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Feller's test for explosions of stochastic Volterra equations*

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Abstract

This paper provides a Feller's test for explosions of one-dimensional continuous stochastic Volterra processes of convolution type. The study focuses on dynamics governed by nonsingular kernels, which preserve the semimartingale property of the processes and introduce memory features through a path-dependent drift. In contrast to the classical path-independent case, the sufficient condition derived in this study for a Volterra process to remain in the interior of an interval is generally more restrictive than the necessary condition. This gap between necessary and sufficient conditions is also present in the case of a sum-of-exponentials kernel, where this work provides alternative Khasminskii-type criteria for explosion based on multi-dimensional Itô diffusions of independent interest. The results are illustrated with three specifications of the dynamics: the Volterra square-root diffusion, the Volterra Jacobi process and the Volterra power-type diffusion. For the Volterra square-root diffusion, also known as the Volterra CIR process, the paper presents a detailed discussion on the approximation of the singular fractional kernel with a sum of exponentials, a method commonly employed in the mathematical finance literature.

Keywords: stochastic Volterra equations; boundary attainment conditions; stochastic invariance; Feller's test for explosions; Khasminskii's theorem.
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1 Introduction

Consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$, where the filtration $\mathbb{F}$ satisfies the usual conditions, endowed with a standard one-dimensional $\mathbb{F}-$Brownian motion

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$W = (W_t)_{t\geq 0}$. Given two (possibly infinite) numbers l, r such that $-\infty \leq l < r \leq \infty$, define an open interval $I = (l, r)$ in the extended real line $\overline{\mathbb{R}}$. Suppose that a continuous process $X = (X_t)_{t\geq 0}$, starting at a deterministic point $X_0 = x_0 \in I$ and taking values in the closed interval $\bar{I}$, has the following property: for every $\lambda, \rho \in I$ such that $x_0 \in (\lambda, \rho)$, defining $\tau_\lambda^\rho = \inf\{t \geq 0 : X_t = \lambda \text{ or } X_t = \rho\}$, the stopped process $X^{\tau_\lambda^\rho} = \left(X_{t\wedge\tau_\lambda^\rho}\right)_{t\geq 0}$ solves the one-dimensional stochastic Volterra equation

$$X_{t\wedge\tau_\lambda^\rho} = x_0 + \int_0^t 1_{\{s\leq\tau_\lambda^\rho\}} K(t-s) b(X_{s\wedge\tau_\lambda^\rho})\,\mathrm{d}s + \int_0^t 1_{\{s\leq\tau_\lambda^\rho\}} K(t-s)\sigma(X_{s\wedge\tau_\lambda^\rho})\,\mathrm{d}W_s,$$
$$t \geq 0,\ \mathbb{P}-\text{a.s.} \tag{1.1}$$

Here, $K\colon \mathbb{R}_+ \to \mathbb{R}$ is a continuously differentiable kernel ($K \in C^1(\mathbb{R}_+)$) that introduces memory into the dynamics, since $X^{\tau_\lambda^\rho}$ is no longer necessarily Markovian. Under additional hypotheses on K, the objective of this paper is to determine boundary attainment conditions for the process X. More specifically, defining the $\mathbb{F}-$stopping time S by

$$S = \inf\{t \geq 0 : X_t \in \{l, r\}\},$$

the aim is to establish a test based on the initial point x_0 and the parameters of the dynamics (1.1) – that is, K, b and σ – to deduce whether $\mathbb{P}(S = \infty) = 1$ or not. Notice that, by the continuity of X, S represents the first time that the trajectories of X attain l or r. In case $l = -\infty$ and $r = \infty$, S is typically called the first blow-up (or explosion) time of X.

In other words, given a continuous $\bar{I}-$valued process X which is a local solution in the random interval $[0, S)$ (in the sense of (1.1)) of the stochastic Volterra equation

$$X_t = x_0 + \int_0^t K(t-s)b(X_s)\,\mathrm{d}s + \int_0^t K(t-s)\sigma(X_s)\,\mathrm{d}W_s, \quad t \in [0, S),\ \mathbb{P}-\text{a.s.},$$

this work proposes to study necessary and sufficient conditions for X to be also a global, $I-$valued solution.

Stochastic Volterra equations appear in various fields of applied mathematics. For example, in biological and chemical models describing the interactions of two substances (called reactant and catalyst), they arise from and are motivated by scaling limits of branching processes, see [2, 30]. They are also employed in volatility modeling to incorporate different levels of regularity of the trajectories and path-dependence structures, see, e.g., [16, 21] and references therein. Moreover, Volterra-type processes have been used in [13] for the stochastic modeling of energy markets, see also [14].

The boundary attainment conditions obtained in this work contribute to expanding the theory of invariance for stochastic Volterra equations. For some time, progress in this research area was limited to [6, Theorem 3.6], which, in a suitable setting, proves the existence of $\mathbb{R}_+^d-$valued solutions and can be applied, e.g., to study Volterra square-root diffusion dynamics (see also [21]). Recently, however, such invariance theory has garnered renewed interest. Notably, under Lipschitz continuity assumptions on the drift and diffusion maps, [9, Theorem 3.2] shows the invariance of closed convex subsets of $\mathbb{R}^d$ for nonsingular kernels K that preserve nonnegativity. This result also covers singular completely monotone kernels. Furthermore, [4, Theorem 5.1] establishes the existence of weak continuous solutions to stochastic Volterra equations with singular kernels taking values in the multidimensional unit ball. Here, weaker regularity assumptions on the coefficients – namely, continuity and linear growth – are sufficient. This result can be applied to certain polynomial Volterra processes, including the Volterra Jacobi process, which is also examined in this paper in Subsection 3.2. The present article is the first study of invariance properties of stochastic Volterra equations over open sets.

It is worth noticing that the arguments of this manuscript focus on one-dimensional dynamics and require the crucial hypothesis that $K(0) < \infty$, i.e., that the kernel K does not blow up at 0. When $K(0+) = \infty$, a local solution X to the corresponding stochastic Volterra equation is not a semimartingale, preempting the use of Itô's formula, which is crucial for the analysis of this paper. It is also unclear whether the $I-$boundary attainment of X can be inferred from the results established in this work in the nonexplosive setting via stability techniques. Such techniques, employed for instance in [9, Theorem 3.2] to handle a singular completely monotone kernel, consist of approximating K with a sequence of nonexplosive kernels $K^{(n)}$ (e.g., in the $L^2_{\mathrm{loc}}(\mathbb{R}_+)-$sense) such that the associated solutions $X^{(n)}$ converge to X in law, in the spirit of [5, Theorem 3.5]. In fact, the limit of a sequence of nonexplosive [resp., explosive] processes may very well be explosive [resp., nonexplosive]; this also depends on the probabilistic behavior of the explosion times along the sequence. This observation renders the stability approach unfeasible without additional results. Focusing on explosions to $\pm\infty$ of local maximal solutions (whose existence is analyzed, for example, in [34, Subsection 3.3]), when K is the fractional kernel $K(t) = \Gamma(\alpha)^{-1}t^{\alpha-1}$ with $\alpha \in (0,1)$, an Osgood-type criterion for stochastic Volterra equations driven by additive (that is, $\sigma = 1$) Riemann-Liouville fractional Brownian motions will appear in [17]. On the other hand, the case of a more general multiplicative noise appears to be completely open and could be the scope of a future research.

In the classical path-independent case $K \equiv K(0)$, when (1.1) reduces to an Itô diffusion, the problem of finding necessary and sufficient conditions for $\mathbb{P}(S = \infty) = 1$ has long been solved thanks to [22], with a strategy which has become known as *Feller's test for explosions*, see also [27, Theorem 5.29, Chapter 5]. In this approach, for a generic $c \in I$, the model parameters $K(0)$, b and σ are used to define an auxiliary function $v_c \colon I \to \mathbb{R}_+$ which is then employed to establish the following equivalence:

$$\mathbb{P}(S = \infty) = 1 \quad \Longleftrightarrow \quad v_c(l+) = v_c(r-) = \infty. \tag{1.2}$$

Perhaps the most famous application of this result is the so-called *Feller condition* for square-root diffusions, which have been extensively used, for instance, in mathematical finance to model interest rates [19] and asset variance [25].

In order to treat the case of a nonconstant convolution kernel $K \in C^1(\mathbb{R}_+)$, this paper generalizes the procedure carried out in the standard Itô framework, making it feasible to analyze the possibly non-Markovian Equation (1.1). The foundation of the arguments developed here is the fact that, since K is continuously differentiable and in particular $K(0) < \infty$, the stopped process $X^{\tau^\rho_\lambda}$ is a continuous semimartingale. In fact, it is possible to express its dynamics (1.1) in a differential form where the Itô diffusive component is separated from the path-dependent one. The path-dependent component is given in terms of the measure resolvent of the first kind L of K, which solves the convolution identity $K * L = 1$. This formulation (described in detail in Section 2) allows the application of Itô's formula and the control of the path-dependency, essential tools for the proposed methodology.

