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10 October 2026

MILD SOLUTIONS OF HJB EQUATIONS ASSOCIATED WITH CYLINDRICAL STABLE LÉVY NOISE IN INFINITE DIMENSIONS

ALESSANDRO BONDI, FAUSTO GOZZI, ENRICO PRIOLA, AND JERZY ZABCZYK

Dedicated to the memory of Giuseppe Da Prato

Abstract. We study the optimal control of an infinite-dimensional stochastic system governed by an SDE in a separable Hilbert space driven by cylindrical stable noise. We establish the existence and uniqueness of a mild solution to the associated HJB equation. This result forms the basis for the proof of the Verification Theorem, which is the subject of ongoing research and will provide a sufficient condition for optimality.

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1. Introduction

Our study is concerned with a stochastic control system

$$dX_s = (AX_s + F(X_s))ds + a_s ds + dZ_s, \quad s \geq t,\ X_t = x \in H, \tag{1}$$

on a Hilbert space H where A is a linear operator and F a Lipschitz continuous and bounded transformation from H into H. Moreover, $(a_s)_{s\geq 0}$ is a predictable control process with values in the closed ball $\{z \in H : |z| \leq R\}$. Perturbations are modeled by a stochastic process Z of pure jump Lévy type. The ultimate goal is to find a control process which minimizes the cost functional

$$J(t,x,a) = \mathbb{E}\bigg[\int_t^T \Big(g(X_s^{t,x,a}) + \frac{1}{2}|a_s|^2\Big)ds \ + \ h(X_T^{t,x,a})\bigg],$$

where $X_s^{t,x,a}$, $s \in [t,T]$, $x \in H$, is a solution of (1) corresponding to the control $(a_s)_{s\geq 0}$.

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The third author is a member of GNAMPA.

To solve the control problem we apply the dynamic programming approach, with the nonlocal parabolic Hamilton-Jacobi-Bellman equation

$$\begin{aligned}\partial_t u(t,x) =& g(x) + \langle Ax + F(x), Du(t,x)\rangle \\ &+ \int_H \{u(t,x+y) - u(t,x) - \langle Du(t,x), y\rangle\}\nu(dy) \\ &+ \inf_{|\lambda|\le R}\Big[\langle \lambda, Du(t,x)\rangle + \frac{1}{2}|\lambda|^2\Big], \quad t \in]0,T], \\ u(0,x) = h(x), &\quad x \in H,\end{aligned} \tag{2}$$

for the value function

$$V(t,x) = \inf_{a\in\mathcal{U}} J(t,x,a)$$

playing a central role. In Equation (2), ν is the so-called intensity measure of the process Z; $\mathcal{U}$ denotes the set of all control processes.

The specific results will be formulated under the additional assumption that A is an unbounded, negative definite, self-adjoint operator on H having inverse A^{-1} which is compact. This allows to cover the case when Z is a cylindrical $\alpha-$stable process with $\alpha \in (1,2)$ formally given by

$$Z_t = \sum_{n\ge 1} Z_t^n e_n, \quad t \ge 0,$$

where $(e_n)_n$ is an orthonormal basis of eigenfunctions of A and $(Z_t^n)_n$ a sequence of independent one-dimensional $\alpha-$stable Lévy processes (see Section 2 for more details). These processes and the corresponding semilinear SPDEs are introduced in [19] and further investigated in, for instance, [18].

Note that the study of a control problem like ours has been initiated in [7] (see also the later paper [11]) in the well-known cylindrical Wiener case, i.e., when $(Z_t^n)_n$ are independent one-dimensional Wiener processes.

The solution will be achieved in two steps.
The first one is a proof of the existence and uniqueness of the so-called mild version of Equation (2) (see Equation (5) below). This is done in the present paper.
The second step will be the subject of a future research, which will focus on the proof of the *fundamental formula*:

$$\begin{aligned}u(T-t,x) = J(t,x,a) + \mathbb{E}\Big[\int_t^T \Big(&\inf_{|\lambda|\le R}\Big[\langle \lambda, Du(T-s, X_s^{t,x,a})\rangle + \frac{1}{2}|\lambda|^2\Big] \\ &- \frac{1}{2}|a_s|^2 - \langle Du(T-s, X_s^{t,x,a}), a_s\rangle\Big) ds\Big],\end{aligned} \tag{3}$$

valid for the mild solution u of (2) and an arbitrary control process $(a_s)_{s\ge 0}$. It leads directly to the solution of the control problem.
It is worth noting that our approach relies on the smoothing effect and gradient estimates of the transition semigroup of the Ornstein-Uhlenbeck (OU, for short) process associated with the random perturbation Z. This allows us to avoid the theory of viscosity solutions, which is particularly delicate for infinite-dimensional pure jump Lévy processes. Such theory requires restrictive conditions on the drift F (see [21, 22, 23]). Moreover, it does not cover the cylindrical Lévy case we consider. We also mention that it is currently an open problem to establish the regularity of such viscosity solutions.

For another example of infinite-dimensional pure jump Lévy process where the strong Feller and regularizing properties of the corresponding OU semigroup are known, we refer to [2]; see also Remark 1.1 for more details.

