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Weighted average-convexity and Shapley values ☆

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ABSTRACT

We generalize the notion of convexity and average-convexity to the notion of weighted average-convexity. We show several results on the relation between weighted average-convexity and cooperative games. Our main result is that if a game is weighted average-convex, then the corresponding weighted Shapley value is in the core.

1. Introduction

We consider a situation where a set of players $N = \{1, \ldots, n\}$ can cooperate to generate some profit. It is usually assumed that any coalition $S \subseteq N$ of players is feasible and that the profit generated by the players in S can be freely distributed among them. This kind of situation corresponds to a cooperative game with transferable utility. One of the key questions in this framework is to allocate to each player his share of the profit generated by the grand coalition N. Any share of the total worth of N between the n players is called a solution of the game.

Lots of solution concepts have been introduced and they can be divided in two different families: single-valued solutions and set-valued solutions. The best-known singled-valued solution in cooperative game theory is the Shapley value introduced and characterized in Shapley (1953b). The Shapley value allocates to each player its average marginal contribution and is always defined. A typical set valued solution is the core (Gillies (1959)). The core is the set of allocations that are efficient and coalitionally rational, two particularly desirable properties in the context of cooperative games. An allocation is efficient if it fully distributes the worth of the grand coalition N among the n players. Coalitional rationality requires that for every coalition S the total payoff obtained by S is at least equal to the profit that it can generate by itself. Hence no coalition can have an incentive to leave the grand coalition N. Unfortunately, the core of a game may be empty but some properties of the game can guarantee the non-emptiness of the core and more specifically that the Shapley value lies in the core.

Shapley (1971) introduced the class of convex games. In a convex game, the marginal contribution of a player is an increasing function with respect to the coalition size. Shapley (1971) showed that if a game is convex, then the Shapley value belongs to the core and therefore the core is non-empty. In fact, the convexity assumption is very strong and can be relaxed to ensure that the Shapley value is a core element. Iñarra and Usategui (1993) introduced a weaker notion than convexity, called average convexity and proved that it is a sufficient condition for the Shapley value to be in the core. In the

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average convexity, one compares averages of marginal contributions instead of comparing directly marginal contributions. More precisely, for any coalition the sum of the marginal contributions of the players belonging to this coalition has to be less or equal to the sum of the marginal contributions of the same players in a larger coalition. Independently, Sprumont (1990) established a result similar to the one of Iñarra and Usategui (1993) but in a different framework as he investigated population monotonic allocation schemes (PMAS). In particular, average convexity is also a sufficient condition for the existence of a PMAS.

By definition, the Shapley value is symmetric between the players. But many situations like for example players representing groups of individuals or players having different bargaining powers are naturally asymmetric. Shapley (1953a) extended the notion of Shapley value to weighted Shapley value by considering asymmetric positive weights on the players. Even more general, Kalai and Samet (1987) introduced the notion of weight system introducing different priorities on the players and provided an axiomatic characterization of the weighted Shapley values. Further results and characterizations of the weighted Shapley values were proven in Hart and Mas-Colell (1989). Monderer et al. (1992) proved that if the game is convex, then the weighted Shapley values are in the core (for any weight system). Moreover, they showed that for any game one can obtain all the elements of the core as weighted Shapley values by varying the weight system.

The main goal of this article is to establish given a weight system a weaker sufficient condition than convexity to ensure that the weighted Shapley value belongs to the core. To do so, we introduce the notion of weighted average-convexity generalizing the average-convexity of Iñarra and Usategui (1993). We prove then that it is indeed a sufficient condition.

The outline of the article is the following. Section 2 provides classical definitions in cooperative game theory, including several equivalent definitions of the Shapley value and sufficient conditions for it to be in the core. In Section 3, we first recall the definition of the weighted Shapley value and introduce the notion of weighted average-convexity. Then we prove our main result that weighted average-convexity is a sufficient condition to ensure that the weighted Shapley value belongs to the core.

2. TU-games and classical Shapley value

2.1. TU-games

A *TU-game* is defined by a set of players N and a characteristic function v from 2^N to $\mathbb{R}$ that satisfies $v(\emptyset) = 0$. An *allocation* is a vector $x \in \mathbb{R}^N$ that represents the respective payoff of each player. It is said to be *efficient* if

$$\sum_{i \in N} x_i = v(N),$$

and *individually rational* if every player receives at least the value that he could guarantee alone

$$\forall i \in N,\ x_i \geqslant v(\{i\}).$$

This last property can be extended to coalitions. An allocation is *coalitionally rational* if for every coalition the sum of payoffs of its members is at least the value of the coalition

$$\forall S \subseteq N,\ \sum_{i \in S} x_i \geqslant v(S).$$

The *core* is the set of efficient and coalitionally rational allocations. Formally,

$$\mathcal{C}(v) = \left\{ x \in \mathbb{R}^N,\ \sum_{i \in N} x_i = v(N),\ \sum_{i \in S} x_i \geqslant v(S), \forall S \subset N \right\}.$$

A player i in N is a *null player* if $v(S \cup \{i\}) = v(S)$ for every coalition $S \subseteq N \setminus \{i\}$.

A game (N, v) is *superadditive* if, for all $A, B \in 2^N$ such that $A \cap B = \emptyset$, $v(A \cup B) \geqslant v(A) + v(B)$.