The introduction, for every $\beta, \gamma \in \mathbb{R}$, of a modified drift function $\tilde{b}_c(\cdot;\beta,\gamma) \colon I \to \mathbb{R}$ defined by

$$\begin{aligned}\tilde{b}_c(x;\beta,\gamma) =& K(0)b(x) + \frac{K'(0)}{K(0)}\,(x - x_0) \\ &+ \left((K' * L)(0) - (K' * L)(\infty)\right)\left(1_{\{x<c\}}\beta + 1_{\{x\geq c\}}\gamma\right), \quad x \in I, \text{ for some } c \in I,\end{aligned}$$

where $(K' * L)(\infty) = \lim_{t\to\infty}(K' * L)(t)$, can be considered as the main idea of this manuscript. Indeed, roughly speaking, a suitable choice of the shift parameters β and

γ, along with monotonicity assumptions on the convolution $K' * L$, enables to control the sign of the path-dependent component of (1.1), which turns out to be crucial for the analysis of $\mathbb{P}(S = \infty)$. Note that, in the classical path-independent case where $K' \equiv 0$ on $\mathbb{R}_+$, $\tilde{b}_c(\cdot;\beta,\gamma)$ reduces to the drift map $K(0)b$ because the dependence on x_0, β, γ and c is lost.

Employing the modified drift $\tilde{b}_c(\cdot;\beta,\gamma)$, by analogy with the classical case, an auxiliary function $v_c(\cdot;\beta,\gamma)\colon I \to \mathbb{R}_+$ can be defined to study necessary and sufficient conditions for $\mathbb{P}(S = \infty) = 1$. More precisely, given $c \in I$, Theorem 2.12 shows that

$$v_c(l+;\beta,\gamma) = v_c(r-;\beta,\gamma) = \infty, \text{ for every } \beta < 0 < \gamma, \tag{1.3}$$

is a necessary condition for $\mathbb{P}(S = \infty) = 1$. Meanwhile, Theorem 2.13 asserts that, given two real-valued sequences $(\lambda_n)_n$ and $(\rho_n)_n$ such that $\lambda_n \downarrow l$ and $\rho_n \uparrow r$ as $n \to \infty$, it is sufficient that

$$\lim_{n\to\infty} v_c(\lambda_n; x_0 - \lambda_n, x_0 - \rho_n) = \lim_{n\to\infty} v_c(\rho_n; x_0 - \lambda_n, x_0 - \rho_n) = \infty \tag{1.4}$$

to infer $\mathbb{P}(S = \infty) = 1$. Contrary to the equivalence in (1.2) which holds when $K \equiv K(0)$, in general, the requirement in (1.4) is stronger than the one in (1.3), see Lemma 2.11. However, the conditions (1.3)-(1.4) coincide when I is bounded (under further assumptions on σ) or K is an exponential kernel, as detailed in Corollary 2.15 and the subsequent Remark 2.16. These additional assumptions make $v_c(\cdot;\beta,\gamma)$ independent of the shift parameters β and γ, bringing the setting back to the standard Feller's test formulation.

Intuition on the structure of the necessary and sufficient conditions When the kernel K driving the dynamics of X is an exponential, namely $K(t) = K(0)e^{-\Lambda t}$, $t \geq 0$, for some $\Lambda > 0$, the stopped process $X^{\tau_\lambda^\rho}$ is a standard Itô diffusion. By the stochastic product rule, the stochastic Volterra equation (1.1) that it satisfies can be rewritten in differential form as follows:

$$\mathrm{d}X_t^{\tau_\lambda^\rho} = 1_{\{t\leq\tau_\lambda^\rho\}}\Big[b(X_t^{\tau_\lambda^\rho}) - \Lambda(X_t^{\tau_\lambda^\rho} - x_0)\Big]\mathrm{d}t + 1_{\{t\leq\tau_\lambda^\rho\}}\sigma(X_t^{\tau_\lambda^\rho})\mathrm{d}W_t, \qquad X_0^{\tau_\lambda^\rho} = x_0. \tag{1.5}$$

In particular, note that the drift term depends on the initial state x_0, and so do the conditions in the standard Feller's test for $\mathbb{P}(S = \infty) = 1$, see (1.2). By extension, in the case of a general kernel K which induces path–dependent dynamics, we expect to determine boundary attainment conditions which depend on x_0, unlike the classical one-dimensional Feller's test for explosions.

Consider now a kernel K_N given by a weighted sum of exponentials, i.e., $K_N(t) = \sum_{n=1}^N m_n e^{-\Lambda_n t}$ for $t \geq 0$ and $m_n, \Lambda_n > 0$, $n = 1, \ldots, N$. Suppose also that b and σ in (1.1) are defined in $\mathbb{R}$, so that the focus is on blow-ups to $\pm\infty$. The stochastic Volterra equation (1.1) corresponding to K_N does not admit a local solution X_N given by an Itô diffusion, as it features a memory component and cannot be expressed as a path-independent one-dimensional SDE. However, a local solution X_N of (1.1) can be constructed by employing a sum of factors solving an $N-$dimensional system of SDEs, in the spirit of [1, 5]. More precisely, if we define X_N by $X_N = x_0 + \sum_{n=1}^N m_n Y_n$ on $[0, S)$, where for every $n = 1, \ldots, N$

$$\begin{aligned}\mathrm{d}Y_{n,t}^{\tau^\rho} &= 1_{\{t\leq\tau^\rho\}}\Big[-\Lambda_n Y_{n,t}^{\tau^\rho} + b\Big(x_0 + \sum_{n=1}^N m_n Y_{n,t}^{\tau^\rho}\Big)\Big]\mathrm{d}t + 1_{\{t\leq\tau^\rho\}}\sigma\Big(x_0 + \sum_{n=1}^N m_n Y_{n,t}^{\tau^\rho}\Big)\mathrm{d}W_t,\\ Y_{n,0} &= 0,\end{aligned} \tag{1.6}$$

then X^N is a local solution of (1.1) with kernel K_N on $[0, S)$. Here,

$$S = \inf\{t \geq 0 : |[Y_{1,t}, \dots, Y_{N,t}]^\top| = \infty\}, \quad \tau^\rho = \inf\{t \geq 0 : |[Y_{1,t}, \dots, Y_{N,t}]^\top| = \rho\},\ \rho > 0.$$

It follows that $\mathbb{P}(S = \infty) = 1$ is equivalent to X_N being a global solution. To study the non-explosion of the stochastic system (1.6), we rewrite it in the form

$$\mathrm{d}Y_t^{\tau^\rho} = 1_{\{t \leq \tau^\rho\}} B(Y_t^{\tau^\rho}; x_0)\mathrm{d}t + 1_{\{t \leq \tau^\rho\}} \Sigma(Y_t^{\tau^\rho}; x_0)\mathrm{d}W_t, \quad Y_0 = 0 \in \mathbb{R}^N,$$

where $Y = [Y_1, \dots, Y_N]^\top$ and, for every $y = [y_1, \dots, y_N]^\top \in \mathbb{R}^N$,

$$\begin{aligned} B(y; x_0) &= \Big[-\Lambda_n y_n + b\Big(x_0 + \sum_{j=1}^N m_j y_j\Big)\Big]^\top_{n=1,\dots N} \in \mathbb{R}^N, \\ \Sigma(y; x_0) &= \Big[\sigma\Big(x_0 + \sum_{j=1}^N m_j y_j\Big)\Big]^\top_{n=1,\dots N} \in \mathbb{R}^N. \end{aligned} \tag{1.7}$$

According to the Khasminskii-type criterion that we provide in Appendix B, which extends classical results in e.g. [29], necessary and sufficient conditions for $\mathbb{P}(S = \infty) = 1$ do not coincide. In particular, the necessary [resp., sufficient] condition involves computing the $\inf$ [resp., $\sup$] over $(N-1)-$dimensional hyperspheres of quantities defined through the coefficients $B(\cdot; x_0)$ and $\Sigma(\cdot; x_0)$; we refer the reader to Theorem B.1 for the precise statements. Consequently, in the case of an arbitrary kernel K, in general we expect a gap between the necessary and sufficients criteria.

The paper is organized as follows. Section 2 sets the stage by imposing suitable assumptions on the kernel K and the coefficients b and σ, and by deducing a differential expression for the dynamics (1.1) of the stopped process $X^{\tau^\rho_\lambda}$, see Lemma 2.1. Moreover, by analogy with the classical path-independent case, it introduces a so-called scale function $p_c(\cdot; \beta, \gamma)\colon I \to \mathbb{R}$ and the map $v_c(\cdot; \beta, \gamma)$ discussed above. Then, Subsection 2.1 shows how to employ $p_c(\cdot; \beta, \gamma)$ to derive information on the $\inf$ and the $\sup$ of X in the random interval $[0, S)$. Subsection 2.2 contains the main contribution of this work, which is a Feller-type test for the $I-$boundary attainment of X. This test is divided into a necessary condition and a sufficient one for $\mathbb{P}(S = \infty) = 1$, which, in general, do not coincide. Such conditions are presented in Theorems 2.12-2.13. Section 3 applies the theoretical results of Section 2 to three specifications of (1.1) which are frequent in the literature. In order of appearance, they are: the Volterra CIR process, the Volterra Jacobi process and the Volterra power-type diffusion process. Each example is investigated in a separate subsection. In particular, Subsection 3.1 treats Volterra square-root diffusions and the approximation of the fractional kernel with nonsingular kernels, a common practice in mathematical finance. Here, we also conduct numerical experiments to investigate parameters sets where the Feller-type test of Subsection 2.2 does not apply. Finally, Appendix A contains the proofs of some technical results about the dependence of $p_c(\cdot; \beta, \gamma)$ and $v_c(\cdot; \beta, \gamma)$ on β, γ and c, while Appendix B provides a Khasminskii-type criterion that offers a different approach to boundary attainment for stochastic Volterra equations when the kernel is a sum of exponentials. On the one hand, this result complements the existing literature on Khasminskii criteria; on the other, it provides conditions that are very difficult to use in practice, thereby highlighting the convenience of our Feller-type test, which relies on the measure resolvent L, see also the concluding Remark B.3.