To define the mild version of (2), denote by P_t, $t \geq 0$, the transition semigroup of the generalized OU process Z_A (cf. [16]):

$$dZ_{A,t} = AZ_{A,t}dt + dZ_t, \quad Z_{A,0} = x. \tag{4}$$

For ϕ in the space of bounded continuous functions $C_b(H)$:

$$P_t\phi(x) = \mathbb{E}\left[\phi(Z^x_{A,t})\right], \quad t \geq 0,\ x \in H,$$

where Z^x_A is the mild solution of (4). The mild version of (2) is of the form

$$u(t,x) = P_t h(x) + \int_0^t P_{t-s}[\mathcal{H}(\cdot, Du(s,\cdot))](x)\, ds, \quad t \in [0,T],\ x \in H, \tag{5}$$

where, for arbitrary $y \in H$ and $p \in H$,

$$\mathcal{H}(y,p) = g(y) + \langle F(y), p\rangle + \inf_{|\lambda|\leq R}\left[\langle \lambda, p\rangle + \frac{1}{2}|\lambda|^2\right].$$

The existence and uniqueness of a regular solution u to (5) in the space $C^1_\gamma(H)$ (see (26) for its definition) is proved in Theorem 4.2. In particular, such solution u is Fréchet differentiable in the space variable x. In Theorem 4.3 we show in addition that the Fréchet derivative $Du(t,\cdot)$ is $\theta-$Hölder continuous from H into H for suitable $\theta \in (0,1)$, $t \in]0,T]$. These theorems are the main results of this paper and are crucial for the proof of the fundamental formula (3).

Even in the finite-dimensional setting, contrary to the Wiener case, which is extensively analyzed, for instance, in the monograph [13], the theory of optimal stochastic control problems with random perturbations of Lévy type is not very developed (especially for the case of multiplicative Lévy noises). We refer to [5, 12, 17] for Bellman's principles involving special dynamics and to the more recent [4] for a dynamic programming principle associated with more general controlled jump diffusions. We also mention the book [15], where several examples of control problems for jump diffusions motivated by applications can be found.

Control systems in infinite dimensions with Wiener-type perturbations are of current interest and discussed in many publications, see, in particular, the comprehensive monograph [10].

Besides the already mentioned [22, 23], however, we are unaware of works on stochastic infinite-dimensional control systems with pure Lévy-type perturbations without Gaussian component. They require essential modifications of the classical techniques, although basic dynamic programming ideas are still applicable.

Remark 1.1. *The paper [2] studies regularizing properties and establishes gradient estimates for the OU transition semigroup corresponding to subordinated cylindrical Wiener processes W_S formally given by*

$$W_{S_t} = \sum_{n\geq 1} B^n_{S_t} e_n, \quad t \geq 0.$$

Here, $(B^n_t)_n$ is a sequence of one-dimensional independent Brownian motions and S is an independent $\alpha-$stable subordinator, with $\alpha \in (\frac{1}{2}, 1)$. The perturbation W_S is $2\alpha-$stable and, unlike Z, is isotropic, i.e., invariant by rotation. Employing the gradient estimates in [2], the machinery devised in this paper can be adapted to analyze a state equation like (1) driven by W_S instead of Z.

2. Preliminary material

In this section, we recall some properties of cylindrical $\alpha-$stable Lévy processes from [19]. Moreover, in Subsection 2.2 we introduce the state equation of the control problem investigated in the following sections.

2.1. **On stable Lévy processes in Hilbert spaces.** Let H be a real separable Hilbert space. Given a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual assumptions, we consider a cylindrical $\alpha-$stable process $Z = (Z_t)_t$, $\alpha \in (1,2)$, formally given by

$$Z_t = \sum_{n\geq 1} \beta_n Z_t^n e_n, \quad t \geq 0.$$

Here, $(e_n)_n$ is a fixed orthonormal basis in H, $(\beta_n)_n \subset (0,\infty)$ is a sequence of positive numbers and $(Z_t^n)_n$ is a sequence of independent one-dimensional $\alpha-$stable Lévy processes defined on the previous stochastic basis such that, for any $n \in \mathbb{N}$ and $t \geq 0$,

$$\mathbb{E}[e^{iZ_t^n h}] = e^{-t|h|^\alpha}, \quad h \in \mathbb{R}. \tag{6}$$

Let $A\colon D(A) \subset H \to H$ be an operator that fulfills the next assumptions taken from [19].

Hypothesis 2.1. *The operator A is self-adjoint. Moreover, the following holds.*

(i) The reference basis $(e_n)_n$ is a basis of eigenvectors of A. More specifically, $(e_n)_n \subset D(A)$ and there exists a sequence $(\gamma_n)_n \subset (0,\infty)$ of positive numbers such that $\gamma_n \to \infty$ as $n \to \infty$ and

$$Ae_n = -\gamma_n e_n, \quad n \in \mathbb{N}.$$

(ii) There exists $\gamma \in [1/\alpha, 1)$ and $\bar{C} > 0$ such that

$$\beta_n \geq \bar{C}\, \gamma_n^{\frac{1}{\alpha}-\gamma}, \quad n \in \mathbb{N}. \tag{7}$$

(iii) The series $\sum_{n\geq 1} \beta_n^\alpha \gamma_n^{-1}$ converges.

Note that when $\sum_{n\geq 1} \gamma_n^{-1}$ converges we can cover in particular the cylindrical case $\sum_{n\geq 1} Z_t^n e_n$. This happens, for instance, when A is the generator of the one-dimensional heat semigroup on a bounded interval with Dirichlet boundary conditions.