For any given subset $\emptyset \neq S \subseteq N$, the unanimity game (N, u_S) is defined by

$$u_S(A) = \begin{cases} 1 & \text{if } A \supseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider a game (N, v). For arbitrary subsets A and B of N, we define the value

$$\Delta v(A, B) := v(A \cup B) + v(A \cap B) - v(A) - v(B).$$

A game (N, v) is *convex* if its characteristic function v is supermodular, *i.e.*, $\Delta v(A, B) \geqslant 0$ for all $A, B \in 2^N$, or equivalently, for all $i \in N$ and all $S \subseteq T \subseteq N$ with $i \in S$

$$v(S) - v(S \setminus \{i\}) \leqslant v(T) - v(T \setminus \{i\}).$$

Shapley (1953b) proved that every cooperative game (N, v) can be written as a unique linear combination of unanimity games, $v = \sum_{S \subseteq N} \lambda_S(v) u_S$, where $\lambda_\emptyset(v) = 0$, and for any $S \neq \emptyset$ the coefficients $\lambda_S(v) \in \mathbb{R}$ are given by $\lambda_S(v) = \sum_{T \subseteq S} (-1)^{s-t} v(T)$. We refer to these coefficients $\lambda_S(v)$ as the *unanimity coefficients*[1] of v.

2.2. Shapley value

The Shapley value of (N, v) is a solution concept. In terms of the unanimity coefficients the Shapley value is given by

$$\Phi_i(v) = \sum_{S \subseteq N : i \in S} \frac{1}{s} \lambda_S(v),$$

for all $i \in N$. It is equivalent to say that the Shapley value associated to a game (N, v) is a linear function and that the allocation associated to the unanimity game of set S is

$$x_i = \begin{cases} \frac{1}{s}, & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the agents in S share the utility of 1 in equal shares.

Another definition is based on the following story. Consider that all the players are in the room. At every stage, one player is chosen uniformly among the remaining players and leaves the room. When leaving, he obtains as payoff its marginal contribution between when he was present and when he was not anymore. The Shapley value is its expected payoff. Formally, let $\mathcal{L}$ be the set of ordered sequences, then define $\mathcal{U}$ to be the uniform distribution over $\mathcal{L}$. Then

$$\Phi_i(v) = \mathbb{E}_{\mathcal{U}} \left(v(\{L \geqslant i\}) - v(\{L > i\}) \right),$$

where $\{L \geqslant i\}$ (resp. $\{L > i\}$) is the set of agents (resp. strictly) after i in the list L, i.e. $\{j \in N, L(j) \geqslant L(i)\}$ (resp. $\{j \in N, L(j) > L(i)\}$).

Let us justify this alternative definition. First, the right-hand side is indeed a linear function of v. Second, let us consider the unanimity game (N, u_S). Given an order L, we define the pivot of S the first occurence of an element of S. Formally, it is the element i of S such that $L(i)$ is minimal. By definition of u_S, the marginal contribution of an agent is equal to 1 if and only if this agent is a pivot. Hence

$$\mathbb{E}_{\mathcal{U}} \left(u_S(\{L \geqslant i\}) - u_S(\{L > i\}) \right) = \mathbb{P}_{\mathcal{U}} \left(i \text{ is pivot of } S \right) = \frac{1}{s}$$

By symmetry of $\mathcal{U}$, the probability for i to be the pivot of S is $\frac{1}{s}$ giving the result.

Shapley (1953b) showed that the Shapley value of a convex game belongs to the core. Looking for a weaker condition insuring that the Shapley value of a game lies in the core, Iñarra and Usategui (1993) relaxed the convexity assumption by introducing the notion of average convexity.

Definition 1. The game (N, v) is *average convex* if for every $S \subset T \subseteq N$,

$$\sum_{i \in S} (v(S) - v(S \setminus \{i\})) \leqslant \sum_{i \in S} (v(T) - v(T \setminus \{i\})).$$

They obtained the following result.

Proposition 1 *(Iñarra and Usategui (1993); Sprumont (1990)[2]). If the game is average convex then the Shapley value is in the core.*

3. Weighted Shapley value and weighted convexity

3.1. Weighted Shapley value

The notion of Shapley value was then extended in Shapley (1953a) and in Kalai and Samet (1987) to weighted Shapley value. In the first one, the weight of each player is strictly positive whereas in the second one, a player may have a null weight. It is then necessary to define a weight system to describe how the players with a zero weight would share their contributions when only them are present.

[1] The λ_S's are also called Harsanyi dividends in cooperative game theory (Harsanyi, 1963).

[2] Sprumont (1990) established the same result. Average convex games are called quasiconvex in Sprumont's work.

Formally, a weight system is a pair (ω, Σ) where $\omega \in \mathbb{R}^N_{++}$ and $\Sigma = (N_1, ..., N_m)$ is an ordered partition of N. It is called simple if Σ is equal to the singleton N. m will be the size of the partition. Given an element i in N, we define its priority denoted by $p(i)$ as the unique integer such that $i \in N_{p(i)}$. Given a set S, we define the priority of S, denoted by $p(S)$, as the largest $k \in \{1, \cdots, m\}$ such that $N_k \cap S \neq \emptyset$. Notice that this coincides with the previous definition if S is a singleton. Moreover, priority is a nondecreasing mapping. We also define $\overline{S}$ as the elements in S with the highest priority, i.e., $\overline{S} = \{i \in S, p(i) = p(S)\}$.

Definition 2. The weighted Shapley value with weight system (ω, Σ) is the unique function from the set of TU-games to allocation such that

- it is linear,
- the allocation of the unanimity game on the set S is defined as follows: for all $i \in N$,

$$x_i = \begin{cases} \frac{\omega_i}{\sum_{i \in \overline{S}} \omega_i}, & \text{if } i \in \overline{S}, \\ 0 & \text{otherwise.} \end{cases}$$

In the previous definition, it is interesting to distinguish three sets of agents: $\overline{S}$, $S - \overline{S}$ and $N - S$. In the unanimity game with set S,

- agents in $N - S$ are not contributing to obtain a positive payoff, hence they obtain 0,
- agents in $S - \overline{S}$ are contributing to obtain a positive payoff but they have low priority, hence they obtain 0,
- agents in $\overline{S}$ are contributing to obtain a positive payoff and have the highest priority inside S, hence they share the total value of 1.

