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REVIEW PAPER

Functional Gaussian fields on hyperspheres with their equivalent Gaussian measures

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Abstract

We develop a general framework for isotropic functional Gaussian fields on the d-dimensional sphere $\mathbb{S}^d$, where the field takes values in a separable Hilbert space $\mathcal{H}$. We establish an operator-valued extension of Schoenberg's theorem and show that the covariance structure of such fields admits a representation in terms of a sequence of trace-class d-Schoenberg operators. This yields an explicit spectral decomposition of the covariance operator on $L^2(\mathbb{S}^d;\mathcal{H})$. We then derive a functional version of the Feldman-Hájek criterion and prove that equivalence of the Gaussian measures induced by two Hilbert-valued spherical fields is determined by a Hilbert summability criterion that involves Schoenberg functional sequences, thereby extending classical results for scalar and vector fields on spheres to the infinite-dimensional setting. We further show how equivalence of all scalar projections is contained within, and dominated by, the functional criterion. The theory is illustrated through two classes of models: (i) a multiquadratic bivariate family on $\mathbb{S}^d$, for which the equivalence region can be expressed in closed form in terms of cross-correlation and geodesic decay parameters, and (ii) an infinite-dimensional Legendre-Matérn construction, where operator-valued spectra lead to explicit identifiability conditions on smoothness and scale parameters. These examples demonstrate how the operator-valued Schoenberg coefficients govern both the geometry and the measure-theoretic behavior of functional spherical fields. Overall, the results provide a unified spectral framework for Gaussian measures on $L^2(\mathbb{S}^d;\mathcal{H})$, bridging harmonic analysis, operator theory, and stochastic geometry on manifolds, and offering foundational tools for functional data analysis, spatial statistics, and kernel methods on spherical domains.

Keywords Functional Gaussian fields · Hilbert-valued random fields · Spherical harmonics · Operator-valued Schoenberg sequences · Equivalence of Gaussian measures · Feldman-Hájek criterion · Hyperspherical analysis · Spatial statistics on spheres · Operator-valued covariance functions

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1 Introduction

1.1 Context

Functional data analysis (FDA) has become a central subject in many branches of theoretical and applied sciences. Within the paradigm of FDA, each datum is an element taking values over infinite dimensional spaces such as separable Hilbert spaces (Ramsay and Silverman 2005; Ferraty and Vieu 2006; Hsing and Eubank 2015; Wang et al. 2016; Aneiros et al. 2022; Cuevas 2014). Typical examples include curves or surfaces indexed by time, space or both, as well as images in remote sensing, trajectories in neuroscience and biomechanics, to mention just a few. Treating data as Hilbert-valued objects allows to connect FDA with solid theory coming from stochastic processes, functional and harmonic analysis within a unified framework. Stochastic processes defined over high dimensional spaces (space or space-time) are customarily called random fields or random functions (Matheron 1965). Functional Gaussian fields, which are random fields having an Hilbert space as an image set, have been largely used within both the statistics and machine learning communities (Berlinet and Thomas-Agnan 2004; Rasmussen and Williams 2006; Sheffield 2007). In particular, the theory of reproducing kernel Hilbert spaces has been especially popular within the realm of high dimensional fields, and perfectly suitable to treat observations that can be conceived as random objects defined over Hilbert spaces. Predictive modeling for functional random fields has attracted large interest in the last 30 years, and the reader is referred to Giraldo et al. (2011); Menafoglio et al. (2013, 2014) and Delicado et al. (2010), with the references therein.

Early contributions include the spatial functional mode estimation approach of Dabo-Niang et al. (2010) and the development of fractal-based spatial filtering methodologies in Ruiz-Medina and Fernández-Pascual (2010). Functional autoregressive Hilbert-space modelling has also been explored for instance in Ruiz-Medina and Salmerón (2010); Kokoszka and Reimherr (2013); Caponera and Panaretos (2022).

A well established line of research has focused on spatial functional kriging, that, on best linear unbiased prediction within the realm of functional data. Universal kriging extensions for functional spatial data were introduced in Caballero et al. (2013), while Aitchison geometry-based kriging for compositional curves was developed in Menafoglio et al. (2014). Advances in prediction and optimal sampling of multivariate functional random fields are presented in Bohorquez et al. (2017), with comparative studies of functional versus spatio-temporal kriging approaches in Strandberg et al. (2019). A more recent ingenious approach based on physics-based residual functional kriging for dynamically evolving random fields have been considered in Peli et al. (2022).

Beyond prediction, functional models have supported regression and inference in environmental applications. Penalized regression for spatial functional data has been proposed by Bernardi et al. (2017), while functional drift-kriging in air quality modelling is discussed in Ignaccolo et al. (2014). Functional ANOVA methods applied to pollutant dynamics during the COVID-19 pandemic were studied by Acal et al. (2022). Finally, for recent developments in multivariate clustering and association for functional air pollution data, the reader is referred to Pulido et al. (2025); Fortuna et al. (2025).

Finally, theoretical foundations relevant to the equivalence of Gaussian measures in the context of environmental multivariate random fields are presented in Ruiz-Medina and Porcu (2015), which provides key results with implications for asymptotic inference and covariance specification.

1.2 State of the art and contribution

Gaussian random fields defined over (hyper) spheres have a long history that traces back to harmonic analysis, probability theory, statistics and machine learning. Recently, they have become very popular as they are the backbone for the construction of statistical models for climate model outputs (Crippa et al. 2016; Porcu et al. 2021; Caponera and Marinucci 2021; Caponera et al. 2022; Marinucci et al. 2021; Caponera et al. xxx). While the theory of Gaussian processes on the sphere is well developed for scalar and vector-valued (finite-dimensional) fields, little is known about the functional case. A key reference in this direction is Caponera (2024), which develops the spectral theory for the 2-dimensional sphere.

For Gaussian fields, the finite dimensional distributions are completely specified through the covariance functions, which are positive definite. A natural assumption for covariance functions over spheres is that of geodesic isotropy (Porcu et al. 2021), that is the covariance between any pair of random variables located over two different points depends exclusively on their geodesic distance, being the arccosine of the dot product between any two points located over the spherical shell. Characterization of scalar positive definite functions over d-dimensional spheres is due to Schoenberg (194203), and subsequent generalizations to Berg and Porcu (2017) for the case of product spaces involving the sphere. Characterization of matrix-valued positive definite functions over spheres is due to Hannan (1980), while the functional case is elusive so far. This theory has allowed for the construction of covariance models that are indispensable for phenomena intrinsically defined on spherical domains, including the Cosmic Microwave Background (CMB) in cosmology and geophysical variables (temperature, pressure, wind, ocean currents) in Earth system science.

A key concept underlying the asymptotic behavior of Gaussian process models is the equivalence of Gaussian measures. Roughly speaking, two stochastic models are equivalent if they generate probability distributions on the space of sample paths that assign zero probability to exactly the same events. In such a situation, even an arbitrarily dense set of observations from a bounded spatial domain cannot distinguish between the two models with probability tending to one (Zhang 2004). In other words, although the models may correspond to different covariance parameters, their induced probability laws on the space of realizations are so similar that they are statistically indistinguishable under fixed-domain asymptotics. This phenomenon is fundamentally infinite-dimensional: while Gaussian distributions in finite-dimensional Euclidean spaces are always mutually absolutely continuous when their covariance matrices are nonsingular, Gaussian processes may produce either equivalent or orthogonal probability measures depending on the structure of their covariance functions. The theory of equivalence and orthogonality of Gaussian measures therefore provides the probabilistic foundation for the notion of *microergodic parameters* in spatial statistics, whereby certain combinations of covariance parameters can be consistently estimated while the individual parameters themselves cannot.

Equivalence of Gaussian measures is of paramount importance in spatial statistics under the paradigm of infill asymptotics. They represent a tool *sine qua non* to evaluate asymptotic effects of misspecified best linear unbiased prediction (kriging), that is when prediction is performed with a wrong covariance function; see (Stein 1999) with the references therein, for a thorough treatment. Further, they cover fundamental importance to evaluate the asymptotic performance of certain likelihood-based estimators for the parameters indexing any parametric class of covariance functions. While this subject was originally born within the realm of abstract probability theory (Skorokhod and Yadrenko 1973; Yadrenko 1983; Feldman 1958), in the last 30 years there has been a massive use of this theory to have new theoretical findings in both spatial statistics and machine learning communities.

Equivalence of Gaussian measures over d-dimensional spheres has been studied by Arafat et al. (2018) for scalar random fields. Equivalence of functional Gaussian fields has been studied over planar surfaces endowed with Euclidean metrics only, and the case of the sphere is once again elusive. Extending these characterizations to Hilbert-valued fields requires an operator-valued analogue of the Schoenberg decomposition and a functional version of the Feldman-Hájek criterion.