For every $x \in H$, the OU process $Z_A^x = (Z_{A,t}^x)_t$ associated with Z is defined by

$$Z_{A,t}^x = e^{tA}x + \sum_{n=1}^{\infty} \beta_n \left(\int_0^t e^{-\gamma_n(t-s)} dZ_s^n \right) e_n, \quad t \geq 0.$$

Thanks to Hypothesis 2.1, by [19, Proposition 4.2], Z_A^x is an $H-$valued process. Moreover, by [19, Theorem 4.4], we can consider a version of Z_A^x which is stochastically continuous, predictable and with $p-$locally integrable trajectories, for every $p \in [1, \alpha)$. In particular, we note that, by [9, Proposition 3.7(ii)], the existence of a predictable modification is a consequence of the stochastic continuity and $\mathcal{F}_t-$adaptedness of Z_A^x. We observe that for our arguments we do not need the càdlàg regularity for the paths of the OU processes. In fact, in our infinite-dimensional setting, Z_A^x may fail to have càdlàg paths; this is the case, for instance, when the sequence $(\beta_n)_n$ does not converge to 0, see Theorem 2.1 and Corollary 2.1 in [6].

We denote by $C_b(H)$ [resp., $C_b(H;H)$] the Banach space of bounded and continuous real-valued [resp., $H-$valued] functions defined in H, endowed with the usual norm $\|\cdot\|_0$. Additionally, $C_b^1(H)$ is the Banach space of continuous, bounded and Fréchet differentiable functions from H into $\mathbb{R}$ with continuous and bounded Fréchet derivative, endowed with the usual norm $\|\cdot\|_1$. Let $P=(P_t)_{t\geq 0}$ be the OU transition semigroup associated with the processes Z_A^x, $x\in H$, i.e.,

$$P_t\phi(x)=\mathbb{E}\left[\phi(Z_{A,t}^x)\right], \quad t\geq 0,\, x\in H,\, \phi\in C_b(H). \tag{8}$$

In the sequel, we denote by $C=C(\alpha,\gamma)>0$ a positive constant allowed to change from line to line. According to [19, Theorem 4.14], for every $t>0$, P_t maps Borel measurable and bounded functions ϕ into $C_b^1(H)$, and the following gradient estimate holds:

$$\|DP_t\phi\|_0=\sup_{x\in H}|DP_t\phi(x)|\leq \frac{C}{t^\gamma}\|\phi\|_0. \tag{9}$$

If additionally $\phi\in C_b(H)$, then the Gâteaux derivative of $P_t\phi$ is given by

$$\langle DP_t\phi(x),h\rangle=\int_H \phi(e^{tA}x+y)\; J_t(h,y)\;\mu_t(dy), \quad h\in H.$$

Here, μ_t is the law of the stochastic convolution $Z_{A,t}^0$ and, for every $h\in H$ and $t>0$, the map $y\mapsto J_t(h,y)$ is defined as follows:

$$J_t(h,y)=\sum_{n=1}^\infty \frac{p'_\alpha}{p_\alpha}\left(\frac{\langle y,e_n\rangle}{c_n(t)}\right)\frac{e^{-\gamma_n t}\langle h,e_n\rangle}{c_n(t)}, \quad \text{with } c_n(t)=\beta_n\left(\frac{1-e^{-\alpha\gamma_n t}}{\alpha\gamma_n}\right)^{1/\alpha},$$

where the convergence of the series is in the $L^2(H,\mu_t)-$sense and p_α denotes the density with respect to the Lebesgue measure of the probability measure induced on $\mathbb{R}$ by Z_1^n, for any $n\in\mathbb{N}$. In particular, $J_t(h,\cdot)\in L^2(H,\mu_t)$ is such that

$$\int_H |J_t(h,y)|^2\;\mu_t(dy)\leq \frac{C}{t^{2\gamma}}|h|^2.$$

We refer to [2, Theorem 7 and Corollary 8] for analogous results in the context of OU processes driven by subordinated cylindrical Brownian noises.

2.2. **The State Equation.** We consider a map

$$F: H\to H \quad \text{Lipschitz continuous and bounded;} \tag{10}$$

we denote by $[F]_{\text{Lip}}$ the Lipschitz continuity constant of F. For a fixed $R>0$, we define the set of processes

$$\mathcal{U}=\mathcal{U}_R=\left\{a\colon [0,\infty)\times\Omega\to B_R(H) \text{ s.t. } a \text{ is predictable}\right\}, \tag{11}$$

where $B_R(H)$ is the closed ball centered at 0 with radius R in H.

Inspired by the cylindrical Wiener case analyzed in [7, 11], for every $a\in\mathcal{U}$, $t\geq 0$ and $x\in H$ we consider the following controlled nonlinear stochastic differential equation:

$$dX_s=(AX_s+F(X_s))ds+a_s ds+dZ_s, \quad s\geq t,\; X_t=x\in H. \tag{12}$$

Starting from Section 3 below, we investigate a control problem featuring (12) as the *state equation*. Indeed, the mild formulation of (12) has a pathwise unique solution, as shown in the next lemma, which is proved similarly to [19, Theorem 5.3].

Lemma 2.2. *For every $a \in \mathcal{U}$, $t \geq 0$ and $x \in H$, (12) admits a unique predictable mild solution with $p-$locally integrable paths for $p \in [1, \alpha)$, that is, there exists a unique predictable process $X = (X_s)_{s \geq t}$ with trajectories in $L^p_{loc}(t, \infty)$ for any $p \in [1, \alpha)$ such that, $\mathbb{P}-a.s.$,*

$$X_s = e^{(s-t)A}x + \int_t^s e^{(s-r)A}\left(F(X_r) + a_r\right) dr + Z^0_{A,s} - e^{(s-t)A}Z^0_{A,t}, \quad s \geq t. \tag{13}$$

Proof. Let $a \in \mathcal{U}$, $t \geq 0$ and $x \in H$; notice that a process $X = (X_s)_{s \geq t}$ satisfies (13) if and only if the process $Y = (Y_s)_{s \geq t}$ defined by $Y_s = X_s - Z^0_{A,s} + e^{(s-t)A}Z^0_{A,t}$ fulfills

$$Y_s = e^{(s-t)A}x + \int_t^s e^{(s-r)A}\left(F(Y_r + Z^0_{A,r} - e^{(r-t)A}Z^0_{A,t}) + a_r\right) dr, \quad s \geq t. \tag{14}$$

We then focus on this equation and demonstrate that it admits a unique predictable solution with continuous trajectories. This is sufficient to deduce the properties of X in the statement of this lemma.