It is convenient for the following results to define the relative weight of the agents in the unanimity game on S. Given a set S, one defines for all $i \in N$

$$\overline{\omega}_i^S = \begin{cases} \omega_i \text{ if } i \in \overline{S}, \\ 0 \text{ if } i \in S \setminus \overline{S}, \\ 0 \text{ if } i \notin S. \end{cases}$$

We also define $\overline{\omega}^S$ as the sum for all $i \in S$ of $\overline{\omega}_i^S$. Notice that it is equal to the sum of the weights of elements in $\overline{S}$. Moreover, the unanimity games on S and $\overline{S}$ lead to the same allocation vector.

Using the decomposition of a game into unanimity games, we obtain that the (ω, Σ)-weighted Shapley value Φ^ω of a game (N, v) is defined for all $i \in N$ by

$$\Phi_i^\omega(v) = \sum_{S \subseteq N : i \in S} \frac{\overline{\omega}_i^S}{\overline{\omega}^S} \lambda_S(v) = \sum_{S \subseteq N : i \in \overline{S}} \frac{\omega_i}{\overline{\omega}^S} \lambda_S(v).$$

The interesting case is if i is in S and not in $\overline{S}$. It means that the priority of i is too low hence, he does not obtain a contribution.

Similarly to the classical Shapley value, a third interpretation can be given to the weighted Shapley value. Recall that $\mathcal{L}$ is the set of ordered sequences on N. Let $\mathbb{P}$ be a probability over $\mathcal{L}$, one can define an allocation as follows: let

$$\Psi_i^{\mathbb{P}}(v) = \mathbb{E}_{\mathbb{P}}\Big(v\big(\{L \geqslant i\}\big) - v\big(\{L > i\}\big)\Big).$$

When $\mathbb{P}$ is the uniform distribution, one obtains the classical Shapley value.

Let us now define a probability distribution $\mathbb{P}_{\omega,\Sigma}$ generating the (ω, Σ)-weighted Shapley value. We define it by induction:

- Let $\overline{N}$ be the set of agents with highest priority in N.
- Then $L(1)$ is equal to i with probability $\overline{\omega}_i^N / \overline{\omega}^N$. In particular, it is strictly positive if and only if i has highest priority or equivalently $i \in \overline{N}$,
- Given $L(1)$ fixed, $(L(2), ..., L(N))$ are defined following the distribution associated to the projection of (ω, Σ) to $N \setminus \{L(1)\}$.

In particular by construction, if $i, j \in N$ such that $p(i) > p(j)$ then the probability under $\mathbb{P}$ of an order L where j appears before i in the order is equal to 0.

Proposition 2. *Given N and $\mathbb{P}_{\omega,\Sigma}$, then $\Psi^{\mathbb{P}_{\omega,\Sigma}}(v)$ is equal to the (ω, Σ)-weighted Shapley value Φ^ω.*

Proof. Proof in Appendix A. □

3.2. Weighted-convexity

We now extend the notion of average convexity by introducing weights.

Definition 3. Let (ω, Σ) be a weight system, we say that the game is (ω, Σ)-convex if for every $S \subset T \subseteq N$,

$$\left(\sum_{i\in S} \overline{\omega_i}^T \left(v(T) - v(T \setminus \{i\}) - v(S) + v(S \setminus \{i\})\right)\right) \geqslant 0 \tag{1}$$

In particular, notice that if $p(S) < p(T)$, then for every $i \in S$, $\overline{\omega}_i^T = 0$ and the corresponding inequality is satisfied by any game. If $\Sigma = \{N\}$ and if all weights are equal, then (ω, Σ)-convexity corresponds to average-convexity.

Remark 1. Let us comment the link between convexity and (ω, Σ)-convexity. It is clear that if a game is convex then it is (ω, Σ)-convex for any weight system (ω, Σ). Moreover, one can observe that the converse is true. Let us consider a game that is (ω, Σ)-convex for any weight system (ω, Σ). For every $i \in N$, Equation (1) applied with $\omega = \{1, \ldots, 1\}$ and $\Sigma = (N \setminus \{i\}, \{i\})$ yields

$$(v(T) - v(T \setminus \{i\}) - v(S) + v(S \setminus \{i\})) \geqslant 0,$$

for every $S \subset T \subseteq N$ with $i \in S$. This corresponds to the definition of convexity.

It is well known that average-convexity implies superadditivity (Iñarra and Usategui (1993)). We prove that (ω, Σ)-convexity also implies superadditivity for any weight system.

Proposition 3. *Let (ω, Σ) be a weight system. Let (N, v) be an (ω, Σ)-convex game, then v is superadditive.*

Proof. Proof in Appendix B. □

3.3. Results

One obtains the following theorem which extends the result of Iñarra and Usategui (1993).

Theorem 1. *Let (ω, Σ) be a weight system. If the game is (ω, Σ)-convex then its (ω, Σ)-weighted Shapley value is in the core.*

Let us insist that the previous theorem is only stating a sufficient condition. It is even possible as shown in the next example that a game is (ω, Σ)-convex for some weight system, is not (ω', Σ')-convex for another weight system while both corresponding weighted Shapley values are in the core.