The plan of the paper is the following. Section 2 contains the necessary background material needed to understand the subsequent mathematical developments, that are provided in Sect. 3. This section contains novel theoretical results in concert with some examples. Section 4 concludes the paper with a discussion.

2 Background

2.1 Scalar and vector Gaussian fields on spheres

Some parts of the paper will make use of Gaussian fields defined over finite dimensional spaces.

Let Z be a Gaussian random field defined over the d-dimensional sphere with unit radius, $\mathbb{S}^d$, embedded in $\mathbb{R}^{d+1}$, $d \geq 1$. We suppose Z to have finite first and second order moments, so that the covariance function is well defined. Additionally, we shall work under the assumption of geodesic isotropy, so that the covariance $R : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ depends on the inner product between any two points located over the spherical shell. Harmonic analysis results originating from Schoenberg (194203) allow one to decompose the function R (whenever it is continuous) as

$$R(x, y) = \sum_{l \geq 0} h(l)\, b_l\, C_l^{(d-1)/2}(x \cdot y), \qquad x, y \in \mathbb{S}^d, \tag{1}$$

where C_l^k denotes the l-th Gegenbauer polynomial of order k (see (Berg and Porcu 2017) and references therein), $h(l)$ denotes the multiplicity, i.e., the dimension of the space of spherical harmonics of degree l (explicitly defined in the subsequent section; see also Arafat et al. (2018); Caponera et al. (2025)), and $\{b_l\}_{l \geq 0}$ is a sequence of nonnegative coefficients satisfying $\sum_{l \geq 0} h(l)\, b_l = \sigma^2$, where σ^2 denotes the variance of Z. We should have actually used the notation $h(l, d)$ and $b_{l,d}$ to emphasize that the multiplicity and the coefficients depend on the dimension of the sphere. However, this fact is not relevant to the exposition following subsequently, and would be an additional source of obfuscation as the findings in subsequent sections are mathematically and notationally involved. Berg and Porcu (2017), mimicking Daley and Porcu (2014), adopt the illustrative nomenclature d-Schoenberg coefficients for the scalars b_l and consequently $\{b_l\}_{l \geq 0}$ is called a d-Schoenberg sequence. The extension for vector valued Gaussian fields $Z \in \mathbb{R}^p$, with p a positive integer, can be found in Hannan (1980), in this case, the covariance R is matrix valued function from $\mathbb{S}^d \times \mathbb{S}^d$ into $\mathbb{R}^{p \times p}$, and the Schoenberg expansion (1) is similarly attained:

$$R(x, y) = \sum_{l \geq 0} h(l)\, B_l\, C_l^{(d-1)/2}(x \cdot y), \qquad x, y \in \mathbb{S}^d, \tag{2}$$

where $\{B_l\}_{l \geq 0}$ is a summable sequence of positive semidefinite matrices.

Let $(\Omega, \mathcal{A}, \mathrm{P}_i)$, $i = 1, 2$ be two probability spaces. Two measures $\mathbf{P}_i$ are called equivalent when they are mutually absolutely continuous. Otherwise, they are called orthogonal. Gaussian measures are the only ones having either equivalence or orthogonality (no indetermination cases). For Gaussian measures associated with two random fields, equivalence is understood in terms of the probability measures induced by the random fields on the space of sample paths (Zhang 2004). Since Gaussian measures are uniquely determined by their mean and covariance, it is not a mystery that any condition regarding equivalence of Gaussian measures translates into conditions for the respective covariance functions, or equivalently their spectrum.

Equivalence of Gaussian measures associated with random fields defined over hyperspheres has been studied by Arafat et al. (2018), who proved that

$$\mathrm{P}_1 \equiv \mathrm{P}_2 \iff \sum_{l\geq 0} h(l)\left(\frac{b_l^{(1)}}{b_l^{(2)}}-1\right)^2 < \infty, \tag{3}$$

where $\{b_i^{(i)}\}_{l\geq 0}$ are the d-Schoenberg sequences associated with the covariance functions R_i, $i=1,2$.

2.2 Functional Gaussian fields on spheres

Notation Let us denote by $\mathcal{H}$ a separable Hilbert space, with its inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and the induced norm $\|\cdot\|_{\mathcal{H}}$. Denote by $x\otimes y$ the operator acting as $(x\otimes y)z=\langle z,y\rangle_{\mathcal{H}}\,x$. The most important case is that when $\|x\|_{\mathcal{H}}=1$, the operator $x\otimes x$ is the orthogonal projection onto the direction of x. The space of Hilbert-Schmidt operators is denoted by $\mathcal{L}_2(\mathcal{H})$, with its norm given by $\|A\|^2_{\mathcal{L}_2(\mathcal{H})}=\sum_{k\geq 1}\|Ae_k\|^2_{\mathcal{H}}$, where $\{e_k\}_{k\geq 1}$ is any orthonormal basis. Let $\mathbb{S}^d\subset\mathbb{R}^{d+1}$ be the unit sphere, with $d\geq 1$. The Euclidean dot product between two vectors $x,y\in\mathbb{R}^{d+1}$ is denoted by $x\cdot y$, and the Euclidean norm by $\|x\|$. Moreover, we denote by $L^2(\mathbb{S}^d)$ the usual space of square-integrable scalar functions on the sphere with the inner product $\langle f,g\rangle_{L^2(\mathbb{S}^d)}:=\int_{\mathbb{S}^d} f(x)g(x)\mathrm{d}\sigma(x)$, where $\mathrm{d}\sigma(\cdot)$ is the surface (Haar) measure on the sphere $\mathbb{S}^d$, normalized so that $\int_{\mathbb{S}^d}\mathrm{d}\sigma(x)=1$. It is well known that $L^2(\mathbb{S}^d)$ decomposes as a direct sum of mutually orthogonal subspaces spanned by the eigenfunctions of the Laplacian on $\mathbb{S}^d$. These eigenfunctions are referred to as (hyper) spherical harmonics. A standard orthonormal basis for the eigenspace of degree $l\geq 0$ is given by the (in this paper, *real-valued*) *fully normalized spherical harmonics* $\{\mathcal{Y}_{l,m}, m=1,\dots,h(l)\}$, with $h(0):=1$ and

$$h(l):=\frac{(2\,l+d-1)\,(l+d-2)!}{l!\,(d-1)!},\qquad l\geq 1$$

We denote with $L^2(\mathbb{S}^d;\mathcal{H})$ the space of square-integrable functions on the sphere taking values on $\mathcal{H}$, i.e., $f:\mathbb{S}^d\to\mathcal{H}$ such that $\int_{\mathbb{S}^d}\|f(x)\|^2_{\mathcal{H}}\,\mathrm{d}\sigma(x)<\infty$. This is also a Hilbert space with the inner product $\langle f,g\rangle_{L^2(\mathbb{S}^d;\mathcal{H})}:=\int_{\mathbb{S}^d}\langle f(x),g(x)\rangle_{\mathcal{H}}\,\mathrm{d}\sigma(x)$. The operators $f\otimes_{L^2(\mathbb{S}^d)} g$ and $f\otimes_{L^2(\mathbb{S}^d;\mathcal{H})} g$ are defined similarly as before, for $f,g\in L^2(\mathbb{S}^d)$ or $L^2(\mathbb{S}^d;\mathcal{H})$ respectively. Given a positive semi-definite operator B in any of the Hilbert spaces described above, we define $B^{1/2}$ as the unique positive semi-definite operator satisfying $B^{1/2}B^{1/2}=B$. If additionally B is injective, B^{-1} is the (possibly unbounded) operator with domain $D(B^{-1})=\mathrm{Range}(B)$, i.e., $B^{-1}\phi=f\iff Bf=\phi$. For $(\Omega,\mathcal{F},\mathbb{P})$ a probability space, a $\mathcal{H}$-valued random variable F is a measurable map from $(\Omega,\mathcal{F})$ onto $(\mathcal{H},\mathfrak{B}(\mathcal{H}))$, $\mathfrak{B}(\mathcal{H})$ denoting the Borel σ-field of $\mathcal{H}$. The map F is Gaussian if, for all w in $\mathcal{H}$, the real-valued random variable $\langle F,w\rangle_{\mathcal{H}}$ is Gaussian.

We consider a collection $\{Z(x),\ x\in\mathbb{S}^d\}$ of $\mathcal{H}$-valued random variables defined on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$ and such that $\mathbb{E}\|Z(x)\|^2_{\mathcal{H}}<\infty$, for any $x\in\mathbb{S}^d$. We also assume that the mapping $Z:\Omega\times\mathbb{S}^d\to\mathcal{H}$ is *jointly measurable*, i.e., measurable with respect to the product σ-field $\mathfrak{B}(\mathbb{S}^d)\times\mathcal{F}$, $\mathfrak{B}(\mathbb{S}^d)$ denoting the Borel σ-field of $\mathbb{S}^d$. We call $\{Z(x),\ x\in\mathbb{S}^d\}$ *$\mathcal{H}$-valued spherical random field*. The mean element is well-defined as Bochner integral and for simplicity we set it to be the zero element of $\mathcal{H}$ (for more details, see Hsing and Eubank (2015)).