2 Boundary attainment conditions for stochastic Volterra equations

We let $\mathbb{R}_+ = [0, \infty)$ and denote by $\delta_a(\mathrm{d}t)$ the Dirac measure on $\mathbb{R}_+$ centered in $a \in \mathbb{R}_+$. For a right-continuous function $f\colon \mathbb{R}_+ \to \mathbb{R}$ of locally bounded variation, we let $\mathrm{d}f(t)$ be the Lebesgue-Stieltjes measure on the $\delta-$ring $\cup_{n\in\mathbb{N}}\mathcal{B}([0,n])$ associated with f. By convention, $\mathrm{d}f(\{0\}) = 0$.

In this paper, we consider a kernel $K\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following requirement.

Hypothesis 1. *The kernel K belongs to $C^1(\mathbb{R}_+)$, with $K' \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+)$, and admits a nonnegative measure resolvent of the first kind L, that is, a nonnegative locally finite measure L solving (almost everywhere) the convolution identity $K * L = 1$.*

Note that the map $t \mapsto L([0,t])$ is right-continuous in $\mathbb{R}_+$.[1] Therefore, the convolution $K * L$ is right-continuous in $\mathbb{R}_+$, cf. [24, Corollary 6.2, Chapter 3], and the resolvent identity yields $L(\{0\}) = K(0)^{-1}$. By definition, L is finite on compact sets, hence $K(0) > 0$.

Define the open interval $I = (l, r)$, with l, $r \in \overline{\mathbb{R}}$ such that $-\infty \le l < r \le \infty$. Take a decreasing sequence $(l_n)_n \subset \mathbb{R}$ and an increasing sequence $(r_n)_n \subset \mathbb{R}$ such that $l_n < r_n$, for every $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} l_n = l, \qquad \lim_{n\to\infty} r_n = r.$$

Letting $I_n = (l_n, r_n)$, it is clear that $I = \cup_n I_n$.

On a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\ge0})$ where the filtration $\mathbb{F}$ satisfies the usual conditions, we consider an $\mathbb{F}-$adapted, $\bar{I} = [l, r]-$valued continuous process $X = (X_t)_{t\ge0}$ starting from $x_0 \in I$. We define $S_n = \inf\{t \ge 0 : X_t \notin I_n\}$ and $S = \inf\{t \ge 0 : X_t \notin I\}$. Observe that $(S_n)_n$ is an increasing sequence of $\mathbb{F}-$stopping times converging to S. Given a measurable drift $b\colon I \to \mathbb{R}$ such that

$$\mathbb{P}\bigg(\int_0^{t\wedge S} |b(X_s)|\mathrm{d}s < \infty\bigg) = 1, \quad t \ge 0, \tag{2.1}$$

and a measurable locally bounded diffusion $\sigma\colon I \to \mathbb{R}$, we suppose that X satisfies the following stochastic Volterra equation (henceforth, SVE):

$$\begin{aligned} X_t = x_0 + \int_0^t 1_{\{s\le S_n\}} K(t-s)b(X_s)\,\mathrm{d}s + \int_0^t 1_{\{s\le S_n\}} K(t-s)\sigma(X_s)\,\mathrm{d}W_s, \\ t \in [0, S_n],\ \mathbb{P}-\text{a.s.}, \end{aligned} \tag{2.2}$$

for every $n \in \mathbb{N}$. Here $W = (W_t)_{t\ge0}$ is a standard one–dimensional Brownian motion. We remark that the integral processes appearing in the right-hand side of (2.2) are well defined and continuous in $\mathbb{R}_+$. Indeed, the continuity of the integral in the Lebesgue measure follows from (2.1), [24, Corollary 2.3, Chapter 2] and the continuity of K in $\mathbb{R}_+$. As for the stochastic integral process, the continuity of its trajectories is given by [6, Lemma 2.4], which can be applied because $1_{\{\cdot\le S_n\}}X$ is bounded and σ is locally bounded in I. Since the equality in (2.2) holds in the random interval $[0, S_n]$, for every $n \in \mathbb{N}$, we say that

$$X \text{ is a local solution of the SVE} \quad X = x_0 + K * b(X) + K * (\sigma(X)\mathrm{d}W) \ \text{ in } [0, S),$$

[1] Fix $\bar{t} \ge 0$ and consider a sequence $(t_n)_n \subset (\bar{t}, T)$ that converges to $\bar{t}$. Since, for every $n \in \mathbb{N}$, $L([0,t_n]) = \int_0^T 1_{\{t\le t_n\}} L(\mathrm{d}t)$, and $1_{\{t\le t_n\}} \to 1_{\{t\le\bar{t}\}}$ for every $t \in [0,T]$ as $n \to \infty$, by the dominated convergence theorem $\lim_{n\to\infty} L([0,t_n]) = L([0,\bar{t}\,])$. Note that the right-continuity of $t \mapsto L([0,t])$ on $\mathbb{R}_+$ follows also from the fact that L – being locally finite – is a Borel regular measure on $\mathcal{B}([0,T])$, $T > 0$.

where $*$ denotes the convolution operation.[2] Please notice that we are only interested in the dynamics of X before it hits the boundary points l or r, that is, on the intervals $[0, S_n]$, $n \in \mathbb{N}$; if $S < \infty$, the values of X_t when $t > S$ are not relevant for our analysis.

Our objective is to determine conditions, both necessary and sufficient, for $\mathbb{P}(S = \infty) = 1$, namely, to ensure that X does not exit from the interval I, see Subsection 2.2. Moreover, in Subsection 2.1, we discuss sufficient conditions that can be employed to derive information on the $\sup$ and the $\inf$ of the solution process X of (2.2) in $[0, S)$.

Since K is continuously differentiable on $\mathbb{R}_+$ and in particular $K(0) \in (0, \infty)$, i.e., the kernel K does not explode at 0, we can express the dynamics (2.2) of X in differential form. This is done in Lemma 2.1, which is an analogue of Lemma 2.6 in [6] under different assumptions on the continuity of the drift and diffusion coefficients. This result is crucial for the subsequent arguments, as it shows that the stopped processes X^{S_n}, $n \in \mathbb{N}$, are continuous semimartingales and enables us to use Itô's formula. Moreover, it expresses the path-dependent drift component in terms of the measure resolvent L, which is fundamental for the arguments in the sequel.

Lemma 2.1. *For every $n \in \mathbb{N}$, the stopped process X^{S_n} satisfies the following stochastic differential equation in $\mathbb{R}_+$:*

$$\begin{aligned} \mathrm{d}X_t^{S_n} =& 1_{\{t \le S_n\}} \left[\left(K(0) b(X_t^{S_n}) + \frac{K'(0)}{K(0)} X_t^{S_n} \right) \mathrm{d}t + K(0) \sigma(X_t^{S_n}) \mathrm{d}W_t \right] \\ &+ 1_{\{t \le S_n\}} \left[\left(\mathrm{d} \left(K' * L \right) * X^{S_n} \right)(t) - x_0 \left(K' * L \right)(t) \right] \mathrm{d}t, \end{aligned} \tag{2.3}$$

with initial condition $X_0^{S_n} = x_0$.

Proof. Fix $n \in \mathbb{N}$. Consider the continuous semimartingale $Z^{(n)}$ starting at 0 defined by

$$Z_t^{(n)} = \int_0^t 1_{\{s \le S_n\}} b(X_s^{S_n}) \mathrm{d}s + \int_0^t 1_{\{s \le S_n\}} \sigma(X_s^{S_n}) \mathrm{d}W_s, \quad t \ge 0.$$

Since $K \in C^1(\mathbb{R}_+)$, by the fundamental theorem of calculus $K = K(0) + K' * 1$. As a result, from (2.2) we infer that, $\mathbb{P}-$a.s.,

$$\begin{aligned} X_t^{S_n} &= x_0 + \left(K * \mathrm{d}Z^{(n)} \right)_t = x_0 + K(0) Z_t^{(n)} + \left((K' * 1) * \mathrm{d}Z^{(n)} \right)_t \\ &= x_0 + K(0) Z_t^{(n)} + \left(1 * \left(K' * \mathrm{d}Z^{(n)} \right) \right)(t), \quad t \in [0, S_n], \end{aligned}$$

where in the last step we also employ the associativity property of the stochastic convolution in [6, Lemma 2.1]. By Hypothesis 1 (see also the subsequent comment) and [24, Corollary 6.2, Chapter 3], the map $t \mapsto (K' * L)(t)$ is right-continuous and of locally bounded variation. Therefore, we can proceed as in the proof of [6, Lemma 2.6] to obtain from the previous equation, $\mathbb{P}-$a.s.,

$$\begin{aligned} X_t^{S_n} =& x_0 + K(0) Z_t^{(n)} + \frac{K'(0)}{K(0)} \left(1 * X^{S_n} \right)(t) - x_0 \left(1 * (K' * L) \right)(t) \\ &+ \left(1 * \left(\mathrm{d}(K' * L) * X^{S_n} \right) \right)(t), \quad t \in [0, S_n]. \end{aligned}$$

In differential form, this is precisely (2.3), hence the proof is complete. □

Remark 2.2. Consider the SVE (2.2) with two kernels, respectively K_1 and K_2, driving the drift and diffusion components, i.e.,

$$X_t = x_0 + \int_0^t 1_{\{s \le S_n\}} K_1(t-s) b(X_s) \, \mathrm{d}s + \int_0^t 1_{\{s \le S_n\}} K_2(t-s) \sigma(X_s) \, \mathrm{d}W_s,$$