Fix $T > 0$ and denote by $C([t, t+T]; H)$ the Banach space of $H-$valued continuous functions on $[t, t+T]$ endowed with the usual supremum norm $\|\cdot\|_0$. For $\mathbb{P}-$a.e. $\omega \in \Omega$, the functional $\Gamma_{t_0^T,\omega}$ given by

$$\begin{aligned}\Gamma_{t_0^T,\omega}(f)(s) &= e^{(s-t)A}x \\ &\quad + \int_t^s e^{(s-r)A}\big(F(f(r) + Z^0_{A,r}(\omega) - e^{(r-t)A}Z^0_{A,t}(\omega)) + a_r(\omega)\big)dr, \\ s \in [t, t+T],\ f &\in C([t, t+T]; H),\end{aligned}$$

is well defined. Indeed, recalling that $a_r \in B_R(H)$ for every $r \in [t, t+T]$ and $Z^0_{A,\cdot} \in L^p(t, t+T)$ for any $p \in [1, \alpha)$, by (10) we have

$$\begin{aligned}|\Gamma_{t_0^T,\omega}(f)(s)| &\leq |x| + \int_t^s \big(|F(0)| + |a_r(\omega)| + [F]_{\mathrm{Lip}}\left(\|f\|_0 + |Z^0_{A,r}(\omega)| + |Z^0_{A,t}(\omega)|\right)\big)dr \\ &< \infty, \quad s \in [t, t+T],\end{aligned}$$

where we also use the fact that $e^{\cdot A} = (e^{uA})_{u \geq 0}$ is a contraction semigroup on H. Since $e^{\cdot A}$ is strongly continuous, by the dominated convergence theorem we infer that $\Gamma_{t_0^T,\omega}$ takes values in $C([t, t+T]; H)$. Moreover, for every $f_1, f_2 \in C([t, t+T]; H)$,

$$\|\Gamma_{t_0^T,\omega}(f_1) - \Gamma_{t_0^T,\omega}(f_2)\|_0 \leq [F]_{\mathrm{Lip}}|T|\|f_1 - f_2\|_0. \tag{15}$$

It then follows that, for T sufficiently small, $\Gamma_{t_0^T,\omega}$ is a contraction in $C([t, t+T]; H)$ which has a unique fixed point $\bar{f}_{t_0^T,\omega}$.

Given that the relation among constants in (15) does not depend on the initial point x, a standard argument by steps based on the semigroup property of $e^{\cdot A}$ enable us to consider the entire half-line $[t, \infty)$. More precisely, thanks to (15), for every $n \in \mathbb{N}$ we can define iteratively the contraction mappings $\Gamma_{t_{nT}^{(n+1)T},\omega}$ in $C([t+nT, t+(n+1)T]; H)$ by

$$\begin{aligned}\Gamma_{t_{nT}^{(n+1)T},\omega}(f)(s) &= e^{(s-t-nT)A}\bar{f}_{t_{(n-1)T}^{nT},\omega}(t+nT) \\ &\quad + \int_{t+nT}^s e^{(s-r)A}\big(F(f(r) + Z^0_{A,r}(\omega) - e^{(r-t)A}Z^0_{A,t}(\omega)) + a_r(\omega)\big)dr, \\ s \in [t+nT, t+(n+1)T],\ f &\in C([t+nT, t+(n+1)T]; H),\end{aligned}$$

where $\bar{f}_{t^{nT}_{(n-1)T},\omega}$ denotes the unique fixed point in $C([t+(n-1)T, t+nT]; H)$ of $\Gamma_{t^{nT}_{(n-1)T},\omega}$. Therefore, the function $f_\omega \in C([t,\infty); H)$ defined by

$$f_\omega(s) = \bar{f}_{t^{nT}_{(n-1)T},\omega}(s), \quad s \in [t+(n-1)T, t+nT],\ n \in \mathbb{N},$$

is the unique continuous mapping in $[t,\infty)$ such that

$$f_\omega(s) = e^{(s-t)A}x + \int_t^s e^{(s-r)A}\left(F(f_\omega(r) + Z^0_{A,r}(\omega) - e^{(r-t)A}Z^0_{A,t}(\omega)) + a_r(\omega)\right) dr,$$
$$s \geq t.$$

Additionally, for every $n \in \mathbb{N}$, denoting by $\Gamma^{(m)}_{t^{nT}_{(n-1)T},\omega}$ the composition of $\Gamma_{t^{nT}_{(n-1)T},\omega}$ with itself $m-$times, we have

$$\lim_{m\to\infty} \left\| \Gamma^{(m)}_{t^{nT}_{(n-1)T},\omega}(0) - f_\omega\big|_{[t+(n-1)T, t+nT]} \right\|_0 = 0.$$

The process $Y = (Y_s)_{s\geq t}$ defined by $Y_s(\omega) = f_\omega(s)$ for $s \in [t,\infty)$ and $\mathbb{P}-$a.s. $\omega \in \Omega$ is then the unique continuous solution of (14). Furthermore, arguing by induction, one can easily see that

$$\left(\Gamma^{(m)}_{t^{nT}_{(n-1)T},\omega}(0)\right)_m \text{ is a sequence of predictable processes, for every } n \in \mathbb{N}.$$