Example 1. We consider the following game (N, v) where $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 2 & \text{if } S = N, \\ 3/2 & \text{if } S = \{1, 2\}, \\ 1 & \text{if } S = \{2, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$

One can check that this game is average convex and therefore the Shapley value is in the core. On the contrary, let (ω, Σ) be equal to $((8, 1, 1), (N))$. The game is not (ω, Σ)-convex. Indeed, if one considers the sets $\{1, 2\} \subset N$, then

$$\begin{aligned} &8\left(v(N) - v(\{2,3\}) - v(\{1,2\}) + v(\{2\})\right) + \left(v(N) - v(\{1,3\}) - v(\{1,2\}) + v(\{1\})\right), \\ &= 8 * (2 - 1 - 3/2 + 0) + (2 - 0 - 3/2 + 0), \\ &= -4 + 1/2 < 0. \end{aligned}$$

Hence Equation (1) is not statisfied and the game is not (ω, Σ)-average convex. Nevertheless, the (ω, Σ)-weighted Shapley value is such that

$$\Phi_1^{\omega}(v) = 56/60,$$
$$\Phi_2^{\omega}(v) = 37/60,$$
$$\Phi_3^{\omega}(v) = 27/60,$$

and, therefore, the (ω, Σ)-weighted Shapley value is in the core.

The first part of the proof is inspired from (Sprumont, 1990). Let (N, v) be a given game. For any non-empty coalition $T \subseteq N$, we denote by v^T the subgame of v induced by T, *i.e.*, $v^T(S) = v(S)$ for any $S \subseteq T$. Let $\Phi^{\omega}(v^T)$ be the (ω, Σ)-weighted Shapley value of the subgame v^T. Define the vector $\Psi^{\omega} = (\Psi_{iT}^{\omega})_{i\in T, T\in\mathcal{P}(N)}$ recursively by

$$\Psi_{iT}^{\omega} = \frac{\overline{\omega}_i^T}{\overline{\omega}^T}(v(T) - v(T \setminus \{i\})) + \sum_{j\in T\setminus\{i\}} \frac{\overline{\omega}_j^T}{\overline{\omega}^T}\Psi_{iT\setminus\{j\}}^{\omega},$$

for all $i \in T$, $T \in \mathcal{P}(N)$, and setting $\Psi_{i\emptyset}^{\omega} = 0$ for all $i \in N$.

Proposition 4. *For all $T \in \mathcal{P}(N)$, $(\Psi_{iT}^{\omega})_{i\in T} = \Phi^{\omega}(v^T)$.*

Proof. Proof in Appendix C. □

As a consequence, one obtains the following result.

Proposition 5. *Let us denote by Φ_T^{ω} the (ω, Σ)-weighted Shapley value of v^T. We have*

$$\Phi_{iT}^{\omega} = \frac{\overline{\omega}_i^T}{\overline{\omega}^T}(v(T) - v(T \setminus \{i\})) + \sum_{j\in T\setminus\{i\}} \frac{\overline{\omega}_j^T}{\overline{\omega}^T}\Phi_{iT\setminus\{j\}}^{\omega},$$

for all $i \in T$.

Let us now prove the main result.

Proof of Theorem 1. We do a proof by induction. If $n = 1$, the result is true. Let $n > 1$ and assume that for all (ω, Σ)-convex games of size $n - 1$, the result is true. Let us consider an (ω, Σ)-convex game v of size n. We check the condition for Φ^{ω} to be in the core. Let T be a subset of N. By Proposition 4 and by definition of Ψ^{ω}, we get

$$\begin{aligned}\sum_{i\in T}\Phi_i^{\omega}(v) &= \sum_{i\in T}\Psi_{iN}^{\omega},\\ &= \sum_{i\in T}\left(\frac{\overline{\omega}_i^N}{\overline{\omega}^N}(v(N) - v(N\setminus\{i\})) + \sum_{k\in N,\, k\neq i}\frac{\overline{\omega}_k^N}{\overline{\omega}^N}\Psi_{iN\setminus k}^{\omega}\right),\\ &= \frac{1}{\overline{\omega}^N}\left(\sum_{i\in T}\overline{\omega}_i^N(v(N) - v(N\setminus\{i\})) + \sum_{k\in N}\overline{\omega}_k^N\left(\sum_{i\neq k,\, i\in T}\Psi_{iN\setminus k}^{\omega}\right)\right).\end{aligned} \tag{2}$$

By Proposition 4, we have $\Phi^{\omega}(v^{N\setminus k}) = (\Psi_{iN\setminus k}^{\omega})_{i\in N\setminus k}$ and by the induction assumption, $\Phi^{\omega}(v^{N\setminus k})$ belongs to the core of $v^{N\setminus k}$. We distinguish two cases for the sum $\sum_{i\neq k,\, i\in T}\Psi_{iN\setminus k}^{\omega}$:

- If $k \in T$, then one sum for all $i \in T \setminus k$ and by induction assumption

$$\sum_{i\neq k,\, i\in T}\Psi_{iN\setminus k}^{\omega} \geqslant v(T\setminus k). \tag{3}$$

  Let us note that (3) is satisfied at equality for $T = N$.
- If $k \notin T$, then one sum for all $i \in T$ and by induction assumption

$$\sum_{i\neq k,\, i\in T}\Psi_{iN\setminus k}^{\omega} \geqslant v(T). \tag{4}$$

If $T=N$, then (2) and (3) imply $\sum_{i\in N}\Phi_i^{\omega}(v)=v(N)$. Let now T be a strict subset of N. By (2), (3), and (4), one obtains

$$\sum_{i\in T}\Phi_i^{\omega}(v)\geqslant\frac{1}{\overline{\omega}^N}\left(\sum_{i\in T}\overline{\omega}_i^N(v(N)-v(N\setminus\{i\}))+\sum_{k\in T}\overline{\omega}_k^N v(T\setminus k)+\sum_{k\in N,\,k\notin T}\overline{\omega}_k^N v(T)\right).$$

