$, we obtain that

$$\lim_{m \rightarrow \infty} \text{Apx}_0(\mathcal{K}(G)) = \frac{k-2}{k}.$$

Thus, for any fixed $\mu > 0$, there exists a sufficiently high value for m yielding the claim. ◀

6 Conclusion

In this paper we have analyzed the performances of mildly greedy players in k -coloring games. First, by exploiting the primal-dual method of Bilò [4], we have established a lower bound of $\frac{k-1}{k(1+\epsilon)-\epsilon}$ to the ϵ -PoA for the entire class of k -coloring games. Moreover, we have shown that such bound is tight, for any $\epsilon \geq 0$, via a particular instance of the game defined on a k -partite complete graph. Second, we have provided lower bounds on the approximation

ratio of the solutions achieved by a $(1 + \epsilon)$ -approximate one-round walk, specifically a lower bound of $\min \left\{ \frac{k-2}{k}, \frac{k-1}{(k-1)\epsilon+k} \right\}$, for any $\epsilon \geq 0$ and $k \geq 5$, and some finer bounds depending on ϵ for the cases of $k = 3$ and $k = 4$. A family of instances of the k -coloring game with $k \geq 3$ matching the lower bound of $\frac{k-2}{k}$ is provided, for the basic case of $\epsilon = 0$. Our results confirm that, if players are willing to relax their selfishness, socially better outcomes are possible. We leave open the problem of determining matching upper bounds for all other missing cases. Our results generalize those known for *cut games* – the special case of 2-coloring games [7].

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