As a result, Y is predictable, as $Y_s(\omega) = \lim_{m\to\infty} \Gamma^{(m)}_{t^{nT}_{(n-1)T},\omega}(0)(s)$ for $\mathbb{P}-$a.s. $\omega \in \Omega$ and $s \in [t+(n-1)T, t+nT]$. The proof is now complete. □

Consider the unique solution $(X_s)_{s\geq t}$ of (13). Recalling that Z^0_A is predictable, if we define $X_s = X_t$ for every $s \in [0,t)$, then the process $X = (X_s)_{s\geq 0}$ is predictable, as well. In the sequel, to stress the dependence of X on the starting point x, the initial time t and the control a, we denote X by $X^{t,x,a} = (X^{t,x,a}_s)_{s\geq 0}$.

Remark 2.3. *Lemma 2.2 still holds when F is only Lipschitz continuous. Indeed, the boundedness of F in (10) is never used in its proof.*

3. The Control Problem and the associated HJB equation

For a fixed $R > 0$, we consider a control problem where the set of admissible controls is $\mathcal{U} = \mathcal{U}_R$, see (11), and the state equation is (12). As discussed in Subsection 2.2, for any $a \in \mathcal{U}$, $t \geq 0$ and $x \in H$, (12) admits a unique predictable mild solution $X^{t,x,a} = (X^{t,x,a}_s)_{s\geq 0}$ satisfying (13).

Given h, $g \in C_b(H)$ and a finite time-horizon $T > 0$, the cost functional $J(t,x,a)$ that we investigate is

$$J(t,x,a) = \mathbb{E}\left[\int_t^T \left(g(X^{t,x,a}_s) + \frac{1}{2}|a_s|^2\right) ds \ + \ h(X^{t,x,a}_T)\right],$$
$$t \in [0,T],\ x \in H,\ a \in \mathcal{U}. \tag{16}$$

The corresponding value function $V : [0,T] \times H \to \mathbb{R}$ is then defined by

$$V(t,x) = \inf_{a\in\mathcal{U}} J(t,x,a). \tag{17}$$

We study this control problem following the Dynamic Programming Approach, focusing on the nonlocal parabolic HJB equation (see, e.g., [7], [10] and [24])

$$\begin{cases} \partial_t u(t,x) = g(x) + \inf_{\lambda \in B_R(H)} [\mathcal{L}^\lambda u(t,x)], \quad t \in]0,T], \\ u(0,x) = h(x), \quad x \in H. \end{cases} \tag{18}$$

Here, for a sufficiently regular cylindrical function ϕ,

$$\mathcal{L}^\lambda \phi(x) = L^{OU}\phi(x) + \langle F(x), D\phi(x)\rangle + \left[\langle \lambda, D\phi(x)\rangle + \frac{1}{2}|\lambda|^2\right], \quad x \in H,$$

where, denoting by $\nu(d\xi)$ the Lévy measure of the processes $(Z^n)_n$ (see also [20, Theorem 31.5])

$$\begin{aligned} L^{OU}\phi(x) =& \langle Ax, D\phi(x)\rangle \\ &+ \sum_{j=1}^{\infty} \int_{\mathbb{R}} \left(\phi(x+\beta_j \xi e_j) - \phi(x) - \beta_j \xi \frac{\partial \phi}{\partial x_j}(x)\right) \nu(d\xi), \quad x \in H. \end{aligned} \tag{19}$$

Notice that, by (6), [20, Theorems 14.3 (ii) and 14.15] and the Lévy-Khintchine formula,

$$\nu(d\xi) = \frac{c_\alpha}{|\xi|^{\alpha+1}} d\xi, \quad \text{where} \quad c_\alpha = \frac{1}{2}\left(-\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}\right)^{-1}, \tag{20}$$

thus, with the change of variables $\xi_j' = \beta_j \xi$, we can rewrite (19) as follows:

$$\begin{aligned} L^{OU}\phi(x) =& \langle Ax, D\phi(x)\rangle \\ &+ c_\alpha \sum_{j=1}^{\infty} \beta_j^\alpha \int_{\mathbb{R}} \left(\phi(x+\xi e_j) - \phi(x) - \xi \frac{\partial \phi}{\partial x_j}(x)\right) \frac{1}{|\xi|^{\alpha+1}} d\xi, \quad x \in H. \end{aligned}$$

We now introduce the Hamiltonian function $\mathcal{H}$ defined by

$$\begin{aligned} \mathcal{H}(x,p) &= \inf_{\lambda \in B_R(H)} \left[\langle \lambda, p\rangle + \frac{1}{2}|\lambda|^2\right] + \langle F(x), p\rangle + g(x), \\ &=: H(p) + \langle F(x), p\rangle + g(x), \quad x, p \in H. \end{aligned} \tag{21}$$

By imposing first order conditions on the Gâteaux differential of the convex map $\lambda \mapsto \langle \lambda, p\rangle + \frac{1}{2}|\lambda|^2$ and applying the Cauchy–Schwarz inequality, we derive an explicit expression for H, namely

$$H(p) = \begin{cases} -\frac{1}{2}|p|^2, & |p| \le R, \\ -R|p| + \frac{1}{2}R^2, & |p| > R. \end{cases} \tag{22}$$

Note also that $\mathcal{H}(\cdot, 0) = g$. Using $\mathcal{H}$, the HJB equation (18) can be rewritten as follows:

$$\begin{cases} \partial_t u(t,x) = \mathcal{H}(x, Du(t,x)) + L^{OU}u(t,x), \quad t \in]0,T], \\ u(0,x) = h(x), \quad x \in H. \end{cases} \tag{23}$$

In Section 4, we study the well-posedness of (23) in a mild formulation.