We say that the collection $\{Z(x),\ x\in\mathbb{S}^d\}$ of zero-mean $\mathcal{H}$-valued random variables is geodesically isotropic if $\mathbb{E}\|Z(x)\|^2_{\mathcal{H}}<\infty$, for all $x\in\mathbb{S}^d$, and for any $x,y\in\mathbb{S}^d$ the covariance function $R(x,y):=\mathbb{E}[Z(x)\otimes_{\mathcal{H}}Z(y)]$ depends only on $x\cdot y$. With some abuse of notation, we write $R(x,y)=R(x\cdot y)$.

Under joint measurability with the additional assumption of isotropy, it is readily seen that $\mathbb{E}\left[\int_{\mathbb{S}^d}\|T(x)\|^2_{\mathcal{H}}\mathrm{d}\sigma(x)\right]<\infty$, and hence $Z(\omega,\cdot)$ is an element of $L^2(\mathbb{S}^d;\mathcal{H})$ for $\mathbb{P}$-almost every $\omega\in\Omega$. In particular, the covariance operator $\mathcal{B}:=\mathbb{E}[Z\otimes_{L^2(\mathbb{S}^d;\mathcal{H})}Z]$ is well defined on $L^2(\mathbb{S}^d;\mathcal{H})$.

The spectral theory for such fields is well developed in Caponera (2024), for the case $d=2$. The proofs for general dimension d follow clearly the same steps and we state the result here without proof.

Theorem 1 *Let $\{Z(x),\ x\in\mathbb{S}^d\}$ be an isotropic $\mathcal{H}$-valued spherical random field. Then the following decomposition holds:*

$$Z(x)=\sum_{l=0}^{\infty}\sum_{m=1}^{h(l)}a_{l,m}\mathcal{Y}_{l,m}(x),\qquad x\in\mathbb{S}^d, \tag{4}$$

where the coefficients are $\mathcal{H}$-valued random variables given by the Bochner integral $a_{l,m}:=\int_{\mathbb{S}^d}Z(x)\mathcal{Y}_{l,m}(x)\mathrm{d}\sigma(x)$. Additionally, for all $l,l'\geq 0$, $1\leq m\leq h(l)$ and $1\leq m'\leq h(l')$, we have that

$$\mathbb{E}[a_{l,m}\otimes_{\mathcal{H}}a_{l',m'}]=b_l\delta_{l,l'}\delta_{m,m'}, \tag{5}$$

where $b_l:\mathcal{H}\to\mathcal{H}$ is a positive semi-definite trace class operator. Furthermore, the convergence in (4) occurs in the following sense:

$$\mathbb{E}\left\|Z-\sum_{l=0}^{L}\sum_{m=1}^{h(l)}a_{l,m}\mathcal{Y}_{l,m}\right\|^2_{L^2(\mathbb{S}^d;\mathcal{H})}=\mathbb{E}\left[\int_{\mathbb{S}^d}\left\|Z(x)-\sum_{l=0}^{L}\sum_{m=1}^{h(l)}a_{l,m}\mathcal{Y}_{l,m}(x)\right\|^2_{\mathcal{H}}\mathrm{d}\sigma(x)\right]\xrightarrow[L\to\infty]{}0$$

and

$$\sup_{x\in\mathbb{S}^d}\mathbb{E}\left\|Z(x)-\sum_{l=0}^{L}\sum_{m=1}^{h(l)}a_{l,m}\mathcal{Y}_{l,m}(x)\right\|^2_{\mathcal{H}}\xrightarrow[L\to\infty]{}0. \tag{6}$$

Remark 1 A careful inspection of the proof in Caponera (2024) shows that the result in (6) holds uniformly over $x\in\mathbb{S}^d$, since the expectation is constant on $\mathbb{S}^d$ as a consequence of isotropy. Moreover, it is shown that the operators $\{b_l\}_{l\geq 0}$ have summable trace norms, i.e., $\sum_{l\geq 0}h(l)\|b_l\|_{\mathrm{tr}}<\infty$.

As an immediate consequence of representation (4) and the orthogonality relations (5) we have the decomposition of the covariance function $R(\cdot,\cdot)$:

$$\begin{aligned}R(x,y)&=\mathbb{E}[Z(x)\otimes_{\mathcal{H}}Z(y)]\\&=\sum_{\substack{l\geq 0\\ l'\geq 0}}\sum_{\substack{1\leq m\leq h(l)\\ 1\leq m'\leq h(l')}}\mathbb{E}[a_{l,m}\otimes_{\mathcal{H}}a_{l',m'}]\mathcal{Y}_{l,m}(x)\mathcal{Y}_{l,m}(y)\\&=\sum_{l\geq 0}b_l\sum_{m=1}^{h(l)}\mathcal{Y}_{l,m}(x)\mathcal{Y}_{l,m}(y),\qquad x,y\in\mathbb{S}^d.\end{aligned}\tag{7}$$

Recalling the summation identity for spherical harmonics in terms of Gegenbauer polynomials (Gneiting 201309; Berg and Porcu 2017), we have

$$h(l)\,C_l^{(d-1)/2}(x\cdot y)=\sum_{m=1}^{h(l)}\mathcal{Y}_{l,m}(x)\,\mathcal{Y}_{l,m}(y),\qquad x,y\in\mathbb{S}^d,$$

where the Gegenbauer polynomials are normalized so that $C_l^{(d-1)/2}(1)=1$. In particular, for $d=1$, we set $C_l^0(\cos\theta)=\cos(l\theta),\theta\in[0,\pi]$ (see Schoenberg (194203)). Plugging this into the last identity in (7), the dependence of R on $x\cdot y$ becomes apparent:

$$R(x,y)=\sum_{l\geq 0}h(l)\,b_l\,C_l^{(d-1)/2}(x\cdot y),\qquad x,y\in\mathbb{S}^d. \tag{8}$$

This identity reveals much more, as it is the natural extension of Schoenberg theorem (Schoenberg 194203) to the functional case. Extending the notations from Berg and Porcu (2017) and Daley and Porcu (2014), it seems natural to call the sequence $\{b_l\}_{l\geq 0}$ a d-functional Schoenberg sequence whose elements map $\mathcal{H}$ onto itself and are additionally trace class with summable trace norms.

We now turn our attention into the covariance operator $\mathcal{B}=\mathbb{E}[Z\otimes_{L^2(\mathbb{S}^d;\mathcal{H})}Z]$. Such operator acts on functions $f\in L^2(\mathbb{S}^d;\mathcal{H})$ as

$$\begin{aligned}(\mathcal{B}f)(x)&=\mathbb{E}[\langle f,Z\rangle_{L^2(\mathbb{S}^d;\mathcal{H})}Z](x)\\&=\mathbb{E}\left[\int_{\mathbb{S}^d}\langle f(y),Z(y)\rangle_{\mathcal{H}}\,\mathrm{d}\sigma(y)Z(x)\right]\\&=\int_{\mathbb{S}^d}\mathbb{E}[\langle f(y),Z(y)\rangle_{\mathcal{H}}\,Z(x)]\mathrm{d}\sigma(y)\\&=\int_{\mathbb{S}^d}\mathbb{E}[(Z(x)\otimes_{\mathcal{H}}Z(y))f(y)]\mathrm{d}\sigma(y)\\&=\int_{\mathbb{S}^d}R(x,y)f(y)\mathrm{d}\sigma(y),\qquad x\in\mathbb{S}^d.\end{aligned}\tag{9}$$

The above chain of identities can then be used in concert with (7) to get

$$\begin{aligned}(\mathcal{B}f)(x)&=\int_{\mathbb{S}^d}\sum_{l\geq 0}b_l\sum_{m=1}^{h(l)}\mathcal{Y}_{l,m}(x)\mathcal{Y}_{l,m}(y)f(y)\mathrm{d}\sigma(y)\\&=\sum_{l\geq 0}b_l\sum_{m=1}^{h(l)}\int_{\mathbb{S}^d}\mathcal{Y}_{l,m}(y)f(y)\mathrm{d}\sigma(y)\mathcal{Y}_{l,m}(x)\\&=\sum_{l\geq 0}b_l(P_lf)(x),\qquad x\in\mathbb{S}^d,\end{aligned}\tag{10}$$

where

$$(P_lf)(x)=\sum_{m=1}^{h(l)}\langle f,\mathcal{Y}_{l,m}\rangle_{L^2(\mathbb{S}^d)}\,\mathcal{Y}_{l,m}(x),\qquad x\in\mathbb{S}^d.$$

If $\mathcal{H}=\mathbb{R}$, and hence $f\in L^2(\mathbb{S}^d)$, P_l is simply the orthogonal projection of f onto the space spanned by $\{\mathcal{Y}_{l,m},m=1,\ldots,h(l)\}$. In the general case, we can decompose f with respect to an orthonormal basis of $\mathbb{H}$, so that its coordinates are functions in $L^2(\mathbb{S}^d)$. In this representation, the operator P_l acts componentwise, that is, it applies the same projection onto spherical harmonics of degree l to each coordinate of f.