Finally, the assumption of (ω,Σ)-convexity implies

$$\sum_{i\in T}\Phi_i^{\omega}(v)\geqslant\frac{1}{\overline{\omega}^N}\left(\sum_{i\in T}\overline{\omega}_i^N(v(T)-v(T\setminus\{i\}))+\sum_{k\in T}\overline{\omega}_k^N v(T\setminus k)+\sum_{k\in N,\,k\notin T}\overline{\omega}_k^N v(T)\right).$$

We now see that the terms $v(T\setminus\{i\})$ are compensated by $v(T\setminus\{k\})$, whereas the terms $v(T)$ are accumulated with a total weight of $\overline{w}^N$, hence

$$\sum_{i\in T}\Phi_i^{\omega}(v)\geqslant v(T).\quad\square$$

Declaration of competing interest

We have no conflict of interest to declare.

Data availability

No data was used for the research described in the article.

Appendix A. Proof of Proposition 2

Proof. Let us check that this definition is indeed coherent with the first definition of the weighted Shapley value. Since it is defined through the expectation of v under the probability distribution $\mathbb{P}_{\omega,\Sigma}$, it is clearly a linear mapping of v.

Fix the unanimity game on the set S, then

- A player not in S will never have a positive marginal contribution, hence his value is 0.
- A player i in S only has a positive marginal contribution, if $S\subseteq\{L\geqslant i\}$ and i is the pivotal element, hence the first element of S in L. What is the probability of this event? If i is not in $\overline{S}$, then this probability is 0, hence he obtains 0. If i is in $\overline{S}$, then its probability is exactly $\overline{\omega}_i^S/\overline{\omega}^S$. The key argument is that the probability of the first element of S to appear to be i conditionally on the fact that an element of S is picked at the current stage is exactly equal to $\overline{\omega}_i^S/\overline{\omega}^S$. Let $i\in S$. For any $t\in\{1,\dots,n\}$, we set $L_{\leqslant t}=\{L(u),u\leqslant t\}$. Formally, we have

$$\begin{aligned}
&\mathbb{P}_{\omega,\Sigma}\big(i\text{ pivot of }S\big)\\
&=\sum_{t=0}^{n-1}\sum_{\substack{T\subset N-S,\\|T|=t}}\mathbb{P}_{\omega,\Sigma}\Big(L(t+1)=i,L_{\leqslant t}=T\Big),\\
&=\sum_{t=0}^{n-1}\sum_{\substack{T\subset N-S,\\|T|=t}}\mathbb{P}_{\omega,\Sigma}\Big(L(t+1)\in S,\ L_{\leqslant t}=T\Big)\mathbb{P}_{\omega,\Sigma}\Big(L(t+1)=i|L(t+1)\in S,\ L_{\leqslant t}=T\Big),\\
&=\sum_{t=0}^{n-1}\sum_{\substack{T\subset N-S,\\|T|=t}}\mathbb{P}_{\omega,\Sigma}\Big(L(t+1)\in S,\ L_{\leqslant t}=T\Big)\frac{\mathbb{P}_{\omega,\Sigma}\Big(\{L(t+1)=i\}\cap\{L(t+1)\in S\}\cap\{L_{\leqslant t}=T\}\Big)}{\mathbb{P}_{\omega,\Sigma}\Big(\{\{L(t+1)\in S\}\cap\{L_{\leqslant t}=T\}\Big)},\\
&=\sum_{t=0}^{n-1}\sum_{\substack{T\subset N-S,\\|T|=t}}\mathbb{P}_{\omega,\Sigma}\Big(L(t+1)\in S,\ L_{\leqslant t}=T\Big)\frac{\mathbb{P}_{\omega,\Sigma}\Big(L(t+1)=i|L_{\leqslant t}=T\Big)}{\mathbb{P}_{\omega,\Sigma}\Big(L(t+1)\in S|L_{\leqslant t}=T\Big)},\\
&=\sum_{t=0}^{n-1}\sum_{\substack{T\subset N-S,\\|T|=t}}\mathbb{P}_{\omega,\Sigma}\Big(L(t+1)\in S,\ L_{\leqslant t}=T\Big)\left(\frac{\frac{\overline{\omega}_i^{N-T}}{\overline{\omega}^{N-T}}}{\sum_{j\in S}\frac{\overline{\omega}_j^{N-T}}{\overline{\omega}^{N-T}}}\right).
\end{aligned}$$

By assumption $T\subset N-S$, hence $N-T$ is a superset of S and therefore $p(N-T)$ is greater or equal to $p(S)$. We distinguish two different cases,

- if $p(N-T) > p(S)$, then $P\Big(L(t) \in S,\ \forall u < t,\ L(u) \notin S\Big) = 0$, since any element of S has still a too low priority.
- if $p(N-T) = p(S)$, then for every $i \in \overline{S}$, $\overline{\omega}_i^{N-T} = \omega_i$ whereas for $i \in S - \overline{S}$, one has $\overline{\omega}_i^{N-T} = 0$.