We conclude this part with a lemma stating some properties of $\mathcal{H}$. Its proof, which relies on the definition in (21), (22) and the fact that F and g are continuous and bounded, is straightforward and therefore omitted.

Lemma 3.1. *The Hamiltonian $\mathcal{H}\colon H \times H \to \mathbb{R}$ is continuous in both variables. Furthermore, there exists a constant $L > 0$ such that*

$$|\mathcal{H}(x,p) - \mathcal{H}(x,q)| \leq L|p-q|, \quad p,\, q,\, x \in H.$$

In particular,

$$|\mathcal{H}(x,p)| \leq L|p| + \|g\|_0, \quad x,\, p \in H. \tag{24}$$

4. Mild solutions of the HJB equation

In this section, we study Equation (23) in mild form. Recall that a suitably regular map $u\colon [0,T] \times H \to \mathbb{R}$ is a *mild solution* of (23) if u fulfills the following convolution equation:

$$u(t,x) = P_t h(x) + \int_0^t P_{t-s}[\mathcal{H}(\cdot, Du(s,\cdot))](x)\, ds, \quad t \in [0,T],\ x \in H. \tag{25}$$

To stress the dependence of u on the given functions g and h, in the sequel we can also write

$$u(t,x) = u^{g,h}(t,x), \quad t \in [0,T],\, x \in H.$$

We search for solutions of (25) in the functional space

$$C^1_\gamma(H) = \left\{ u\colon [0,T]\times H \to \mathbb{R} \text{ cont., bounded} \;\middle|\; \begin{array}{c} u(t,\cdot) \in C^1_b(H),\ t \in]0,T] \\ \sup\limits_{t\in]0,T]} t^\gamma \|Du(t,\cdot)\|_0 < \infty \end{array} \right\}, \tag{26}$$

where γ is given in Hypothesis 2.1. As in [24, Section 9.2] (see also [3], [9, Section 9.5] and [10, Section 4.2.2], where similar spaces are introduced), we consider the norm

$$\|u\|_{C^1_\gamma} = \sup_{t\in[0,T]} \|u(t,\cdot)\|_0 + \sup_{t\in]0,T]} t^\gamma \|Du(t,\cdot)\|_0, \quad u \in C^1_\gamma(H);$$

the couple $(C^1_\gamma(H), \|\cdot\|_{C^1_\gamma})$ constitutes a Banach space.
In Theorem 4.2 below we demonstrate the well-posedness of (25) in $C^1_\gamma(H)$. We refer the reader to [3, Theorem 2.1], [9, Theorem 9.38], [24, Section 9.2] and [10, Section 4.4.1] for similar results in different settings.

Before that, we present a preliminary lemma on the regularity of the map $(t,x) \mapsto P_t\phi(x)$ on $[0,\infty) \times H$, for a given $\phi \in C_b(H)$.

Lemma 4.1. *For every $\phi \in C_b(H)$, the function $(t,x) \mapsto P_t\phi(x)$ is continuous in $[0,\infty) \times H$. Furthermore, given a direction $p \in H$, the Gâteaux derivative $(t,x) \mapsto \langle DP_t\phi(x), p\rangle$ along p is continuous in $(0,\infty) \times H$.*

Proof. Fix $\phi \in C_b(H)$. The joint continuity of the map $(t,x) \mapsto P_t\phi(x)$ in $[0,\infty) \times H$ is a consequence of [1, Lemma 2.1] and the stochastic continuity of Z^0_A.

As regards the Gâteaux derivative, notice that, for every $t > 0$, by (8),

$$P_t\phi(x+y) = \mathbb{E}\left[\phi\left(e^{tA}x + e^{tA}y + Z^0_{A,t}\right)\right] = P_t(\phi(\cdot + e^{tA}y))(x), \quad x,y \in H. \tag{27}$$

Then, given $p \in H$, for every $\epsilon > 0$, by the semigroup property of P we write

$$\begin{aligned}\langle DP_t\phi(x), p\rangle &= \lim_{h\to 0} P_{t-\epsilon}\left(\frac{P_\epsilon(\phi(\cdot + he^{tA}p)) - P_\epsilon\phi}{h}\right)(x)\\ &= \lim_{h\to 0} P_{t-\epsilon}\left(\frac{P_\epsilon\phi(\cdot + he^{(t-\epsilon)A}p)) - P_\epsilon\phi}{h}\right)(x)\\ &= P_{t-\epsilon}\langle DP_\epsilon\phi(\cdot), e^{(t-\epsilon)A}p\rangle(x), \quad t \geq \epsilon,\ x \in H. \end{aligned} \tag{28}$$

Here, for the second equality, we apply (27) to the operator P_ϵ choosing $y = he^{(t-\epsilon)A}p$, together with the semigroup property of $(e^{uA})_{u\geq 0}$. The last equality follows from the dominated convergence theorem, which can be applied because, by (9), the mean value theorem and the fact that $(e^{uA})_{u\geq 0}$ is a contraction semigroup on H,

$$|P_\epsilon\phi(y + he^{(t-\epsilon)A}p) - P_\epsilon\phi(y)| \leq C\frac{1}{\epsilon^\gamma}\|\phi\|_0|h||p|, \quad y \in H.$$