It is readily seen that $P_lP_{l'}=P_l\delta_{l,l'}$, $P_{l'}\mathcal{Y}_{l,m}=\delta_{l,l'}\mathcal{Y}_{l,m}$, and

$$\|f\|^2_{L^2(\mathbb{S}^d;\mathcal{H})}=\sum_{l\geq 0}\|P_lf\|^2_{L^2(\mathbb{S}^d;\mathcal{H})}\,. \tag{11}$$

Additionally, since $\mathcal{Y}_{l,m}$ are scalar functions, we have $b_lP_lf=P_lb_lf$.

3 Results

3.1 A functional Feldman-Hajek form

Now consider two such random fields $Z^{(j)}$ with associated sequences of operators $\{b_l^{(j)}\}_{l\geq 0}$, $j=1,2$, which we assume to be all strictly positive. We are interested in necessary and sufficient conditions for their respective distributions $\mathbf{P}_1$ and $\mathbf{P}_2$ to be equivalent (one being absolutely continuous with respect to the other) in terms of $\{b_l^{(j)}\}_{l\geq 0}$, $j=1,2$. The result following subsequently extends the condition (3) to the case where Z is a functional Gaussian field as previously described. We anticipate that we provide a constructive proof that is a consequence of Feldman-Hájek Theorem (Feldman 1958; Hájek 1958), which in turn states that the two Gaussian measures are equivalent if and only if $\mathcal{B}_2^{-1/2}\mathcal{B}_1\mathcal{B}_2^{-1/2}-I$ is a Hilbert-Schmidt operator on $L^2(\mathbb{S}^d;\mathcal{H})$. We start by studying the representation for the operator $\mathcal{B}_2^{-1/2}$. In the scalar case ($\mathcal{H}=\mathbb{R}$), for each $l\geq 0$, $b_l\in\mathbb{R}$ is the eigenvalue associated with the eigenspace spanned by $\{\mathcal{Y}_{l,m}, m=1,\dots,h(l)\}$ and for $\alpha\in\mathbb{R}$ we have $B^\alpha f=\sum_{l\geq 0} b_l^\alpha P_l f$, provided that the sum converges in the L^2 norm. Below we show an analogous decomposition holds.

Proposition 2 *Let $\mathcal{B}$ admit a decomposition as in (10), with positive definite operators $\{b_l\}_{l\geq 0}$. Then, the domain of $\mathcal{B}^{-1/2}$ is given by:*

$$D(\mathcal{B}^{-1/2})=\left\{u\in L^2(\mathbb{S}^d;\mathcal{H});\sum_{l\geq 0}\left\|b_l^{-1/2}P_l u\right\|^2_{L^2(\mathbb{S}^d;\mathcal{H})}<\infty\right\},$$

and $\mathcal{B}^{-1/2}$ admits the representation

$$\mathcal{B}^{-1/2}u=\sum_{l\geq 0} b_l^{-1/2}P_l u. \tag{12}$$

Proof We first verify that $\mathcal{B}^{1/2}=\sum_{l\geq 0} b_l^{1/2}P_l$. Indeed, direct inspection proves

$$\Big(\sum_{l\geq 0} b_l^{1/2}P_l\Big)^2=\sum_{l,l'\geq 0} b_l^{1/2}b_{l'}^{1/2}P_lP_{l'}=\sum_{l,l'\geq 0} b_l^{1/2}b_{l'}^{1/2}P_l\delta_{l,l'}=\sum_{l\geq 0} b_lP_l=\mathcal{B}.$$

To characterize $D(\mathcal{B}^{-1/2})$, we first take $u\in D(\mathcal{B}^{-1/2})=\mathrm{Range}(\mathcal{B}^{1/2})$. Thus, there exists an element $f\in L^2(\mathbb{S}^d;\mathcal{H})$ such that

$$u=\mathcal{B}^{1/2}f=\sum_{l\geq 0} b_l^{1/2}P_l f=\sum_{l\geq 0} P_l b_l^{1/2} f.$$

Thus, for any $l\geq 0$, $P_l u=P_l b_l^{1/2} f$, and hence $b_l^{-1/2}P_l u=P_l f$, from which we obtain

$$\sum_{l\geq 0}\left\|b_l^{-1/2}P_l u\right\|^2_{L^2(\mathbb{S}^d;\mathcal{H})}=\sum_{l\geq 0}\|P_l f\|^2_{L^2(\mathbb{S}^d;\mathcal{H})}=\|f\|^2_{L^2(\mathbb{S}^d;\mathcal{H})}<+\infty.$$

Conversely, if $u\in L^2(\mathbb{S}^d;\mathcal{H})$ is such that $\sum_{l\geq 0}\left\|b_l^{-1/2}P_l u\right\|^2_{L^2(\mathbb{S}^d;\mathcal{H})}<+\infty$, then $f:=\sum_{l\geq 0} b_l^{-1/2}P_l u$ is well defined, and using the representation for $\mathcal{B}^{1/2}$ above we get:

$$\begin{aligned}\mathcal{B}^{1/2}f&=\sum_{l'\geq 0} b_{l'}^{1/2}P_{l'}f=\sum_{l,l'\geq 0} b_{l'}^{1/2}b_l^{-1/2}P_{l'}P_l u\\&=\sum_{l,l'\geq 0} b_{l'}^{1/2}b_l^{-1/2}P_l\delta_{l,l'}u\\&=\sum_{l\geq 0} b_l^{1/2}b_l^{-1/2}P_l u=\sum_{l\geq 0}P_l u=u.\end{aligned}$$

Thus $u\in D(\mathcal{B}^{-1/2})$, with $\mathcal{B}^{-1/2}u=f$ satisfying (12), which concludes the proof. □

The characterization above for the operator $\mathcal{B}^{1/2}$ is the crux of the argument for providing equivalence conditions for Gaussian measures associated with $\mathcal{H}$-valued fields. To state the following result, we introduce some additional notation. Throughout, I is the identity operator on $\mathcal{H}$.

Theorem 3 *For $j=1,2$, let $Z^{(j)}$ be an isotropic $\mathcal{H}$-valued Gaussian random field defined on $\mathbb{S}^d$, where its distribution $\mathbf{P}_j$ has covariance operator $\mathcal{B}_j$ with its associated sequence of positive definite operators $\{b_l^{(j)}\}_{l\geq 0}$ according to (10). Then, $\mathbf{P}_1\equiv\mathrm{P}_2$ if and only if:*

$$\sum_{l\geq 0} h(l)\left\|(b_l^{(2)})^{-1/2}b_l^{(1)}(b_l^{(2)})^{-1/2}-I\right\|^2_{\mathcal{L}_2(\mathcal{H})}<+\infty. \tag{13}$$

Proof By combining the representations for $\mathcal{B}_1^{1/2}$ and $\mathcal{B}_2^{-1/2}$ as in Theorem 2, we obtain

$$D:=\mathcal{B}_2^{-1/2}\mathcal{B}_1\mathcal{B}_2^{-1/2}-I=\sum_{l\geq 0}[(b_l^{(2)})^{-1/2}b_l^{(1)}(b_l^{(2)})^{-1/2}-I]P_l. \tag{14}$$

By Feldman-Hájek Theorem, $\mathbf{P}_1\equiv\mathrm{P}_2$ if and only if D is a Hilbert-Schmidt operator on $L^2(\mathbb{S}^d;\mathcal{H})$. To conclude the proof we hence need to verify that the Hilbert-Schmidt norm of D is given by the left hand side of (13).