[2] Given a process $f \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}$ and a function $g \colon \mathbb{R}_+ \to \mathbb{R}$ [resp., a locally-finite measure μ on $\mathbb{R}_+$, or a Brownian motion W], the convolution operator $*$ is defined by $(f * g)(t) = \int_0^t f(t-s) g(s) \mathrm{d}s$, $t \ge 0$ [resp., $(f * \mu)(t) = \int_{[0,t]} f(t-s) \mu(\mathrm{d}s)$, or $(f * \mathrm{d}W)_t = \int_0^t f(t-s) \mathrm{d}W_s$], if this integral exists.

which holds for every $t \in [0, S_n]$ and $n \in \mathbb{N}$, $\mathbb{P}-$a.s. If K_1 and K_2 satisfy Hypothesis 1, following the arguments of Lemma 2.1, one can show that the stopped processes X^{S_n} are semimartingales that admit the following decomposition on $\mathbb{R}_+$:

$$\begin{aligned}\mathrm{d}X_t^{S_n} =& 1_{\{t\le S_n\}}\left[K_1(0)b(X_t^{S_n})\mathrm{d}t + K_2(0)\sigma(X_t^{S_n})\mathrm{d}W_t\right]\\ &+ 1_{\{t\le S_n\}}\Bigg[\frac{K_1'(0)}{K_1(0)}\int_0^t K_1(t-s)b(X_s^{S_n})\mathrm{d}s + \frac{K_2'(0)}{K_2(0)}\int_0^t K_2(t-s)\sigma(X_s^{S_n})\mathrm{d}W_s\\ &\quad + \left(\mathrm{d}\left(K_1'*L_1\right)*\left(K_1*b(X^{S_n})\right) + \mathrm{d}\left(K_2'*L_2\right)*\left(K_2*\sigma(X^{S_n})\mathrm{d}W\right)\right)(t)\Bigg]\mathrm{d}t.\end{aligned}$$

Here L_1 [resp., L_2] is the resolvent of the first kind of K_1 [resp., K_2]. In contrast to (2.3), the path-dependent drift component on the second and third lines of the previous equation cannot be expressed as a linear combination between X and a path-by-path convolution of a measure (or a function) with X. This is a fundamental difference, which makes the shifting approach that we develop in the sequel unfeasible. For an analogous reason, it is not immediately clear how to treat the following non-convolution framework with our methods (even in the case of a single kernel $K(t,s)$):

$$X_t = x_0 + \int_0^t 1_{\{s\le S_n\}}K(t,s)b(X_s)\,\mathrm{d}s + \int_0^t 1_{\{s\le S_n\}}K(t,s)\sigma(X_s)\,\mathrm{d}W_s, \quad n\in\mathbb{N}.$$

Indeed, although it is well known that – under suitable hypotheses on K – a solution process X is a semimartingale simply by the fundamental theorem of calculus and the stochastic Fubini theorem, see [31, Theorem 3.3], the expression of the path-dependent drift does not allow to carry out the analysis we explore below. Note, in particular, that the concept of measure resolvent is not available in the non-convolution set-up. Moreover, in this case the coefficients in the Itô diffusion component are time-dependent, which is another important difference compared to the standard Feller test.

We define the modified drift $\tilde{b}(\cdot;x_0)\colon I\to\mathbb{R}$ and the modified diffusion $\tilde{\sigma}\colon I\to\mathbb{R}$ by

$$\tilde{b}(x;x_0) = K(0)b(x) + \frac{K'(0)}{K(0)}(x-x_0), \qquad \tilde{\sigma}(x) = K(0)\sigma(x), \quad x\in I.$$

In this way, for every $n\in\mathbb{N}$, Equation (2.3) reads

$$\begin{aligned}\mathrm{d}X_t^{S_n} =& 1_{\{t\le S_n\}}\left[\tilde{b}(X_t^{S_n};x_0)\,\mathrm{d}t + \tilde{\sigma}(X_t^{S_n})\,\mathrm{d}W_t\right]\\ &+ 1_{\{t\le S_n\}}\left[\left(\mathrm{d}\left(K'*L\right)*\left(X^{S_n}-x_0\right)\right)(t)\right]\mathrm{d}t, \quad X_0^{S_n}=x_0.\end{aligned} \tag{2.4}$$

Notice that the first two addends in the right-hand side of (2.4) describe an Itô diffusion, while the last drift term takes into account the path-dependent part of the dynamics. In the sequel, we denote the modified drift $\tilde{b}(\cdot;x_0)$ by $\tilde{b}$, omitting the dependence on x_0 to keep the notation simple.

Remark 2.3. If $K\in C^1(\mathbb{R}_+)$ is a completely monotone kernel on $(0,\infty)$ not identically equal to 0, then it satisfies Hypothesis 1. In particular, by [24, Theorem 5.4, Chapter 5], its resolvent of the first kind L admits the decomposition $L(\mathrm{d}t) = K(0)^{-1}\delta_0(\mathrm{d}t) + \rho(t)\,\mathrm{d}t$, for some $\rho\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ completely monotone on $(0,\infty)$. From the resolvent identity $K*L=1$ and [24, Corollary 7.3, Chapter 3], it then follows that $K'*L = -K(0)\rho$ in $(0,\infty)$. Taking the limit as $t\to 0+$, by [24, Corollary 6.2, Chapter 3] we deduce that $K(0)\rho(0+) = -\frac{K'(0)}{K(0)}$. Thus, for every $n\in\mathbb{N}$, (2.3) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}X_t^{S_n}}{K(0)} =& 1_{\{t\le S_n\}}\left[\left(b(X_t^{S_n}) - \rho(0+)(X_t^{S_n}-x_0)\right)\mathrm{d}t + \sigma(X_t^{S_n})\,\mathrm{d}W_t\right]\\ &- 1_{\{t\le S_n\}}\left[\left(\rho'*(X^{S_n}-x_0)\right)(t)\right]\mathrm{d}t,\end{aligned}$$

with $X_0^{S_n} = x_0$.

By analogy with the classical case treated, for instance, in [27, Section 5.5.C], we impose the next assumptions of nondegeneracy and local integrability on the coefficients $\tilde{b}$ and $\tilde{\sigma}$ of (2.4).

Hypothesis 2. *The map $\tilde{\sigma} \neq 0$ almost everywhere in I. Moreover, the functions $\tilde{\sigma}^{-2}$ and $|\tilde{b}|\tilde{\sigma}^{-2}$ are locally integrable in I.*

Under Hypotheses 1 and 2, if we suppose that there exists finite $(K' * L)(\infty) = \lim_{t\to\infty}(K' * L)(t) \in \mathbb{R}$ (see also Hypothesis 3 and Remark 2.7 below), for every $c \in I$ and $\beta, \gamma \in \mathbb{R}$, we consider the map $\tilde{b}_c(\cdot;\beta,\gamma)\colon I \to \mathbb{R}$ given by

$$\tilde{b}_c(x;\beta,\gamma) = \tilde{b}(x) + \left(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\right)\left(1_{\{x<c\}}\beta + 1_{\{x\geq c\}}\gamma\right), \quad x \in I. \tag{2.5}$$

Such $\tilde{b}_c(\cdot;\beta,\gamma)$ can be interpreted as a shifted modified drift, where the shift parameters are β and γ. Clearly $\tilde{b}_c(\cdot;\beta,\gamma)$ satisfies Hypothesis 2, as well. Notice that, when $x \in (l,c)$ [resp., $x \in [c,r)$], the value of $\tilde{b}_c(x;\beta,\gamma)$ depends only on β [resp., γ]. In what follows, we will then write $\tilde{b}_c(x;\beta)$ [resp., $\tilde{b}_c(x;\gamma)$] for $x \in (l,c)$ [resp., $x \in [c,r)$] to simplify the notation.

We employ $\tilde{b}_c(\cdot;\beta,\gamma)$ and $\tilde{\sigma}$ to define the *scale function* $p_c(\cdot;\beta,\gamma)\colon I \to \mathbb{R}$ by

$$p_c(x;\beta,\gamma) = \int_c^x \exp\left\{-2\int_c^y \frac{\tilde{b}_c(z;\beta,\gamma)}{\tilde{\sigma}^2(z)}\mathrm{d}z\right\}\mathrm{d}y, \quad x \in I. \tag{2.6}$$

Also in this case, we only write $p_c(x;\beta)$ [resp. $p_c(x;\gamma)$] when $x \in (l,c)$ [resp., $x \in [c,r)$]. Observe that $p_c(\cdot;\beta,\gamma)$ is null at $x = c$ and (strictly) increasing in I. As a result, the limits $p_c(l+;\beta)$ and $p_c(r-;\gamma)$ always exist, finite or infinite. The mapping $p_c(\cdot;\beta,\gamma)$ belongs to $C^1(I)$, and a direct computation shows that

$$p_c''(x;\beta,\gamma) = -2\frac{\tilde{b}_c(x;\beta,\gamma)}{\tilde{\sigma}^2(x)}p_c'(x;\beta,\gamma), \quad \text{for a.e. } x \in I. \tag{2.7}$$

As in the classical path-independent case, we use the scale function $p_c(\cdot;\beta,\gamma)$ to define, by recursion, the sequence $(u_{c,n}(\cdot;\beta,\gamma))_{n\in\mathbb{N}_0}$ of $\mathbb{R}_+-$valued functions on I where $u_{c,0}(\cdot;\beta,\gamma) \equiv 1$ and

$$u_{c,n}(x;\beta,\gamma) = 2\int_c^x p_c'(y;\beta,\gamma)\left(\int_c^y \frac{u_{c,n-1}(z;\beta,\gamma)}{p_c'(z;\beta,\gamma)\tilde{\sigma}^2(z)}\mathrm{d}z\right)\mathrm{d}y, \quad x \in I,\ n \in \mathbb{N}. \tag{2.8}$$

Of particular importance is the map

$$v_c(x;\beta,\gamma) = u_{c,1}(x;\beta,\gamma), \quad x \in I. \tag{2.9}$$

Indeed, for suitable values of β and γ, $v_c(\cdot;\beta,\gamma)$ will be employed to establish necessary and sufficient conditions for $\mathbb{P}(S = \infty) = 1$, see Theorems 2.12-2.13 in Subsection 2.2. Notice that $v_c(\cdot;\beta,\gamma)$ is decreasing in (l,c), increasing in (c,r) and equals to 0 at $x = c$. As usual, we will write $v_c(x;\beta)$, $u_{c,n}(x,\beta)$ [resp., $v_c(x;\gamma)$, $u_{c,n}(x;\gamma)$] when $x \in (l,c)$ [resp., $x \in [c,r)$]. Recalling that the modified drift $\tilde{b}$ depends on x_0, we observe that $\tilde{b}_c(\cdot;\beta,\gamma)$, and consequently $p_c(\cdot;\beta,\gamma)$ and $u_{c,n}(\cdot;\beta,\gamma)$, depend on the initial condition x_0, as well.