If we now consider $t \geq \epsilon$, $x \in H$ and two sequences $(t_n)_n \subset [\epsilon, \infty)$, $(x_n)_n \subset H$ such that $t_n \to t$ and $x_n \to x$ as $n \to \infty$, then

$$\begin{aligned}|\langle DP_{t_n}\phi(x_n), p\rangle - \langle DP_t\phi(x), p\rangle| &\leq |P_{t_n-\epsilon}(\langle DP_\epsilon\phi(\cdot), e^{(t_n-\epsilon)A}p - e^{(t-\epsilon)A}p\rangle)(x_n)|\\ &+|P_{t_n-\epsilon}(\langle DP_\epsilon\phi(\cdot), e^{(t-\epsilon)A}p\rangle)(x_n) - P_{t-\epsilon}(\langle DP_\epsilon\phi(\cdot), e^{(t-\epsilon)A}p\rangle)(x)| =: \mathbf{I}_n + \mathbf{II}_n.\end{aligned}$$

Observe that, by (9) and the strong continuity of the semigroup $(e^{uA})_{u\geq 0}$ on H,

$$\mathbf{I}_n \leq \frac{C}{\epsilon^\gamma}\|\phi\|_0\left|e^{(t_n-\epsilon)A}p - e^{(t-\epsilon)A}p\right| \underset{n\to\infty}{\longrightarrow} 0,$$

and, considering that $\langle DP_\epsilon\phi(\cdot), e^{(t-\epsilon)A}p\rangle \in C_b(H)$, by the first result of this lemma

$$\lim_{n\to\infty} \mathbf{II}_n = 0.$$

Therefore we conclude that $(t, x) \mapsto \langle DP_t\phi(x), p\rangle$ is continuous in $[\epsilon, \infty) \times H$. Since $\epsilon > 0$ is arbitrary, the proof is complete. □

Theorem 4.2. *There exists a unique solution $u = u^{g,h}$ of (25) in $C^1_\gamma(H)$.*

Proof. Define the mapping S in $C^1_\gamma(H)$ by

$$\begin{aligned}S(u)(t, x) &= P_t h(x) + \int_0^t P_{t-s}[\mathcal{H}(\cdot, Du(s, \cdot))](x)\, ds,\\ &t \in [0, T],\ x \in H,\ u \in C^1_\gamma(H).\end{aligned} \tag{29}$$

Observe that S takes values in $C^1_\gamma(H)$. Indeed, since $h \in C_b(H)$, by (9) and Lemma 4.1, the function $(t, x) \mapsto P_t h(x)$ belongs to $C^1_\gamma(H)$. Additionally, for every $u \in C^1_\gamma(H)$, the map $\Gamma(u)(t, x) := \int_0^t P_{t-s}[\mathcal{H}(\cdot, Du(s, \cdot))](x)\, ds$ is continuous on $[0, T] \times H$ by the dominated convergence theorem and Lemmas 3.1 - 4.1. Furthermore, once again, by the dominated converge theorem and (9), $\Gamma u(t, \cdot) \in C^1_b(H)$ for any $t \in]0, T]$, and, by (24),

$$\begin{aligned}\|D\Gamma(u)(t, \cdot)\|_0 &\leq C\int_0^t \frac{1}{(t-s)^\gamma}\left(L\frac{\|u\|_{C^1_\gamma}}{s^\gamma} + \|g\|_0\right) ds\\ &\leq \frac{C}{1-\gamma}T^{1-\gamma}\left(4^\gamma L\|u\|_{C^1_\gamma}\frac{1}{t^\gamma} + \|g\|_0\right), \quad t \in]0, T],\end{aligned}$$

where we use [3, Equation (2.12)] for the last inequality. Thus, $S : C^1_\gamma(H) \to C^1_\gamma(H)$. Since, by estimates similar to those above and Lemma 3.1,

$$\|S(u_1) - S(u_2)\|_{C^1_\gamma} \leq \frac{T^{1-\gamma}}{1-\gamma} L(1 + 4^\gamma C)\|u_1 - u_2\|_{C^\gamma_1}, \quad u_1, u_2 \in C^1_\gamma(H),$$

we deduce that S is a contraction in $C^1_\gamma(H)$ for T small enough. Consequently, (25) admits a unique solution $u \in C^1_\gamma(H)$ for a sufficiently small T.
This conclusion continues to hold even in the case of a general T, which can be demonstrated by a standard step method relying on the semigroup property of P. The theorem is now completely proved. □

We conclude the paper with a regularity result on the solution $u = u^{g,h}$ to (25) – Theorem 4.3 – that seems to be new even in the limiting Brownian case $\alpha = 2$. We focus on the Hölder continuity of $Du(t,\cdot)$, $t \in]0,T]$. For every $\theta \in (0,1]$, we denote by $C^{0,\theta}_b(H)$ [resp., $C^{0,\theta}_b(H;H)$] the space of $\mathbb{R}-$valued [resp., $H-$valued] bounded and $\theta-$Hölder continuous functions l endowed with the usual norm

$$\|l\|_{C^{0,\theta}_b} := \|l\|_0 + [l]_\theta, \quad \text{where } [l]_\theta = \sup_{x,y\in H,\, x\neq y} \frac{|l(x) - l(y)|}{|x-y|^\theta}.$$

In particular, $C^{0,1}_b(H)$ [resp., $C^{0,1}_b(H;H)$] is the space of $\mathbb{R}-$[resp., $H-$]valued Lipschitz continuous and bounded functions. As we have done for F in (10), we write $[l]_{\mathrm{Lip}}$ for $[l]_1$. We also consider the space

$$C^{1,\theta}_b(H) := \{f \in C^1_b(H) \text{ s.t. } Df : H \to H \text{ is } \theta-\text{Hölder continuous}\},$$
$$\text{endowed with the norm } \|f\|_{C^{1,\theta}_b} := \|f\|_1 + [Df]_\theta.$$