Let $\psi_{k,l,m}:=e_k\mathcal{Y}_{l,m}\in L^2(\mathbb{S}^d;\mathcal{H})$, where $\{e_k\}_{k\geq 0}$ is an orthonormal basis of $\mathcal{H}$. Since the $\mathcal{Y}_{l,m}$'s form an orthonormal basis of $L^2(\mathbb{S}^d)$, it follows that the $\psi_{k,l,m}$'s form an orthonormal basis of $L^2(\mathbb{S}^d;\mathcal{H})$. Denoting $d_l:=(b_l^{(2)})^{-1/2}b_l^{(1)}(b_l^{(2)})^{-1/2}-I$, we can write:

$$
\begin{aligned}
\|D\|^2_{\mathcal{L}_2(L^2(\mathbb{S}^d;\mathcal{H}))} &= \sum_{k,l,m} \|D\psi_{k,l,m}\|^2_{L^2(\mathbb{S}^d;\mathcal{H})} \\
&= \sum_{k,l,m} \left\| \sum_{l'} (d_{l'} P_{l'})(e_k \mathcal{Y}_{l,m}) \right\|^2_{L^2(\mathbb{S}^d;\mathcal{H})} \\
&= \sum_{k,l,m} \left\| \sum_{l'} (d_{l'} e_k)\delta_{l,l'} \mathcal{Y}_{l,m} \right\|^2_{L^2(\mathbb{S}^d;\mathcal{H})} \qquad (\text{ because } P_{l'}\mathcal{Y}_{l,m} = \delta_{l,l'}\mathcal{Y}_{l,m}) \\
&= \sum_{k,l,m} \|(d_l e_k)\mathcal{Y}_{l,m}\|^2_{L^2(\mathbb{S}^d;\mathcal{H})} \\
&= \sum_{k,l,m} \int_{\mathbb{S}^d} \|(d_l e_k)\mathcal{Y}_{l,m}(x)\|^2_{\mathcal{H}} \, \mathrm{d}\sigma(x) \\
&= \sum_{l,m} \int_{\mathbb{S}^d} \mathcal{Y}_{l,m}(x)^2 \mathrm{d}\sigma(x) \sum_k \|d_l e_k\|^2_{\mathcal{H}} \\
&= \sum_{l\geq 0} \sum_{m=1}^{h(l)} \|d_l\|^2_{\mathcal{L}_2(\mathcal{H})} \quad (\text{using } \|\mathcal{Y}_{l,m}\|_{L^2(\mathbb{S}^d)} = 1 \text{ and the definition of } \|\cdot\|_{\mathcal{L}_2(\mathcal{H})}) \\
&= \sum_{l\geq 0} h(l) \left\| (b_l^{(2)})^{-1/2} b_l^{(1)} (b_l^{(2)})^{-1/2} - I \right\|^2_{\mathcal{L}_2(\mathcal{H})}.
\end{aligned}
\tag{15}
$$

This concludes the proof. □

3.2 Scalar marginalization

Next, we discuss how to relate the problems of equivalence between the vector fields Z_1 and Z_2 with the equivalence of its scalar components $\langle Z_1, u\rangle_{\mathcal{H}}$ and $\langle Z_2, u\rangle_{\mathcal{H}}$, $u \in \mathcal{H}$. For that purpose $Z_u := \langle Z, u\rangle_{\mathcal{H}}$. Its covariance function R_u at any two points $x, y \in \mathbb{R}^d$ is given by

$$
\begin{aligned}
R_u(x, y) &= \mathbb{E}[Z_u(x) Z_u(y)] = \mathbb{E}[\langle Z(x), u\rangle_{\mathcal{H}} \langle u, Z(y)\rangle_{\mathcal{H}}] \\
&= \mathbb{E}[\langle Z(x) \otimes_{\mathcal{H}} Z(y) u, u\rangle_{\mathcal{H}}] \\
&= \langle R(x, y) u, u\rangle_{\mathcal{H}}.
\end{aligned}
$$

The covariance operator $\mathcal{B}_u$ of Z_u acts on scalar functions $f \in L^2(\mathbb{S}^d)$ as

$$
\begin{aligned}
(\mathcal{B}_u f)(x) &= \int_{\mathbb{S}^d} R_u(x, y) f(y) \mathrm{d}\sigma(y) \\
&= \int_{\mathbb{S}^d} \langle R(x, y) u, u\rangle_{\mathcal{H}} f(y) \mathrm{d}\sigma(y) \\
&= \left\langle \int_{\mathbb{S}^d} R(x, y) u f(y) \mathrm{d}\sigma(y), u \right\rangle_{\mathcal{H}} \\
&= \langle \mathcal{B}(uf)(x), u\rangle_{\mathcal{H}}.
\end{aligned}
$$

Using the series representation of $\mathcal{B}$ from (10) we get

$$
\mathcal{B}_u f = \left\langle \sum_{l\geq 0} b_l P_l(uf), u \right\rangle_{\mathcal{H}} = \sum_{l\geq 0} \langle b_l u, u\rangle_{\mathcal{H}} P_l f, \tag{16}
$$

which shows that the scalar Schoenberg coefficients of $\mathcal{B}_u$ are $\langle b_l u, u\rangle_{\mathcal{H}}$, where b_l are the operator valued Schoenberg coefficients of $\mathcal{B}$. Let us now consider a pair of such random fields $Z_u^{(j)}$, $j = 1, 2$ as defined above. By using (16) we get that the operator $D_u := [(\mathcal{B}_u^{(2)})^{1-2} \mathcal{B}_u^{(1)} (\mathcal{B}_u^{(2)})^{1-2} - I]$ can be written as:

$$
D_u = \sum_{l\geq 0} \left(\frac{\left\langle b_l^{(2)} u, u \right\rangle_{\mathcal{H}}}{\left\langle b_l^{(1)} u, u \right\rangle_{\mathcal{H}}} - 1 \right) P_l.
$$

Thus, the necessary and sufficient condition for the distributions $\mathbf{P}_u^{(j)}$ of $Z_u^{(j)}$ to be equivalent is

$$
\sum_{l\geq 0} h(l) \left(\frac{\left\langle b_l^{(2)} u, u \right\rangle_{\mathcal{H}}}{\left\langle b_l^{(1)} u, u \right\rangle_{\mathcal{H}}} - 1 \right)^2 < +\infty. \tag{17}
$$

It is straightforward from the definition of equivalence that if $\mathbf{P}^{(1)} \equiv \mathrm{P}^{(2)}$ then $\mathbf{P}_u^{(1)} \equiv \mathrm{P}_u^{(2)}$ for all $u \in \mathcal{H}$. The opposite is not always true and an example is given in Proposition 6 in Sect. 3.3. However, we prove below that the left hand side of (17) is bounded by the one in (13), which is useful in the next section when identifying parameters that ensure equivalence of measures in some classes of Gaussian fields. First we prove the following inequality.

Proposition 4 *Let A, B be linear operators on $\mathcal{H}$ such that B is a positive definite, self-adjoint and compact. Then, for any $u \in \mathcal{H}$,*

$$
\left| \frac{\langle (A - B)u, u\rangle_{\mathcal{H}}}{\langle Bu, u\rangle_{\mathcal{H}}} \right| \leq \left\| B^{-1/2} A B^{-1/2} - I \right\|_{\mathcal{L}_2(\mathcal{H})}. \tag{18}
$$

Proof Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis of eigenvectors of B, with $Be_n = \lambda_n e_n$. Let $u \in \mathcal{H}$. By writing $u = \sum_{n\geq 0} u_n e_n$, where $u_n = \langle u, e_n\rangle_{\mathcal{H}}$ we easily get

$$
\langle Bu, u\rangle_{\mathcal{H}} = \sum_{n\geq 0} \lambda_n u_n^2. \tag{19}
$$

Next, we bound $\langle (A - B)u, u\rangle_{\mathcal{H}}$ by doing:

$$
\begin{aligned}
\langle (A - B)u, u\rangle_{\mathcal{H}} &= \sum_{m,n} u_m u_n (\langle A e_m, e_n\rangle_{\mathcal{H}} - \langle B e_m, e_n\rangle_{\mathcal{H}}) \\
&= \sum_{m,n} u_m u_n (\langle A e_m, e_n\rangle_{\mathcal{H}} - \left\langle B^{1/2} e_m, B^{1/2} e_n \right\rangle_{\mathcal{H}}) \\
&= \sum_{m,n} u_m u_n (\langle A e_m, e_n\rangle_{\mathcal{H}} - \left\langle \lambda_m^{1/2} e_m, \lambda_n^{1/2} e_n \right\rangle_{\mathcal{H}}) \\
&= \sum_{m,n} \lambda_m^{1/2} u_m \lambda_n^{1/2} u_n (\left\langle A(\lambda_m^{-1/2} e_m), \lambda_n^{-1/2} e_n \right\rangle_{\mathcal{H}} - \langle e_m, e_n\rangle_{\mathcal{H}}) \\
&= \sum_{m,n} \lambda_m^{1/2} u_m \lambda_n^{1/2} u_n (\left\langle A B^{-1/2} e_m, B^{-1/2} e_n \right\rangle_{\mathcal{H}} - \langle e_m, e_n\rangle_{\mathcal{H}}) \\
&= \sum_{m,n} \lambda_m^{1/2} u_m \lambda_n^{1/2} u_n \left\langle (B^{-1/2} A B^{-1/2} - I) e_m, e_n \right\rangle_{\mathcal{H}} \\
&\leq \left(\sum_{m,n} \lambda_m u_m^2 \lambda_n u_n^2 \right)^{1/2} \left(\sum_{m,n} \left\langle (B^{-1/2} A B^{-1/2} - I) e_m, e_n \right\rangle_{\mathcal{H}}^2 \right)^{1/2} \\
&= \langle Bu, u\rangle_{\mathcal{H}} \left\| B^{-1/2} A B^{-1/2} - I \right\|_{\mathcal{L}_2(\mathcal{H})},
\end{aligned}
\tag{20}
$$

where in the identities above we used in the third and fourth lines that $B^{\pm 1/2}u_n = \lambda^{\pm 1/2}u_n$, in the seventh line we used Cauchy-Schwarz for the double sum and in the last line we employed identity (19). □

A direct implication of the result above is the following.