Hence,

$$\begin{aligned}\mathbb{P}_{\omega,\Sigma}\Big(S \subseteq \{L \geqslant i\}\Big) &= \sum_{t=0}^{n-1} \sum_{T \subset N-S, |T|=t} \mathbb{P}_{\omega,\Sigma}\Big(L(t+1) \in S,\ \{L(u), u \leqslant t\} = T\Big) \frac{\omega_i}{\sum_{j \in \overline{S}} \omega_i}, \\ &= \frac{\overline{\omega}_i^S}{\overline{\omega}^S} \left(\sum_{t=0}^{n-1} \sum_{T \subset N-S, |T|=t} \mathbb{P}_{\omega,\Sigma}\Big(L(t+1) \in S,\ \{L(u), u \leqslant t\} = T\Big) \right), \\ &= \frac{\overline{\omega}_i^S}{\overline{\omega}^S} \left(\sum_{t=0}^{n-1} \mathbb{P}_{\omega,\Sigma}\Big(L(t+1) \in S,\ \ \forall u \leqslant t,\ L(u) \notin S\}\Big) \right), \\ &= \frac{\overline{\omega}_i^S}{\overline{\omega}^S}. \quad \square\end{aligned}$$

Appendix B. Proof of Proposition 3

Proposition 3. *Let (ω, Σ) be a weight system. Let (N, v) be an (ω, Σ)-convex game, then v is superadditive.*

The proof of the proposition relies on the following lemma.

Lemma 1. *Let (N, v) be an (ω, Σ)-convex game. Let us consider $S, T, U \subseteq N$ such that $S \cap T = \emptyset$, $U \subseteq T$ and for all $s \in S$, $p(s) > p(U)$ and $p(s) \geqslant p(T)$ then*

$$v(S \cup T) - v(T) \geqslant v(S \cup U) - v(U). \tag{5}$$

Proof. The proof is by induction on $|S|$. If $|S| = 1$, then we set $S = \{i\}$ and (ω, Σ)-convexity applied to $S' = \{i\} \cup U \subseteq T' = \{i\} \cup T$ implies

$$\overline{\omega}_i^{T'}(v(\{i\} \cup T) - v(T)) \geqslant \overline{\omega}_i^{T'}(v(\{i\} \cup U) - v(U)).$$

By assumption on the priorities, we have $p(i) = p(T')$ hence $\overline{\omega}_i^{T'} > 0$ and one can divide the inequality by $\overline{\omega}_i^{T'} > 0$. We obtain that (5) is satisfied for $|S| = 1$.

Let k be a given integer with $k \geqslant 1$. Let us assume (5) satisfied for any triple $S, T, U \subseteq N$ such that $S \cap T = \emptyset$, $U \subseteq T$, and for all $s \in S$, $p(s) > p(U)$, $p(s) \geqslant p(T)$, and $|S| = k$. Let us consider $S, T, U \subseteq N$ such that $S \cap T = \emptyset$, $U \subseteq T$, and for all $s \in S$, $p(s) > p(U)$, $p(s) \geqslant p(T)$, and $|S| = k+1$. Let us set w.l.o.g. $S = \{1, 2, \ldots, k+1\}$. (ω, Σ)-convexity applied to $S \cup U \subseteq S \cup T$ implies

$$\sum_{i=1}^{k+1} \overline{\omega}_i^{T \cup S} \left(v(S \cup T) - v((S \setminus \{i\}) \cup T)\right) \geqslant \sum_{i=1}^{k+1} \overline{\omega}_i^{T \cup S} \left(v(S \cup U) - v((S \setminus \{i\}) \cup U)\right). \tag{6}$$

We do not need to consider other element in $S \cup U$ since all elements of U have too low priorities. By induction hypothesis, we have

$$v((S \setminus \{i\}) \cup T) - v(T) \geqslant v((S \setminus \{i\}) \cup U) - v(U),$$

for all $i \in \{1, 2, \ldots, k+1\}$. Multiplying each inequality by $\overline{\omega}_i^{T \cup S}$ and adding them to (6), we get

$$\sum_{i=1}^{k+1} \overline{\omega}_i^{T \cup S}(v(S \cup T) - v(T)) \geqslant \sum_{i=1}^{k+1} \overline{\omega}_i^{T \cup S}(v(S \cup U) - v(U)). \tag{7}$$

This implies (5) and proves the result at the next step of the induction. By the principle of induction, the result is true for every set S. $\square$

Proof of Proposition 3. Let us consider $S, T \subseteq N$ such that $S \cap T = \emptyset$. We have to prove

$$v(S \cup T) - v(T) \geqslant v(S) - v(\emptyset). \tag{8}$$

The proof is by induction on $|S|$.

Let us first assume $|S|=1$. We set $S=\{i\}$. If $p(i)=p(T\cup\{i\})$, then (ω,Σ)-convexity applied to $\{i\}\subseteq\{i\}\cup T$ implies

$$v(\{i\}\cup T)-v(T)\geqslant v(\{i\})-v(\emptyset), \tag{9}$$

and (8) is satisfied. Otherwise, we necessarily have $p(i)<p(T\cup\{i\})$. Let us consider the subset $T^*=\{j\in T,\ p(j)>p(i)\}$. As $p(i)\geqslant p(T\setminus T^*)$, (ω,Σ)-convexity applied to $\{i\}\subseteq\{i\}\cup(T\setminus T^*)$ implies

$$v(\{i\}\cup(T\setminus T^*))-v(T\setminus T^*)\geqslant v(\{i\})-v(\emptyset), \tag{10}$$

As $p(j)>p(i)\geqslant p(T\setminus T^*)$ for any j in T^*, Proposition 1 applied with T^* and $T\setminus T^*\subseteq\{i\}\cup(T\setminus T^*)$ implies

$$v(\{i\}\cup T)-v(\{i\}\cup(T\setminus T^*))\geqslant v(T)-v(T\setminus T^*). \tag{11}$$

(10) and (11) imply (9) and therefore (8) is still satisfied.