Theorem 4.3. *For every $\theta \in (0,1)$ such that $\gamma + \theta\gamma < 1$, the unique solution $u = u^{g,h}$ of (25) in $C^1_\gamma(H)$ satisfies*

$$u(t,\cdot) \in C^{1,\theta}_b(H), \quad t \in]0,T], \tag{30}$$

that is, the Fréchet derivative $Du(t,\cdot)$ is $\theta-$Hölder continuous from H into H. Furthermore, there exists a constant $\tilde{L} > 0$ such that

$$[Du(t,\cdot)]_\theta \leq \tilde{L}\frac{1}{t^{\gamma+\gamma\theta}}, \quad t \in]0,T]. \tag{31}$$

Proof. Fix $t \in]0,T]$ and consider $\theta \in (0,1)$ such that $\gamma + \theta\gamma < 1$. For every $\phi \in C_b(H)$ and $k, p \in H$, denoting by $D^2_{kp}P_t\phi$ the Gâteaux derivative of $\langle DP_t\phi(\cdot), p\rangle$ along the direction k, by the semigroup property of P (see also (28) in the proof of Lemma 4.1) we infer that

$$D^2_{kp}P_t\phi(x) = \langle DP_{t/2}(\langle DP_{t/2}\phi(\cdot), e^{\frac{t}{2}A}p\rangle)(x), k\rangle, \quad x \in H. \tag{32}$$

Then, by (9), since the $(e^{uA})_{u\geq 0}$ is a contraction semigroup on H,

$$\|D\langle DP_t\phi(\cdot), p\rangle\|_0 \leq \frac{C}{t^{2\gamma}}\|\phi\|_0|p|. \tag{33}$$

By [8, Theorem 2.3.3] (see also the monograph [14]), the following characterization for the interpolation space $(UC_b(H), UC^1_b(H))_{\theta,\infty}$ holds:

$$(UC_b(H), UC^1_b(H))_{\theta,\infty} = C^{0,\theta}_b(H).$$

Here, $UC_b(H)$ is the Banach space of uniformly continuous bounded $\mathbb{R}-$valued maps, and $UC_b^1(H)$ is the space of functions in $UC_b(H)$ with uniformly continuous and bounded Fréchet derivative. Denoting by $C_1 = C_1(\theta, \alpha, \gamma, T) > 0$ a positive constant allowed to change from line to line, since $\langle DP_t\phi(\cdot), p\rangle \in UC_b^1(H)$, [8, Example 2.3.4] yields

$$\begin{aligned}\|\langle DP_t\phi(\cdot), p\rangle\|_{C_b^{0,\theta}(H)} &\le C_1\|\langle DP_t\phi(\cdot), p\rangle\|_0^{1-\theta}\|\langle DP_t\phi(\cdot), p\rangle\|_1^{\theta}\\ &\le C_1\frac{1}{t^{\gamma(1-\theta)}}\left(\frac{1}{t^{\gamma\theta}} + \frac{1}{t^{2\gamma\theta}}\right)\|\phi\|_0|p|,\end{aligned}$$

where we use (9), (33) and the subadditivity of $x \mapsto x^\theta$ on $\mathbb{R}_+$ for the second estimate. Therefore

$$\begin{aligned}|\langle DP_t\phi(x) - DP_t\phi(y), p\rangle| &\le \|\langle DP_t\phi(\cdot), p\rangle\|_{C_b^{0,\theta}}|x-y|^\theta\\ &\le C_1\frac{1}{t^{\gamma(1-\theta)}}\left(\frac{1}{t^{\gamma\theta}} + \frac{1}{t^{2\gamma\theta}}\right)|p|\|\phi\|_0|x-y|^\theta\\ &\le C_1\frac{|p|}{t^{\gamma+\gamma\theta}}\|\phi\|_0|x-y|^\theta, \quad x, y \in H. \end{aligned} \tag{34}$$

Differentiating (25), by (24) and (34) we compute, for every $x, y \in H$,

$$\begin{aligned}|\langle Du(t,x) - Du(t,y), p\rangle| &\le |\langle DP_th(x) - DP_th(y), p\rangle|\\ &+ \int_0^t |\langle DP_{t-s}[\mathcal{H}(\cdot, Du(s,\cdot))](x) - DP_{t-s}[\mathcal{H}(\cdot, Du(s,\cdot))](y), p\rangle|\, ds\\ &\le C_1\left(\int_0^t\left(L\frac{\|u\|_{C_\gamma^1}}{s^\gamma} + \|g\|_0\right)\frac{1}{(t-s)^{\gamma+\gamma\theta}}ds + \frac{1}{t^{\gamma+\gamma\theta}}\|h\|_0\right)|p||x-y|^\theta. \end{aligned} \tag{35}$$

Taking the supremum over $p \in H$ such that $|p| \le 1$, this estimate shows that $Du(t,\cdot)$ is $\theta-$Hölder continuous from H into H (i.e., (30)). By [3, Equation (2.12)], we infer that

$$\int_0^t \frac{1}{s^\gamma}\frac{1}{(t-s)^{\gamma+\gamma\theta}}ds \le C_1 t^{1-2(\gamma+\theta\gamma)},$$

hence, recalling that $1 - \gamma - \theta\gamma > 0$ by assumption, (31) readily follows from (35), as well. Given that $t \in]0, T]$ is arbitrary, the proof is complete. □

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