Corollary 4.1 *Let $Z^{(1)}$ and $Z^{(2)}$ be two isotropic $\mathcal{H}$-valued Gaussian random fields defined on $\mathbb{S}^d$ and $\{b_l^{(j)}\}_{l\geq 0}$, $j=1,2$ be their respective Schoenberg operator coefficient sequences. Then:*

$$\sum_{l\geq 0} h(l)\left(\frac{\left\langle b_l^{(2)}u,u\right\rangle_{\mathcal{H}}}{\left\langle b_l^{(1)}u,u\right\rangle_{\mathcal{H}}}-1\right)^2 \leq \sum_{l\geq 0} h(l)\left\|(b_l^{(2)})^{-1/2}b_l^{(1)}(b_l^{(2)})^{-1/2}-I\right\|^2_{\mathcal{L}_2(\mathcal{H})}.$$

Proof For each $l\geq 0$ we apply Proposition 4 to $A=b_l^{(2)}$ and $B=b_l^{(1)}$ to get $\left(\frac{\left\langle b_l^{(2)}u,u\right\rangle_{\mathcal{H}}}{\left\langle b_l^{(1)}u,u\right\rangle_{\mathcal{H}}}-1\right) \leq \left\|(b_l^{(2)})^{-1/2}b_l^{(1)}(b_l^{(2)})^{-1/2}-I\right\|_{\mathcal{L}_2(\mathcal{H})}$ and then perform the sum. □

Some interesting facts can be noted by considering the finite dimensional case. Here we highlight as a particular case of the previous result when we take the Hilbert space to be a finite dimensional Euclidean space: $\mathcal{H}=\mathbb{R}^d$. In this case, the coefficients b_l are positive definite $\mathbb{R}^{d\times d}$ matrices. The Hilbert-Schmidt norm $\mathcal{L}_2(\mathcal{H})$ becomes the familiar Frobenius norm, but since all norms are equivalent in a finite-dimensional space, we can take any of them in the expression (13).

Corollary 4.2 *For $j=1,2$, let $Z^{(j)}$ be isotropic $\mathbb{R}^p$-valued Gaussian random fields defined on $\mathbb{S}^d$ with distributions $\mathbf{P}_j$, whose covariance functions $\mathcal{B}_j$ admit decomposition (10). Then $\mathbf{P}_1\equiv \mathrm{P}_2$ if and only if:*

$$\sum_{l\geq 0} h(l)\left\|(b_l^{(2)})^{-1/2}b_l^{(1)}(b_l^{(2)})^{-1/2}-I\right\|^2 < +\infty, \tag{21}$$

where $\|\cdot\|$ can be any of the equivalent matrix norms.

3.3 Example: multiquadric bivariate model

We illustrate the use of the main results in this work by introducing some bivariate families of random fields and finding the parameter conditions for equivalency of their distributions. The following findings are an extension of the work done for the scalar case in Arafat et al. (2018). We start by discussing the multiquadric family, introduced in Gneiting (201309).

Throughout, we abuse of notation and we write $R(x,y)$, for $R:\mathbb{S}^d\times\mathbb{S}^d\to\mathbb{R}^{2\times 2}$, as $\varphi(\theta)$, for $\varphi:[0,\pi]\to\mathbb{R}^{2\times 2}$ and θ being the arccosine of the inner product between x and y.

Proposition 5 *(Multiquadric bivariate covariance) Let $d\geq 2$ and let $\varphi_M(\theta|\alpha,\rho,\sigma)$ be the matrix valued function with entries:*

$$\begin{aligned}&\varphi_{i,j,M}(\theta|\alpha,\rho,\sigma)\\ &=\rho_{i,j}\sigma_i\sigma_j\frac{(1-\alpha_{i,j})^{d-1}}{\left(1+\alpha_{i,j}^2-2\alpha_{i,j}\cos\theta\right)^{(d-1)/2}}, \qquad i,j=1,2, \quad \theta\in[0,\pi],\end{aligned} \tag{22}$$

where $\rho_{1,1}=\rho_{2,2}=1$, $\rho_{1,2}=\rho_{2,1}\in(0,1)$ and $\alpha_{i,j}\in(0,1)$, $\alpha_{1,2}=\alpha_{2,1}$. If the parameters (α,ρ,σ) satisfy the conditions:

$$\begin{aligned}&\alpha_{1,2}\leq\sqrt{\alpha_{1,1}\alpha_{2,2}},\\ &\rho_{1,2}<\left(\frac{(1-\alpha_{1,1})(1-\alpha_{2,2})}{(1-\alpha_{1,2})^2}\right)^{(d-1)/2},\end{aligned} \tag{23}$$

then the function $R(x,y)=\varphi_M(\theta|\alpha,\rho,\sigma)$ is a valid covariance function for an isotropic Gaussian random field $Z=(Z_1,Z_2)$ on $\mathbb{S}^d$.

Proof From Gneiting (201309); Arafat et al. (2018) we see that the Schoenberg coefficients of R are the 2×2 matrices b_l with entries $b_l(i,j)$, $i,j=1,2$, satisfying

$$h(l)b_l(i,j)=\rho_{i,j}\sigma_i\sigma_j\begin{pmatrix} d+l-2\\ l\end{pmatrix}\alpha_{i,j}^l(1-\alpha_{i,j})^{d-1}. \tag{24}$$

In order for R to be a positive definite covariance function, each b_l must be positive definite. For 2×2 matrices, this is equivalent to requiring $b_l(1,1)>0$ and $\det(b_l)>0$ for all $l\geq 0$. Indeed, $h(l)>0$ and $\sigma_1^2\begin{pmatrix} d+l-2\\ l\end{pmatrix}\alpha_{1,1}^l(1-\alpha_{1,1})^{d-1}>0$, for $d\geq 2$ and $l\geq 0$. Furthermore:

$$\begin{aligned}&\det(b_l)>0\\ &\iff \sigma_1^2\sigma_2^2\left(\alpha_{1,1}^l\alpha_{2,2}^l(1-\alpha_{1,1})^{d-1}(1-\alpha_{2,2})^{d-1}-\rho_{1,2}^2\alpha_{1,2}^{2l}(1-\alpha_{1,2})^{2(d-1)}\right)>0\\ &\iff \rho_{1,2}<\left(\frac{\alpha_{1,1}\alpha_{2,2}}{\alpha_{1,2}^2}\right)^{l/2}\left(\frac{(1-\alpha_{1,1})(1-\alpha_{2,2})}{(1-\alpha_{1,2})^2}\right)^{(d-1)/2}.\end{aligned}$$

Thus we must have $\rho_{1,2}<\inf_{l\geq 0}\left(\frac{\alpha_{1,1}\alpha_{2,2}}{\alpha_{1,2}^2}\right)^{l/2}\left(\frac{(1-\alpha_{1,1})(1-\alpha_{2,2})}{(1-\alpha_{1,2})^2}\right)^{(d-1)/2}$. Observe that if $\alpha_{1,2}^2>\alpha_{1,1}\alpha_{2,2}$, then the infimum is zero and we get $\rho_{1,2}=0$. This is the trivial case where R is diagonal and the corresponding field Z has independent components Z_1 and Z_2, which we are not considering here. Now, if we assume $\alpha_{1,2}^2 \le \alpha_{1,1}\alpha_{2,2}$, then the expression inside the infimum is non-decreasing in l and is minimized at $l = 0$. Therefore $\rho_{1,2} < \left(\frac{(1-\alpha_{1,1})(1-\alpha_{2,2})}{(1-\alpha_{1,2})^2}\right)^{(d-1)/2}$ ensures that all b_l are positive definite, which completes the proof. □

Proposition 6 *Let $Z^{(j)}$, $j = 1, 2$, be two $\mathbb{R}^2$-valued isotropic Gaussian random fields defined on $\mathbb{S}^d$, $d \ge 2$, with respective multiquadric covariance functions $\varphi(\theta|\alpha^{(j)}, \rho^{(j)}, \sigma^{(j)})$ as defined in Proposition 5, which satisfy (23). Then, they have equivalent distributions if and only if one of the following holds:*

- $\sigma^{(1)} = \sigma^{(2)}, \alpha_{i,i}^{(1)} = \alpha_{i,i}^{(2)} =: \alpha_{i,i}$ and $\alpha_{1,2}^{(i)} < \sqrt{\alpha_{1,1}\,\alpha_{2,2}}$, for $i = 1, 2$.
- $(\alpha^{(1)}, \rho^{(1)}, \sigma^{(1)}) = (\alpha^{(2)}, \rho^{(2)}, \sigma^{(2)})$, and $\alpha_{1,2}^{(i)} = \sqrt{\alpha_{1,1}\,\alpha_{2,2}}$ for at least one $i \in \{1, 2\}$.