We now present two technical results which analyze the dependence of the maps $p_c(\cdot;\beta,\gamma)$ and $v_c(\cdot;\beta,\gamma)$ on the parameter $c \in I$. Their proofs, which essentially rely on the definitions in (2.6) and (2.9), are postponed to Appendix A.

Lemma 2.4. *Let β, $\gamma \in \mathbb{R}$ and consider $c_1, c_2 \in I$ such that $c_1 < c_2$. Then, for every $x \in (l, c_1)$,*

$$\begin{aligned} p_{c_2}(x;\beta) &= p_{c_2}(c_1;\beta) + p'_{c_2}(c_1;\beta)p_{c_1}(x;\beta); \\ v_{c_2}(x;\beta) &= v_{c_2}(c_1;\beta) + p_{c_1}(x;\beta)v'_{c_2}(c_1;\beta) + v_{c_1}(x;\beta), \end{aligned} \tag{2.10}$$

and, for every $x \in (c_2, r)$,

$$\begin{aligned} p_{c_1}(x;\gamma) &= p_{c_1}(c_2;\gamma) + p'_{c_1}(c_2;\gamma)p_{c_2}(x;\gamma); \\ v_{c_1}(x;\gamma) &= v_{c_1}(c_2;\gamma) + p_{c_2}(x;\gamma)v'_{c_1}(c_2;\gamma) + v_{c_2}(x;\gamma). \end{aligned} \tag{2.11}$$

Corollary 2.5. *Let β, $\gamma \in \mathbb{R}$. Then the finiteness of $v_c(l+;\beta)$ or $v_c(r-;\gamma)$ does not depend on the choice of $c \in I$, namely, for every c_1, $c_2 \in I$,*

$$\begin{aligned} &v_{c_1}(l+;\beta) = \infty \text{ [resp., } v_{c_1}(r-;\gamma) = \infty] \quad \textit{if and only if} \\ &v_{c_2}(l+;\beta) = \infty \text{ [resp., } v_{c_2}(r-;\gamma) = \infty]. \end{aligned}$$

We conclude this preliminary part by formulating an additional requirement on the kernel K and its resolvent of the first kind L.

Hypothesis 3. *The map $t \mapsto (K' * L)(t)$ is nondecreasing and bounded above on $\mathbb{R}_+$.*

Under Hypothesis 3, the Lebesgue-Stieltjes measure $\mathrm{d}(K' * L)(t)$ is nonnegative. This fact will be an essential tool for the arguments of Subsections 2.1-2.2, as it will allow to handle the path-dependent term in the dynamics (2.4). Moreover, Hypothesis 3 ensures the existence (and finiteness) of the limit $(K' * L)(\infty)$; recall that this quantity is required to define the shifted modified drift $\tilde{b}_c(\cdot;\beta,\gamma)$ in (2.5).

Remark 2.6. Thanks to the computations in Remark 2.3, a kernel $K \in C^1(\mathbb{R}_+)$ which is completely monotone on $(0,\infty)$ satisfies Hypothesis 3.

More generally, Hypothesis 3 is met when $K' \leq 0$ and $t \mapsto L([0,t])$ is concave in $\mathbb{R}_+$. Indeed, $K' \leq 0$ together with the fact that L is a nonnegative measure implies $K' * L \leq 0$. The concavity of $L([0,\cdot])$ ensures that such a function is differentiable a.e. in $\mathbb{R}_+$, with a nonincreasing derivative. Differentiating the resolvent identity we obtain

$$(K' * L)(t) = -K(0)\frac{\mathrm{d}}{\mathrm{d}t}L([0,t]), \quad \text{a.e. } t \geq 0,$$

which shows that the right-continuous map $K' * L$ (cf. [24, Corollary 6.2, Chapter 3]) is nondecreasing in $\mathbb{R}_+$.

Remark 2.7. As in Remark 2.3, if we consider a kernel $K \in C^1(\mathbb{R}_+)$ which is completely monotone on $(0,\infty)$ and not identically equal to 0, then we can compute $(K' * L)(\infty)$ explicitly. More precisely, since $L(\mathrm{d}t) = K(0)^{-1}\delta_0(\mathrm{d}t) + \rho(t)\,\mathrm{d}t$ for some $\rho \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ completely monotone on $(0,\infty)$, the resolvent identity implies that

$$\lim_{t\to\infty}\left(\int_0^t K(t-s)\rho(s)\mathrm{d}s\right) = 1 - \frac{K(\infty)}{K(0)} \in [0,1].$$

We now distinguish two cases. If $K \in L^1(\mathbb{R}_+)$, i.e., K is integrable on $\mathbb{R}_+$, then $K(\infty) = 0$ and

$$1 - \frac{K(t)}{K(0)} = \int_0^t K(s)\rho(t-s)\mathrm{d}s, \quad t > 0. \tag{2.12}$$

Taking the limit as $t \to \infty$ and recalling that ρ is nonincreasing, the dominated convergence theorem yields $\rho(\infty) = \|K\|_{L^1(\mathbb{R}_+)}{}^{-1}$. Since $K' * L = -K(0)\rho$ on $(0,\infty)$ (see Remark 2.3), it follows that

$$(K' * L)(\infty) = -\frac{K(0)}{\|K\|_{L^1(\mathbb{R}_+)}}. \tag{2.13}$$

If instead K is not integrable on $\mathbb{R}_+$, then necessarily $(K' * L)(\infty) = 0$. Indeed, by the resolvent identity (see (2.12))

$$1 - \frac{K(t)}{K(0)} \geq \rho(t)\|K\|_{L^1(0,t)}, \quad t > 0,$$

where we use that ρ is nonincreasing and K nonnegative. The limit as $t \to \infty$ in the previous estimate gives $\rho(\infty) = 0$, which entails – thanks to $K' * L = -K(0)\rho$ on $(0, \infty)$ – that $(K' * L)(\infty) = 0$.

2.1 Sufficient conditions via $p_c(\cdot; \beta, \gamma)$

In this subsection, we show that the values $p_c(l+; \beta)$ and $p_c(r-; \gamma)$ can be employed – for a suitable choice of $\beta, \gamma \in \mathbb{R}$ – to infer properties of the $\inf$ or the $\sup$ of the solution process X of (2.2) in the random interval $[0, S)$. The results that we obtain are shown in Proposition 2.9, whose proof also relies on the technical Lemma 2.8. Finally, Remark 2.10 explains why, for kernels K such that $(K' * L)(\infty) > (K' * L)(0)$ (see also Remark 2.16, where we characterize the kernels for which this inequality is in fact an equality), Proposition 2.9 can only be applied to half-bounded intervals.

Lemma 2.8. *Assume Hypotheses 1-2-3. Fix $c \in I$ and consider $\beta_1,\ \beta_2 \in \mathbb{R}$ and $\gamma_1,\ \gamma_2 \in \mathbb{R}$ such that $\beta_1 \geq \beta_2$ and $\gamma_1 \geq \gamma_2$. Then $p_c(\cdot; \beta_1, \gamma_1) \geq p_c(\cdot; \beta_2, \gamma_2)$ on I.*

Proof. Fix $c \in I$ and real parameters β_i, γ_i according to the statement of this lemma. By the definition in (2.6),

$$p_c(x; \beta_1) = -\int_x^c \exp\left\{2\int_y^c \frac{\tilde{b}(z) - |(K' * L)(0) - (K' * L)(\infty)|\beta_1}{\tilde{\sigma}^2(z)}\mathrm{d}z\right\}\mathrm{d}y \geq p_c(x; \beta_2)$$

for $x \in (l, c)$, and

$$p_c(x; \gamma_1) = \int_c^x \exp\left\{2\int_c^y \frac{-\tilde{b}(z) + |(K' * L)(0) - (K' * L)(\infty)|\gamma_1}{\tilde{\sigma}^2(z)}\mathrm{d}z\right\}\mathrm{d}y \geq p_c(x; \gamma_2)$$

for $x \in [c, r)$. These inequalities complete the proof. □

Proposition 2.9. *Let $c \in I = (l, r)$ and $p_c(\cdot; \beta, \gamma)$, $\beta, \gamma \in \mathbb{R}$, be as in* (2.6)*. Assume Hypotheses 1-2-3.*