Let k be a given integer with $k\geqslant 1$. Let us assume (8) satisfied for any pair $S,T\subseteq N$ with $S\cap T=\emptyset$ and $|S|=k$. Let us consider $S,T\subseteq N$ with $S\cap T=\emptyset$ and $|S|=k+1$. Let us first assume $p(S)=p(S\cup T)$. Then (ω,Σ)-convexity applied with $S\subseteq S\cup T$ implies

$$\sum_{j\in\overline{S}}\omega_j(v(S\cup T)-v((S\setminus\{j\})\cup T)\geqslant\sum_{j\in\overline{S}}\omega_j(v(S)-v(S\setminus\{j\})). \tag{12}$$

By induction hypothesis, we have

$$v((S\setminus\{j\})\cup T)\geqslant v(S\setminus\{j\})+v(T),\ \forall j\in S. \tag{13}$$

Multiplying each inequality by ω_j for all j in $\overline{S}$ and adding them to (12), we get

$$\sum_{j\in\overline{S}}\omega_j v(S\cup T)\geqslant\sum_{j\in\overline{S}}\omega_j(v(S)+v(T)). \tag{14}$$

This implies (8).

Let us now assume $p(S)<p(S\cup T)$. Let us consider the subset $T^*=\{j\in T,\ p(j)>p(S)\}$. As $p(S)\geqslant p(T\setminus T^*)$, (ω,Σ)-convexity applied to $S\subseteq S\cup(T\setminus T^*)$ implies

$$\sum_{j\in\overline{S}}\omega_j(v(S\cup(T\setminus T^*))-v((S\setminus\{j\})\cup(T\setminus T^*)))\geqslant\sum_{j\in\overline{S}}\omega_j(v(S)-v(S\setminus\{j\})). \tag{15}$$

As $p(k)>p(S)\geqslant p(T\setminus T^*)$ for any k in T^*, Proposition 1 applied with T^* and $(S\setminus\{j\})\cup T\setminus T^*\subseteq S\cup(T\setminus T^*)$ implies

$$v(S\cup T)-v(S\cup(T\setminus T^*))\geqslant v((S\setminus\{j\})\cup T)-v((S\setminus\{j\})\cup(T\setminus T^*)),\ \forall j\in S. \tag{16}$$

By induction hypothesis, (13) is also satisfied. (13) and (16) imply

$$v(S\cup T)-v(S\cup(T\setminus T^*))\geqslant v(S\setminus\{j\})+v(T)-v((S\setminus\{j\})\cup(T\setminus T^*)),\ \forall j\in S. \tag{17}$$

Finally, multiplying each inequality by ω_j for all j in $\overline{S}$ and adding them to (15), we get (14) and this implies (8). □

Appendix C. Proof of Proposition 4

We first establish a formula for the (ω,Σ)-weighted Shapley value in terms of the marginal contributions.

Lemma 2. *The (ω,Σ)-weighted Shapley value can be defined as follows*

$$\Phi_i^{\omega}(v)=\sum_{\substack{S\subseteq N\\ i\in S}}\gamma_{S,i}^{N,w}(v(S)-v(S\setminus\{i\}))$$

where

$$\gamma_{S,i}^{N,w}=\begin{cases}\displaystyle\sum_{\substack{T\subseteq N:\\ T\supseteq S,\ p(T)=p(S)}}(-1)^{t-s}\frac{\omega_i}{\overline{\omega}^T} & \text{if } i\in\overline{S},\\ 0 & \text{if } i\in S\setminus\overline{S},\end{cases}$$

for all $S\subseteq N$ with $S\neq\emptyset$.

Proof. By definition, we have

$$\Phi_i^{\omega}(v) = \sum_{S\subseteq N:\, i\in S} \frac{\overline{\omega}_i^S}{\overline{\omega}^S}\lambda_S(v) = \sum_{S\subseteq N:\, i\in S}\sum_{T\subseteq S}(-1)^{s-t}v(T)\frac{\overline{\omega}_i^S}{\overline{\omega}^S}.$$

We get

$$\begin{aligned}\Phi_i^{\omega}(v) &= \sum_{S\subseteq N:\, i\in S}\left[\sum_{T\subseteq S:\, i\in T}(-1)^{s-t}(v(T)-v(T\setminus\{i\}))\right]\frac{\overline{\omega}_i^S}{\overline{\omega}^S}\\ &= \sum_{T\subseteq N:\, i\in T}\left[\sum_{S\subseteq N:\, S\supseteq T}(-1)^{s-t}\frac{\overline{\omega}_i^S}{\overline{\omega}^S}\right](v(T)-v(T\setminus\{i\}))\end{aligned} \tag{18}$$

For any pair S, T occurring in (18) we have $i\in T\subseteq S$ and thus $p(i)\leqslant p(T)\leqslant p(S)$. As $\overline{\omega}_i^S=\omega_i$ (resp. $\overline{\omega}_i^S=0$) for all $i\in S$ with $p(i)=p(S)$ (resp. $p(i)\neq p(S)$), we get the result. □

Let $\Phi^{\omega}(v^T)$ be the (ω,Σ)-weighted Shapley value of the subgame v^T. We have

$$\Phi_i^{\omega}(v^T) = \sum_{\substack{S\subseteq T\\ i\in S}}\gamma_{S,i}^{T,\omega}(v(S)-v(S\setminus\{i\})). \tag{19}$$

Define the vector $\Psi^{\omega}=(\Psi_{iT}^{\omega})_{i\in T,\,T\in\mathcal{P}(N)}$ recursively by

$$\Psi_{iT}^{\omega} = \frac{\overline{\omega}_i^T}{\overline{\omega}^T}(v(T)-v(T\setminus\{i\})) + \sum_{j\in T\setminus\{i\}}\frac{\overline{\omega}_j^T}{\overline{\omega}^T}\Psi_{iT\setminus\{j\}}^{\omega},$$

for all $i\in T$, $T\in\mathcal{P}(N)$ and setting $\Psi_{i\emptyset}^{\omega}=0$ for all $i\in N$.
We recall Proposition 4.