Proof We see that if $Z^{(1)}, Z^{(2)}$ have equivalent distributions, then Corollary 4.1 implies:

$$\sum_{l\ge 0} h(l)\left(\frac{b_l^{(2)}(i,i)}{b_l^{(1)}(i,i)} - 1\right)^2 < +\infty, \qquad \text{for i=1,2,}$$

that is, the marginals are equivalent. However, in Arafat et al. (2018) it is shown that a necessary and sufficient condition for this sum to be finite is that $\sigma_i^{(1)} = \sigma_i^{(2)}$ and $\alpha_{i,i}^{(1)} = \alpha_{i,i}^{(2)} =: \alpha_{i,i}$, $i = 1, 2$. Under these constraints, $b_l^{(2)}$ can be written in terms of $b_l^{(1)}$ and the difference between the off-diagonal terms, i.e.,

$$b_l^{(2)} = b_l^{(1)} + (b_l^{(2)}(1,2) - b_l^{(1)}(1,2))\overline{I}, \qquad \text{with } \overline{I} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence, we can write

$$\begin{aligned} &\left\| (b_l^{(1)})^{-1/2} b_l^{(2)} (b_l^{(1)})^{-1/2} - I \right\|^2 \\ &= (b_l^{(2)}(1,2) - b_l^{(1)}(1,2))^2 \left\| (b_l^{(1)})^{-1/2}\overline{I}(b_l^{(1)})^{-1/2} \right\|^2. \end{aligned}$$

The matrix $(b_l^{(1)})^{-1/2}\overline{I}(b_l^{(1)})^{-1/2}$ is symmetric and, if we consider the Frobenius norm, we can write

$$\begin{aligned} &\left\| (b_l^{(1)})^{-1/2}\overline{I}(b_l^{(1)})^{-1/2} \right\|^2 \\ &= \operatorname{trace}\left((b_l^{(1)})^{-1/2}\overline{I}(b_l^{(1)})^{-1/2}(b_l^{(1)})^{-1/2}\overline{I}(b_l^{(1)})^{-1/2} \right) \\ &= \operatorname{trace}\left((b_l^{(1)})^{-1}\overline{I}(b_l^{(1)})^{-1}\overline{I} \right) \\ &= 2\frac{b_l^{(1)}(1,1)b_l^{(1)}(2,2) + (b_l^{(1)}(1,2))^2}{(b_l^{(1)}(1,1)b_l^{(1)}(2,2) - (b_l^{(1)}(1,2))^2)^2}, \end{aligned}$$

since

$$(b_l^{(1)})^{-1}\overline{I}(b_l^{(1)})^{-1}\overline{I} = \frac{1}{\left(\det(b_l^{(1)})\right)^2}\begin{bmatrix} -b_l^{(1)}(1,2) & b_l^{(1)}(2,2) \\ b_l^{(1)}(1,1) & -b_l^{(1)}(1,2) \end{bmatrix}\begin{bmatrix} -b_l^{(1)}(1,2) & b_l^{(1)}(2,2) \\ b_l^{(1)}(1,1) & -b_l^{(1)}(1,2) \end{bmatrix},$$

and $\det(b_l^{(1)}) = b_l^{(1)}(1,1)b_l^{(1)}(2,2) - (b_l^{(1)}(1,2))^2 > 0$. Thus,

$$\left\| (b_l^{(1)})^{-1/2} b_l^{(2)} (b_l^{(1)})^{-1/2} - I \right\|^2 = 2(\xi_l^{(2)} - \xi_l^{(1)})^2 \frac{1 + (\xi_l^{(1)})^2}{(1 - (\xi_l^{(1)})^2)^2},$$

with

$$\xi_l^{(j)} := \frac{b_l^{(j)}(1,2)}{\sqrt{b_l^{(j)}(1,1)b_l^{(j)}(2,2)}}, \qquad j = 1, 2.$$

Note that each $\xi_l^{(j)}$ is either a strictly positive constant in l or tends to 0, depending on whether $\alpha_{1,2}^{(j)} = \sqrt{\alpha_{1,1}\,\alpha_{2,2}}$ or $\alpha_{1,2}^{(j)} < \sqrt{\alpha_{1,1}\,\alpha_{2,2}}$, respectively. As a consequence, equivalence holds, i.e.,

$$\sum_{l=0}^{\infty} h(l) \left\| (b_l^{(1)})^{-1/2} b_l^{(2)} (b_l^{(1)})^{-1/2} - I \right\|^2 < +\infty,$$

if and only if $\sigma_i^{(1)} = \sigma_i^{(2)}$ and $\alpha_{i,i}^{(1)} = \alpha_{i,i}^{(2)} =: \alpha_{i,i}$, $i = 1, 2$, and

$$\sum_{l=0}^{\infty} h(l)(\xi_l^{(2)} - \xi_l^{(1)})^2 = \sum_{l=0}^{\infty} h(l)\left(\frac{\xi_l^{(2)}}{\xi_l^{(1)}} - 1\right)^2 (\xi_l^{(1)})^2 < +\infty.$$

If $\alpha_{1,2}^{(1)} = \alpha_{1,2}^{(2)}$ and $\rho_{1,2}^{(1)} = \rho_{1,2}^{(2)}$, clearly the series converges. Moreover, we can exclude the cases in which $\alpha_{1,2}^{(j)} = \sqrt{\alpha_{1,1}\,\alpha_{2,2}}$ for at least one $j \in \{1, 2\}$, since it is readily seen that $(\xi_l^{(2)} - \xi_l^{(1)})^2$ dos not converge to zero, unless $\alpha_{1,2}^{(1)} = \alpha_{1,2}^{(2)}$ and $\rho_{1,2}^{(1)} = \rho_{1,2}^{(2)}$.

Now assume $\alpha_{1,2}^{(1)} < \sqrt{\alpha_{1,1}\,\alpha_{2,2}}$ and $\alpha_{1,2}^{(2)} < \sqrt{\alpha_{1,1}\,\alpha_{2,2}}$, and consider $\alpha_{1,2}^{(1)} \ne \alpha_{1,2}^{(2)}$, regardless of $\rho_{1,2}^{(1)}$ and $\rho_{1,2}^{(2)}$. To study the behavior of the series, we use Raabe's test which reduces to study the following limit

$$L:=\lim_{l\to+\infty} l\left(\frac{h(l)}{h(l+1)}\frac{\left(\frac{\rho^{(2)}_{1,2}}{\rho^{(1)}_{1,2}}\left(\frac{\alpha^{(2)}_{1,2}}{\alpha^{(1)}_{1,2}}\right)^{l}\left(\frac{1-\alpha^{(2)}_{1,2}}{1-\alpha^{(1)}_{1,2}}\right)^{d-1}-1\right)^2}{\left(\frac{\rho^{(2)}_{1,2}}{\rho^{(1)}_{1,2}}\left(\frac{\alpha^{(2)}_{1,2}}{\alpha^{(1)}_{1,2}}\right)^{l+1}\left(\frac{1-\alpha^{(2)}_{1,2}}{1-\alpha^{(1)}_{1,2}}\right)^{d-1}-1\right)^2}\left(\frac{\sqrt{\alpha_{1,1}\alpha_{2,2}}}{\alpha^{(1)}_{1,2}}\right)^2-1\right).$$

Observe that

$$\frac{h(l)}{h(l+1)}=\frac{(2\,l+d-1)(l+1)}{(2\,l+d+1)(l+d-1)}=1-\frac{d-1}{l}+O(l^{-2}),\qquad l\to\infty,$$

and

$$\lim_{l\to\infty}\frac{\left(\frac{\rho^{(2)}_{1,2}}{\rho^{(1)}_{1,2}}\left(\frac{\alpha^{(2)}_{1,2}}{\alpha^{(1)}_{1,2}}\right)^{l}\left(\frac{1-\alpha^{(2)}_{1,2}}{1-\alpha^{(1)}_{1,2}}\right)^{d-1}-1\right)^2}{\left(\frac{\rho^{(2)}_{1,2}}{\rho^{(1)}_{1,2}}\left(\frac{\alpha^{(2)}_{1,2}}{\alpha^{(1)}_{1,2}}\right)^{l+1}\left(\frac{1-\alpha^{(2)}_{1,2}}{1-\alpha^{(1)}_{1,2}}\right)^{d-1}-1\right)^2}\left(\frac{\sqrt{\alpha_{1,1}\alpha_{2,2}}}{\alpha^{(1)}_{1,2}}\right)^2$$
$$=\begin{cases}\left(\frac{\sqrt{\alpha_{1,1}\alpha_{2,2}}}{\alpha^{(1)}_{1,2}}\right)^2 & amp;\text{if } \alpha^{(2)}_{1,2} \quad lt;\alpha^{(1)}_{1,2}\\ \left(\frac{\sqrt{\alpha_{1,1}\alpha_{2,2}}}{\alpha^{(2)}_{1,2}}\right)^2 & amp;\text{if } \alpha^{(1)}_{1,2} \quad lt;\alpha^{(2)}_{1,2}\end{cases}.$$

Hence, $L=+\infty$ and the series converges. Similarly it holds for the case $\alpha^{(1)}_{1,2}=\alpha^{(2)}_{1,2}$ and $\rho^{(1)}_{1,2}\neq\rho^{(2)}_{1,2}$, and the proof is concluded. □

3.4 Example: Legendre-Matérn model

We consider now the use of the results on an infinite-dimensional example. For this purpose, let $L^2([0,1])$ be the space of square-integrable periodic functions on [0, 1]. We choose then the $Z^{(j)}$'s to be isotropic $L^2([0,1])$-valued random fields, i.e., $\mathcal{H}=L^2([0,1])$.