(i) If $p_c(x; x_0 - r_n, x_0 - r_n)$ is bounded below, that is, $\inf_{n\in\mathbb{N}} p_c(l+; x_0 - r_n) > -\infty$, and $\lim_{n\to\infty} p_c(r_n; x_0 - r_n) = \infty$, then

$$\mathbb{P}\left(\sup_{0\leq t<S} X_t < r\right) = 1.$$

If $r < \infty$, then the assumption $\inf_{n\in\mathbb{N}} p_c(l+; x_0 - r_n) > -\infty$ is equivalent to $p_c(l+; x_0 - r) > -\infty$, while $\lim_{n\to\infty} p_c(r_n; x_0 - r_n) = \infty$ is implied by $p_c(r-; x_0 - r) = \infty$;

(ii) If $p_c(x; x_0 - l_n, x_0 - l_n)$ is bounded above, that is, $\sup_{n\in\mathbb{N}} p_c(r-; x_0 - l_n) < \infty$, and $\lim_{n\to\infty} p_c(l_n; x_0 - l_n) = -\infty$, then

$$\mathbb{P}\left(\inf_{0\leq t<S} X_t > l\right) = 1.$$

If $l > -\infty$, then the assumption $\sup_{n\in\mathbb{N}} p_c(r-; x_0 - l_n) < \infty$ is equivalent to $p_c(r-; x_0 - l) < \infty$, while $\lim_{n\to\infty} p_c(l_n; x_0 - l_n) = -\infty$ is implied by $p_c(l+; x_0 - l) = -\infty$.

Proof. We only prove Point (i), as Point (ii) can be obtained by similar arguments. Fix $c \in (l, r)$ and define, for every m, $n \in \mathbb{N}$ such that $l_m < x_0 < r_n$ and $l_m < c < r_n$, the stopping times

$$\tau_m = \inf\{t \geq 0 : X_t \leq l_m\}, \qquad \tau^n = \inf\{t \geq 0 : X_t \geq r_n\}.$$

Letting $\tau_m^n = \tau_m \wedge \tau^n$, from (2.4)-(2.5) we infer that

$$\begin{aligned}\mathrm{d}X_t^{\tau_m^n} &= 1_{\{t \leq \tau_m^n\}}\left[\tilde{b}_c(X_t^{\tau_m^n}; x_0 - r_n, x_0 - r_n)\,\mathrm{d}t + \tilde{\sigma}(X_t^{\tau_m^n})\,\mathrm{d}W_t\right] \\ &\quad + 1_{\{t \leq \tau_m^n\}}\left[\Big(\mathrm{d}\,(K' * L) * (X^{\tau_m^n} - r_n)\Big)(t) + (r_n - x_0)\big((K' * L)(t) - (K' * L)(\infty)\big)\right]\mathrm{d}t,\end{aligned}$$

with $X_0^{\tau_m^n} = x_0$. Applying a generalized Itô's formula to the process $(p_c(X_{t\wedge\tau_m^n}; x_0 - r_n, x_0 - r_n))_{t\geq 0}$, see, e.g., [27, Theorem 7.1 and Problem 7.3, Chapter 3], by (2.7) it follows that

$$\begin{aligned}\mathrm{d}p_c\Big(X_t^{\tau_m^n}; x_0 - r_n, x_0 - r_n\Big) &= 1_{\{t \leq \tau_m^n\}} p_c'\left(X_t^{\tau_m^n}; x_0 - r_n, x_0 - r_n\right)\Big[\tilde{\sigma}(X_t^{\tau_m^n})\,\mathrm{d}W_t \\ &\quad + \Big(\Big(\mathrm{d}\,(K' * L) * (X^{\tau_m^n} - r_n)\Big)(t) + (r_n - x_0)\big((K' * L)(t) - (K' * L)(\infty)\big)\Big)\,\mathrm{d}t\Big],\end{aligned}$$

with $p_c\big(X_0^{\tau_m^n}; x_0 - r_n, x_0 - r_n\big) = p_c(x_0; x_0 - r_n, x_0 - r_n)$. In the sequel, we are going to write $p_c^{(n)}(\cdot)$ instead of $p_c(\cdot; x_0 - r_n, x_0 - r_n)$ to shorten the notation. The right-hand side of the previous equation is the sum of a martingale (as $(p_c^{(n)})'$ and $\tilde{\sigma}$ are locally bounded in I) and, thanks to Hypothesis 3 and recalling that $(p_c^{(n)})' > 0$ in I, of a nonpositive absolutely continuous process. As a result, taking expectations, we deduce that

$$\mathbb{E}\Big[p_c^{(n)}\Big(X_t^{\tau_m^n}\Big)\Big] \leq p_c^{(n)}(x_0) \leq p_c(x_0; 0, 0), \quad t \geq 0,$$

where we apply Lemma 2.8 for the last inequality, observing that $x_0 - r_n \leq 0$. Hence, also using the fact that $p_c^{(n)}$ is (strictly) increasing in I,

$$\begin{aligned}\mathbb{E}\Big[p_c^{(n)}\Big(X_t^{\tau_m^n}\Big)1_{\{\tau_m^n < \infty\}}\Big] &\leq p_c(x_0; 0, 0) - \mathbb{E}\left[p_c^{(n)}\left(X_t^{\tau_m^n}\right)1_{\{\tau_m^n = \infty\}}\right] \\ &\leq p_c(x_0; 0, 0) - p_c^{(n)}(l_m), \quad t \geq 0.\end{aligned}$$

Letting $t \to \infty$, the dominated convergence theorem and the continuity of X in $\mathbb{R}_+$ yield

$$p_c^{(n)}(l_m)\mathbb{P}\left(\tau_m^n = \tau_m < \infty\right) + p_c^{(n)}(r_n)\mathbb{P}\left(\tau_m^n = \tau^n < \infty\right) \leq p_c(x_0; 0, 0) - p_c^{(n)}(l_m).$$

In particular, since $p_c^{(n)}(r_n) > 0$ and $p_c^{(n)}(l_m) < 0$,

$$\mathbb{P}\left(\tau_m^n = \tau^n < \infty\right) \leq p_c^{(n)}(r_n)^{-1}(p_c(x_0; 0, 0) - 2p_c^{(n)}(l_m)).$$

Thus, defining $L = \inf_{n\in\mathbb{N}} p_c^{(n)}(l+)$, which is finite by assumption,

$$\mathbb{P}\left(X_t = r_n, \text{ for some } t \in [0, S)\right) = \lim_{m\to\infty} \mathbb{P}\left(\tau_m^n = \tau^n < \infty\right) \leq \frac{p_c(x_0; 0, 0) - 2L}{p_c^{(n)}(r_n)},$$

from which we conclude that

$$\mathbb{P}\left(\sup_{0\leq t<S} X_t = r\right) \leq \liminf_{n\to\infty} \mathbb{P}\left(X_t = r_n, \text{ for some } t \in [0, S)\right) = 0.$$

This is equivalent to $\mathbb{P}\left(\sup_{0\leq t<S} X_t < r\right) = 1$, which is the asserted equality.

It remains to prove the simpler formulation of the hypotheses when $r < \infty$. Based on the definitions in (2.5) and (2.6),

$$p_c(l+; x_0 - r_n) = -\int_l^c \exp\left\{2\int_x^c \frac{\tilde{b}(y)}{\tilde{\sigma}^2(y)}\mathrm{d}y\right\} \times \exp\left\{2|(K' * L)(\infty) - (K' * L)(0)|(r_n - x_0)\int_x^c \frac{1}{\tilde{\sigma}^2(y)}\mathrm{d}y\right\}\mathrm{d}x; \quad (2.14)$$

by Lemma 2.8, the sequence $p_c(l+; x_0 - r_n)$ is nonincreasing, hence taking the limit as $n \to \infty$ in this equality yields, by Beppo Levi's monotone convergence theorem,

$$\begin{aligned}\inf_{n\in\mathbb{N}} p_c(l+; x_0 - r_n) &= -\int_l^c \exp\left\{2\int_x^c \frac{\tilde{b}(y)}{\tilde{\sigma}^2(y)}\mathrm{d}y\right\} \\ &\quad \times \exp\left\{2|(K' * L)(\infty) - (K' * L)(0)|(r - x_0)\int_x^c \frac{1}{\tilde{\sigma}^2(y)}\mathrm{d}y\right\}\mathrm{d}x \\ &= p_c(l+; x_0 - r).\end{aligned}$$

Observing that, again by Lemma 2.8,

$$p_c(r_n; x_0 - r_n) \geq p_c(r_n; x_0 - r) \quad \Longrightarrow \quad \liminf_{n\to\infty} p_c(r_n; x_0 - r_n) \geq p_c(r-; x_0 - r),$$

the proof is complete. □

Remark 2.10. If $(K' * L)(0) < (K' * L)(\infty)$ (see also Remark 2.16 below for more details on this), Proposition 2.9 can only be applied to half-bounded intervals $I = (l, r)$, where $l > -\infty$ or $r < \infty$. More precisely, Point (i) [resp., Point (ii)] does not apply when $r = \infty$ [resp., $l = -\infty$]. Indeed, when $r = \infty$ (the case $l = -\infty$ can be argued analogously), for every fixed $c \in I$ and $\bar{x} \in (l, c)$, since $p_c(\cdot; x_0 - r_n, x_0 - r_n)$ is increasing on I,

$$p_c(l+; x_0 - r_n) \leq -\int_{\bar{x}}^c \exp\left\{2\int_x^c \frac{\tilde{b}(y)}{\tilde{\sigma}^2(y)}\mathrm{d}y\right\} \times \exp\left\{2|(K' * L)(\infty) - (K' * L)(0)|(r_n - x_0)\int_x^c \frac{1}{\tilde{\sigma}^2(y)}\mathrm{d}y\right\}\mathrm{d}x,$$

which holds for any $n \in \mathbb{N}$. Given that $r_n \to r = \infty$ as $n \to \infty$, by Fatou's lemma the $\limsup$ of the right-hand side of the previous inequality is $-\infty$, hence $\inf_{n\in\mathbb{N}} p_c(l+; x_0 - r_n) = -\infty$.