Proposition 4. *For all $T\in\mathcal{P}(N)$, $(\Psi_{iT}^{\omega})_{i\in T}=\Phi^{\omega}(v^T)$.*

In order to prove Proposition 4, we first establish the following lemma.

Lemma 3. *For all $S, T\in\mathcal{P}(N)$ with $\emptyset\neq S\subset T$, and for all $i\in\overline{S}$*

$$\sum_{j\in T\setminus S}\frac{\overline{\omega}_j^T}{\overline{\omega}^T}\gamma_{S,i}^{T\setminus\{j\},\omega} = \gamma_{S,i}^{T,\omega}.$$

Proof. Let us consider $S\subset T\subseteq N$ and $i\in\overline{S}$. We have

$$\begin{aligned}\sum_{j\in T\setminus S}\overline{\omega}_j^T\gamma_{S,i}^{T\setminus\{j\},\omega} &= \sum_{j\in T\setminus S}\overline{\omega}_j^T\left[\sum_{\substack{R\subseteq T\setminus\{j\}:\\ R\supseteq S,\ p(R)=p(S)}}(-1)^{r-s}\frac{\omega_i}{\overline{\omega}^R}\right]\\ &= \sum_{\substack{R\subset T:\, R\supseteq S,\\ p(R)=p(S)}}(-1)^{r-s}\frac{\omega_i}{\overline{\omega}^R}\sum_{j\in T\setminus R}\overline{\omega}_j^T\\ &= \sum_{\substack{R\subset T:\, R\supseteq S,\\ p(R)=p(S)}}(-1)^{r-s}\frac{\omega_i}{\overline{\omega}^R}(\overline{\omega}^T-\overline{\omega}_R^T).\end{aligned} \tag{20}$$

If $p(T)\neq p(S)$, then any $R\subset T$ with $p(R)=p(S)$ satisfies $\overline{\omega}_R^T=0$ and (20) is equivalent to

$$\sum_{j\in T\setminus S}\overline{\omega}_j^T\gamma_{S,i}^{T\setminus\{j\},\omega} = \overline{\omega}^T\sum_{\substack{R\subseteq T:\, R\supseteq S,\\ p(R)=p(S)}}(-1)^{r-s}\frac{\omega_i}{\overline{\omega}^R} = \overline{\omega}^T\gamma_{S,i}^{T,\omega}.$$

If $p(T)=p(S)$, then any $R\subseteq T$ with $R\supseteq S$ satisfies $p(R)=p(S)$ and $\overline{\omega}^R=\overline{\omega}_R^T$ and (20) is equivalent to

$$\sum_{j\in T\setminus S}\overline{\omega}_j^T\gamma_{S,i}^{T\setminus\{j\},\omega}=\overline{\omega}^T\sum_{\substack{R\subseteq T:\,R\supseteq S,\\ p(R)=p(S)}}(-1)^{r-s}\frac{\omega_i}{\overline{\omega}^R}-(-1)^{t-s}\omega_i-\sum_{R\subset T:\,R\supseteq S}(-1)^{r-s}\omega_i$$

$$=\overline{\omega}^T\gamma_{S,i}^{T,\omega}-\omega_i\left((-1)^{t-s}+\sum_{k=0}^{t-s-1}C_{t-s}^k(-1)^k\right). \tag{21}$$

Finally, as $\sum_{k=0}^{t-s-1}C_{t-s}^k(-1)^k=(1-1)^{t-s}-(-1)^{t-s}$, (21) implies the result. □

Proof of Proposition 4. If $t=1$, then the result is satisfied. Let us consider $T\in\mathcal{P}(N)$ with $t>1$ and $i\in T$. We assume $\Psi_S^\omega=\Phi^\omega(v^S)$ for all $S\in\mathcal{P}(N)$ with $s=t-1$. We get

$$\Psi_{iT}^\omega=\frac{\overline{\omega}_i^T}{\overline{\omega}^T}(v(T)-v(T\setminus\{i\}))+\sum_{j\in T\setminus\{i\}}\frac{\overline{\omega}_j^T}{\overline{\omega}^T}\Phi_i^\omega(v^{T\setminus\{j\}})$$

$$=\frac{\overline{\omega}_i^T}{\overline{\omega}^T}(v(T)-v(T\setminus\{i\}))+\sum_{j\in T\setminus\{i\}}\frac{\overline{\omega}_j^T}{\overline{\omega}^T}\sum_{\substack{S\subseteq T\setminus\{j\}\\ i\in S}}\gamma_{S,i}^{T\setminus\{j\},\omega}(v(S)-v(S\setminus\{i\}))$$

$$=\frac{\overline{\omega}_i^T}{\overline{\omega}^T}(v(T)-v(T\setminus\{i\}))+\sum_{\substack{S\subset T\\ i\in S}}\sum_{j\in T\setminus S}\frac{\overline{\omega}_j^T}{\overline{\omega}^T}\gamma_{S,i}^{T\setminus\{j\},\omega}(v(S)-v(S\setminus\{i\})). \tag{22}$$

Then, (22) and Lemma 3 imply

$$\Psi_{iT}^\omega=\frac{\overline{\omega}_i^T}{\overline{\omega}^T}(v(T)-v(T\setminus\{i\}))+\sum_{\substack{S\subset T\\ i\in S}}\gamma_{S,i}^{T,\omega}(v(S)-v(S\setminus\{i\})). \tag{23}$$

As $\gamma_{T,i}^{T,\omega}=\frac{\overline{\omega}_i^T}{\overline{\omega}^T}$, (23) implies the result. □

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