We additionally model the coefficients $a^{(j)}_{l,m}$'s as real stationary processes on $L^2([0,1])$, in order to obtain the following expansion for the corresponding covariance operators

$$b^{(j)}_l=\sum_{k=-\infty}^{\infty}\gamma^{(j)}_{l,k}\,e^{i2\pi k\cdot}\otimes e^{i2\pi k\cdot}.$$

The $\gamma^{(j)}_{l,k}$'s are real numbers such that $\gamma^{(j)}_{l,k}=\gamma^{(j)}_{l,-k}$,

$$\gamma^{(j)}_{l,k}>0,\qquad \sum_{l\geq0}h(l)\sum_{k=-\infty}^{\infty}\gamma^{(j)}_{l,k}$$
$$=\sum_{l\geq0}h(l)\left(\gamma^{(j)}_{l,0}+2\sum_{k=1}^{\infty}\gamma^{(j)}_{l,k}\right)<\infty.$$

Then, the equivalence condition is given by

$$\sum_{l\geq0}\sum_{k\geq0}h(l)\left(\frac{\gamma^{(2)}_{l,k}}{\gamma^{(1)}_{l,k}}-1\right)^2<+\infty.$$

We restrict our attention to the 2-dimensional sphere $\mathbb{S}^2$ and consider an extension of the Legendre-Matérn covariance function (see Guinness and Fuentes (2016)) by specifying

$$h(l)\gamma_{l,k}=\frac{\sigma^2}{(\alpha+k^2+l^2)^{\nu+1/2}} \tag{25}$$

with $\alpha,\nu,\sigma>0$.

Proposition 7 *Let $Z^{(j)}$, $j=1,2$, be two isotropic $L^2([0,1])$-valued Gaussian random fields on $\mathbb{S}^2$ with Legendre-Matérn coefficients (25) of parameters $\alpha^{(j)},\nu^{(j)},\sigma^{(j)}>0$, $j=1,2$. Then, they have equivalent distributions if and only if $\sigma^{(1)}=\sigma^{(2)}$ and $\nu^{(1)}=\nu^{(2)}$.*

Proof The equivalence condition for $Z^{(1)}$ and $Z^{(2)}$ under the Legendre-Matérn model is given by

$$\sum_{l\geq0}\sum_{k\geq0}h(l)\left(\left(\frac{\sigma^{(2)}}{\sigma^{(1)}}\right)^2\frac{(\alpha^{(1)}+k^2+l^2)^{\nu^{(1)}+1/2}}{(\alpha^{(2)}+k^2+l^2)^{\nu^{(2)}+1/2}}-1\right)^2<\infty.$$

Clearly, if $\sigma^{(1)}=\sigma^{(2)},\ \alpha^{(1)}=\alpha^{(2)},\ \nu^{(1)}=\nu^{(2)}$, the series converges. When $\nu^{(1)}\neq\nu^{(2)}$, the series diverges: indeed, by considering for instance the case $k=0$, we observe that

$$\left(\frac{\gamma^{(2)}_{l,0}}{\gamma^{(1)}_{l,0}}-1\right)^2=\left(\left(\frac{\sigma^{(2)}}{\sigma^{(1)}}\right)^2\frac{(\alpha^{(1)}+l^2)^{\nu^{(1)}+1/2}}{(\alpha^{(2)}+l^2)^{\nu^{(2)}+1/2}}-1\right)^2$$

does not converge to 0 as $l\to\infty$. Similarly, if $\nu^{(1)}=\nu^{(2)}$ and $\sigma^{(1)}\neq\sigma^{(2)}$.

As a consequence, we restrict our attention to the the case $\sigma^{(1)}=\sigma^{(2)}$ and $\nu^{(1)}=\nu^{(2)}=:\nu$, with $\alpha^{(1)}\neq\alpha^{(2)}$, which gives

$$\frac{\gamma^{(2)}_{l,k}}{\gamma^{(1)}_{l,k}}=\left(\frac{\alpha^{(1)}+k^2+l^2}{\alpha^{(2)}+k^2+l^2}\right)^{\nu+1/2}=\left(1+\frac{\alpha^{(1)}-\alpha^{(2)}}{\alpha^{(2)}+k^2+l^2}\right)^{\nu+1/2}.$$

For l and k large enough, we can use the Binomial/Taylor expansion and get

$$\left(1+\frac{\alpha^{(1)}-\alpha^{(2)}}{\alpha^{(2)}+k^2+l^2}\right)^{\nu+1/2}$$
$$=1+(\nu+1/2)\frac{\alpha^{(1)}-\alpha^{(2)}}{\alpha^{(2)}+k^2+l^2}+O((k^2+l^2)^{-2}).$$

Hence, we have

$$\left(\left(\frac{\alpha^{(1)}+k^2+l^2}{\alpha^{(2)}+k^2+l^2}\right)^{\nu+1/2}-1\right)^2$$
$$=(\nu+1/2)^2\frac{(\alpha^{(1)}-\alpha^{(2)})^2}{(k^2+l^2)^2}+O((k^2+l^2)^{-3})$$
$$=O((k^4+l^4)^{-1}).$$

Since $h(l)=2l+1$, the double series is finite and the proof is concluded.

□

4 Conclusions

This paper develops a comprehensive framework for the study of functional Gaussian random fields on hyperspheres, extending classical results on scalar and vector-valued fields to the infinite-dimensional setting. By introducing an operator-valued analogue of the Schoenberg representation, we have characterized the covariance structure of Hilbert-valued isotropic spherical fields in terms of trace-class d-Schoenberg sequences. The provided representation sets a natural harmonic-analytic decomposition of the field and provides the background for studying the probabilistic properties in terms of equivalence of Gaussian measures.

The main theoretical contribution of this paper is the functional version of the Feldman–Hájek criterion for Gaussian measures on functional spaces. The resulting characterization establishes that equivalence of Gaussian measures is governed by a square-integrability condition involving the operator-valued Schoenberg coefficients. This condition is the exact infinite-dimensional analogue of previously known results for scalar fields and yields a precise spectral description of when two functional spherical models generate mutually absolutely continuous laws.

Our findings have then been supported through marginalization, for which we show that the functional criterion controls the equivalence of all scalar projections of the field. This provides important intuition: although the functional field lives in an infinite-dimensional space, its measure-theoretic behavior is encoded in a structured family of scalar Schoenberg coefficients obtained through directional evaluations. The inequalities we derive demonstrate that the functional criterion dominates its scalar counterparts, ensuring consistency across all one-dimensional projections. The examples we present illustrate the practical consequences of the theory. For multiquadric bivariate models, the equivalence conditions reduce to sharp constraints on the parameters governing cross-correlation and angular dependence. For the infinite-dimensional Legendre–Matérn family, the spectral characterization leads to a clear description of the role of smoothness and scale parameters in determining equivalence. These examples show that the abstract theory yields concrete and interpretable results for widely used covariance families. Beyond the theoretical contribution, the results have implications for several application domains. In spatial and spatio-temporal statistics, the equivalence framework is central to understanding asymptotics under infill sampling and the robustness of kriging under covariance misspecification. In functional data analysis, these results provide a basic foundation for modeling Hilbert-valued spherical data such as climate model fields, remote sensing images, neural activity patterns on spherical domains, and manifold-valued machine learning kernels. More broadly, the operator-valued Schoenberg framework opens new research avenues for the study of Gaussian processes on non-Euclidean and high-dimensional structures, particularly in the context of reproducing kernel Hilbert spaces and operator-valued kernels on manifolds. Future work may explore extension to non-Gaussian fields, the characterization of equivalence under space–time or product manifolds, connections with Bayesian inverse problems on spheres, and the design of statistical procedures that explicitly exploit the operator-valued structure described here. The theory developed in this paper provides a rigorous mathematical foundation for such developments and, we hope, will stimulate further research at the interface of probability, functional analysis, and spatial statistics.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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