2.2 Necessary and sufficient conditions via $v_c(\cdot; \beta, \gamma)$

In this subsection, more precisely in Theorems 2.12-2.13, we extend the classical Feller test for explosions in [22] (see also [27, Theorem 5.29, Chapter 5]) to the path-dependent case. To do this, we employ the function $v_c(\cdot; \beta, \gamma)$ defined in (2.8)-(2.9).

We start by presenting the following preparatory lemma, which compares the values of such $v_c(\cdot; \beta, \gamma)$ as β and γ vary in $\mathbb{R}$. Its proof is deferred to Appendix A.

Lemma 2.11. *Assume Hypotheses 1-2-3. Fix $c \in I$ and consider β_1, $\beta_2 \in \mathbb{R}$ and γ_1, $\gamma_2 \in \mathbb{R}$ such that $\beta_1 \geq \beta_2$ and $\gamma_1 \leq \gamma_2$. Then $v_c(\cdot; \beta_1, \gamma_1) \leq v_c(\cdot; \beta_2, \gamma_2)$ in I.*

Furthermore, if $\int_l^r \tilde{\sigma}^{-2}(z)\,\mathrm{d}z < \infty$, then $v_c(\cdot; \beta_2, \gamma_2) \leq C v_c(\cdot; \beta_1, \gamma_1)$ in I for some constant $C > 0$.

Consider $c \in I$ and β, $\gamma \in \mathbb{R}$. Recalling (2.8), since $\tilde{b}_c(\cdot; \beta, \gamma)$ and $\tilde{\sigma}$ satisfy Hypothesis 2, by [27, Lemma 5.26, Chapter 5], the series

$$u_c(x; \beta, \gamma) = \sum_{n=0}^{\infty} u_{c,n}(x; \beta, \gamma) \quad (2.15)$$

converges locally uniformly on I to a differentiable function with locally absolutely continuous derivative. Such map $u_c(\cdot;\beta,\gamma)$ is decreasing in (l,c), increasing in (c,r) and solves the second order ODE

$$\begin{cases} u_c(x;\beta,\gamma) = \frac{1}{2}\tilde{\sigma}^2(x)u_c''(x;\beta,\gamma) + \tilde{b}_c(x;\beta,\gamma)u_c'(x;\beta,\gamma), & \text{for a.e. } x \in I, \\ u_c(c;\beta,\gamma) = 1,\ u_c'(c;\beta,\gamma) = 0. \end{cases} \tag{2.16}$$

In addition, the following bounds hold (cf. [27, Equation (5.69), Chapter 5]):

$$1 + v_c(x;\beta,\gamma) \le u_c(x;\beta,\gamma) \le e^{v_c(x;\beta,\gamma)}, \quad x \in I. \tag{2.17}$$

As before, we will write $u_c(x;\beta)$ [resp., $u_c(x;\gamma)$] when $x \in (l,c)$ [resp., $x \in [c,r)$]. We remark that the above construction of the map $u_c(\cdot;\beta,\gamma)$ coincides with the one in the path-independent case, once we introduce, according to Hypothesis 2 and the condition $(K' * L)(\infty) \in \mathbb{R}$, the shifted modified drift $\tilde{b}_c(\cdot;\beta,\gamma)$, see (2.5), and the modified diffusion $\tilde{\sigma}$.

The next theorem gives a necessary condition for the solution process X of (2.2) not to exit from the interval I, i.e., for $\mathbb{P}(S = \infty) = 1$.

Theorem 2.12. *Fix $c \in I$ and assume Hypotheses 1-2-3. Consider the stopping times $S_- = \inf\{t \ge 0 : X_t = l\}$ and $S_+ = \inf\{t \ge 0 : X_t = r\}$, so that $S = \min\{S_-, S_+\}$.*

(i) If $\mathbb{P}(S_- = \infty) = 1$, then $v_c\big(l+;\beta\big) = \infty$, for every $\beta < 0$;

(ii) if $\mathbb{P}(S_+ = \infty) = 1$, then $v_c\big(r-;\gamma\big) = \infty$, for every $\gamma > 0$.

Consequently,

$$v_c\big(l+;\beta\big) = \infty, \text{ for every } \beta < 0, \quad \text{and} \quad v_c\big(r-;\gamma\big) = \infty, \text{ for every } \gamma > 0,$$

is a necessary condition for $\mathbb{P}(S = \infty) = 1$.

Before the proof, we provide an intuitive interpretation of the conditions in Theorem 2.12 in terms of the initial condition x_0. We recall, indeed, that $v_c(\cdot;\beta,\gamma)$ also depends on x_0. If we suppose that $K'(0) \le 0$, then, by the same steps as in the proof of Lemma 2.11, using the expression of the modified drift $\tilde{b}$, we deduce that the map $x_0 \mapsto v_c(x;\beta,\gamma)$ in nondecreasing on (l,r) for $x \in (l,c)$, and nonincreasing for $x \in [c,r)$. Therefore, the requirement $v_c\big(l+;\beta\big) = \infty$, for all $\beta < 0$, becomes more stringent as x_0 approaches l. On the other hand, the closer x_0 to r, the stricter the requirement $v_c\big(r-;\gamma\big) = \infty$, for all $\gamma > 0$.

Proof. We first prove Point (i). Given $x_0 \in I$, we consider $c \in (x_0, r)$ and $\tau_c = \inf\{t \ge 0 : X_t \ge c\}$. Since $K' * L$ is nondecreasing in $\mathbb{R}_+$ by Hypothesis 3, for every continuous function $x\colon \mathbb{R}_+ \to (-\infty, c]$ we have

$$\begin{aligned} \left(\mathrm{d}\left(K' * L\right) * (x - x_0)\right)(h) &= \left(\mathrm{d}\left(K' * L\right) * (x - c)\right)(h) + (c - x_0)\left((K' * L)(h) - (K' * L)(0)\right) \\ &\le (x_0 - c)\left(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\right), \quad h \ge 0. \end{aligned} \tag{2.18}$$

From (2.4)-(2.5), for every $n \in \mathbb{N}$, the dynamics in $\mathbb{R}_+$ of the $[l_n, c]-$valued stopped process $X^{S_n \wedge \tau_c}$ – which is a continuous semimartingale – are

$$\begin{aligned} \mathrm{d}X_t^{S_n\wedge\tau_c} &= 1_{\{t \le S_n\wedge\tau_c\}}\left[\tilde{b}_c\Big(X_t^{S_n\wedge\tau_c}; x_0 - c, x_0 - c\Big)\,\mathrm{d}t + \tilde{\sigma}(X_t^{S_n\wedge\tau_c})\,\mathrm{d}W_t\right] \\ &\quad + 1_{\{t \le S_n\wedge\tau_c\}}\left[\left(\mathrm{d}\left(K' * L\right) * \left(X^{S_n\wedge\tau_c} - x_0\right)\right)(t) + \left(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\right)(c - x_0)\right]\mathrm{d}t. \end{aligned}$$

Define $Y_t^{(n)} = e^{-(t\wedge S_n\wedge\tau_c)}u_c\big(X_t^{S_n\wedge\tau_c}; x_0-c, x_0-c\big)$, $t\geq 0$. By the stochastic product rule, the ODE (2.16) and Itô's lemma – in a generalized form, since $u_c'\big(\cdot; x_0-c, x_0-c\big) \in \mathrm{AC}_{\mathrm{loc}}((l,c])$, see for instance [27, Theorem 7.1 and Problem 7.3, Chapter 3] – we obtain

$$\begin{aligned}
Y_t^{(n)} &= \int_0^t 1_{\{h\leq S_n\wedge\tau_c\}}e^{-h}u_c'\Big(X_h^{S_n\wedge\tau_c}; x_0-c, x_0-c\Big) \\
&\quad\times\left[\big(\mathrm{d}\,(K'*L)*\big(X^{S_n\wedge\tau_c}-x_0\big)\big)(h) + \left(\frac{K'(0)}{K(0)} - (K'*L)(\infty)\right)(c-x_0)\right]\mathrm{d}h \\
&\quad+\int_0^t 1_{\{h\leq S_n\wedge\tau_c\}}e^{-h}u_c'\big(X_h^{S_n\wedge\tau_c}; x_0-c, x_0-c\big)\tilde\sigma(X_h^{S_n\wedge\tau_c})\,\mathrm{d}W_h + u_c(x_0; x_0-c) \\
&=: \mathbf{I}_t^{(n)} + \mathbf{II}_t^{(n)} + u_c(x_0; x_0-c),
\end{aligned} \tag{2.19}$$

which holds for every $t\geq 0$, $\mathbb{P}-$a.s. Notice that $\mathbf{I}^{(n)}\geq 0$ up to indistinguishability because of (2.18) and the fact that $u_c'(\cdot; x_0-c)\leq 0$ in (l,c). Moreover, the stochastic integral process $\mathbf{II}^{(n)}$ is a true martingale. Indeed, by the local boundedness of $\tilde\sigma$ and the continuity of $u_c'(\cdot; x_0-c, x_0-c)$ in $[l_n, c]$, the integrand process is bounded. Therefore, taking expectations in (2.19),

$$u_c(x_0; x_0-c)\leq \mathbb{E}\Big[Y_t^{(n)}\Big], \quad t\geq 0,\, n\in\mathbb{N}. \tag{2.20}$$

Suppose that $\mathbb{P}(S_-=\infty)=1$ and, arguing by contradiction, that $v_c\big(l+; x_0-c\big)<\infty$. By (2.17),

$$u_c(x; x_0-c, x_0-c)\leq C, \quad x\in(l,c], \text{ for some } C>0.$$

In this setting, since $S\wedge\tau_c = S_+\wedge\tau_c = \tau_c$ almost surely, recalling the definition of $Y_t^{(n)}$, by the dominated convergence theorem we compute the limit as $n\to\infty$ in (2.20) to deduce that

$$\begin{aligned}
u_c(x_0; x_0-c) &\leq \mathbb{E}\Big[e^{-(t\wedge S\wedge\tau_c)}u_c(X_{t\wedge S\wedge\tau_c}+; x_0-c, x_0-c)\Big] \\
&= \mathbb{E}\Big[e^{-(t\wedge\tau_c)}u_c(X_{t\wedge\tau_c}; x_0-c, x_0-c)\Big] \\
&= \mathbb{E}\Big[e^{-(t\wedge\tau_c)}u_c(X_{t\wedge\tau_c}; x_0-c, x_0-c)\left(1_{\{\tau_c=\infty\}}+1_{\{\tau_c<\infty\}}\right)\Big] \\
&\leq Ce^{-t} + \mathbb{E}\Big[e^{-t\wedge\tau_c}u_c(X_{t\wedge\tau_c}; x_0-c, x_0-c)1_{\{\tau_c<\infty\}}\Big],
\end{aligned}$$

which holds for every $t\geq 0$. Notice that in the first inequality we write $u_c(X_{t\wedge S\wedge\tau_c}+; x_0-c, x_0-c)$, where the plus sign denotes the limit of $u_c(x; x_0-c, x_0-c)$ as x tends to $X_{t\wedge S\wedge\tau_c}$ from the right. Letting $t\to\infty$, once again by dominated convergence we obtain the inequality

$$u_c(x_0; x_0-c)\leq \mathbb{E}\Big[e^{-\tau_c}u_c(X_{\tau_c}; x_0-c)1_{\{\tau_c<\infty\}}\Big]\leq u_c(c; x_0-c),$$

which is absurd because $u_c(\cdot; x_0-c, x_0-c)$ is (strictly) decreasing in $(l,c]$. Thus,

$$v_c(l+; x_0-c)=\infty.$$

Since we chose a generic $c\in(x_0, r)$, by Corollary 2.5 and Lemma 2.11 we conclude that

$$v_c(l+;\beta)=\infty, \quad \beta<0,\, c\in I,$$

proving Point (i).

As for Point (ii), the necessity of the condition

$$v_c(r-;\gamma)=\infty, \quad \gamma>0,\, c\in I,$$

for $\mathbb{P}(S_+ = \infty) = 1$ can be inferred similarly, this time arguing with $c \in (l, x_0)$, defining $\tau_c = \inf\{t \geq 0 : X_t \leq c\}$ and using the fact that $u_c(\cdot; x_0 - c)$ is (strictly) increasing in $[c, r)$. The proof is then complete. □

In the following theorem, we use the maps $v_c(\cdot; \beta, \gamma)$, for certain $\beta, \gamma \in \mathbb{R}$, to establish sufficient conditions guaranteeing that $\mathbb{P}(S = \infty) = 1$. The hypotheses that we formulate in Theorem 2.13 are stronger than the necessary conditions in Theorem 2.12 and differ from the classical ones in [22]. We refer to Remarks 2.14-2.16 and Corollary 2.15 for more details.

Theorem 2.13. *Fix $c \in I$ and assume Hypotheses 1-2-3. Consider the stopping times $S_- = \inf\{t \geq 0 : X_t = l\}$ and $S_+ = \inf\{t \geq 0 : X_t = r\}$, so that $S = \min\{S_-, S_+\}$.*

(i) If $\lim_{n\to\infty} v_c(l_n; x_0 - l_n) = \infty$, then $\mathbb{P}(\{S < S_-\} \cup \{S_- = \infty\}) = 1$;

(ii) if $\lim_{n\to\infty} v_c(r_n; x_0 - r_n) = \infty$, then $\mathbb{P}(\{S < S_+\} \cup \{S_+ = \infty\}) = 1$.

Consequently, the condition

$$\lim_{n\to\infty} v_c(l_n; x_0 - l_n) = \lim_{n\to\infty} v_c(r_n; x_0 - r_n) = \infty$$

is sufficient for $\mathbb{P}(S = \infty) = 1$.

Proof. Consider $c \in I$, where $I = (l, r)$ for some $-\infty \leq l < r \leq \infty$. Recalling (2.4) and (2.5), the dynamics of the stopped process X^{S_n} in $\mathbb{R}_+$ read, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{d}X_t^{S_n} =& 1_{\{t\leq S_n\}}\left[\tilde{b}_c(X_t^{S_n}; x_0 - l_n, x_0 - r_n)\,\mathrm{d}t + \tilde{\sigma}(X_t^{S_n})\,\mathrm{d}W_t\right] \\
&+ 1_{\{t\leq S_n\}}\Bigg[\left(\mathrm{d}\,(K' * L) * \left(X^{S_n} - x_0\right)\right)(t) \\
&+ \left(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\right)\left((l_n - x_0)1_{\{X_t^{S_n} < c\}} + (r_n - x_0)1_{\{X_t^{S_n}\geq c\}}\right)\Bigg]\mathrm{d}t, \quad (2.21)
\end{aligned}$$

with $X_0^{S_n} = x_0$. Define the continuous semimartingale $\overline{Z}_t^{(n)} = u_c(X_t^{S_n}; x_0 - l_n, x_0 - r_n)$, $t \geq 0$. An application of the generalized Itô's lemma in [27, Theorem 7.1 and Problem 7.3, Chapter 3], along with the ODE (2.16) yields

$$\begin{aligned}
\mathrm{d}\overline{Z}_t^{(n)} =& 1_{\{t\leq S_n\}}\left(u_c\big(X_t^{S_n}; x_0 - l_n, x_0 - r_n\big)\mathrm{d}t + u_c'\big(X_t^{S_n}; x_0 - l_n, x_0 - r_n\big)\tilde{\sigma}(X_t^{S_n})\,\mathrm{d}W_t\right) \\
&+ 1_{\{t\leq S_n\}}u_c'\big(X_t^{S_n}; x_0 - l_n, x_0 - r_n\big)\Bigg[\left(\mathrm{d}\,(K' * L) * \left(X^{S_n} - x_0\right)\right)(t) \\
&+ \left(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\right)\left((l_n - x_0)1_{\{X_t^{S_n} < c\}} + (r_n - x_0)1_{\{X_t^{S_n}\geq c\}}\right)\Bigg]\mathrm{d}t, \quad (2.22)
\end{aligned}$$

with initial condition $\overline{Z}_0^{(n)} = u_c(x_0; x_0 - l_n, x_0 - r_n)$. Then, if we introduce the process $\overline{Y}_t^{(n)} = e^{-t\wedge S_n}\overline{Z}_t^{(n)}$, $t \geq 0$, the stochastic product rule implies that

$$\begin{aligned}
\overline{Y}_t^{(n)} =& \int_0^t 1_{\{h\leq S_n\}}e^{-h}u_c'(X_h^{S_n}; x_0 - l_n, x_0 - r_n)\Bigg[\left(\mathrm{d}\,(K' * L) * \left(X^{S_n} - x_0\right)\right)(h) \\
&+ \left(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\right)\left((l_n - x_0)1_{\{X_h^{S_n} < c\}} + (r_n - x_0)1_{\{X_h^{S_n}\geq c\}}\right)\Bigg]\mathrm{d}h \\
&+ \int_0^t 1_{\{h\leq S_n\}}e^{-h}u_c'(X_h^{S_n}; x_0 - l_n, x_0 - r_n)\tilde{\sigma}(X_h^{S_n})\,\mathrm{d}W_h + u_c(x_0; x_0 - l_n, x_0 - r_n) \\
=:& \overline{\mathbf{I}}_t^{(n)} + \overline{\mathbf{II}}_t^{(n)} + u_c(x_0; x_0 - l_n, x_0 - r_n), \quad t \geq 0, \mathbb{P}-\text{a.s.} \quad (2.23)
\end{aligned}$$

We concentrate on $\bar{\mathbf{I}}_t^{(n)} =: \int_0^t g^{(n)}(h)\,\mathrm{d}h$, $t \geq 0$, with the aim of demonstrating that it is a nonpositive supermartingale when $n \in \mathbb{N}$ is large enough. To do this, it suffices to show that the integrand $g^{(n)} \leq 0$ for n large enough. We then consider $\bar{n} \in \mathbb{N}$ sufficiently big such that $x_0 \in I_n = (l_n, r_n)$, for every $n \geq \bar{n}$. By Hypothesis 3, $K' * L$ is nondecreasing in $\mathbb{R}_+$, whence, for every $n \geq \bar{n}$ and every continuous map $x\colon \mathbb{R}_+ \to I_n$,

$$\begin{aligned}
&\left(\mathrm{d}\left(K' * L\right) * \left(x - x_0\right)\right)(h) \\
&\quad = \left(\mathrm{d}\left(K' * L\right) * \left(x - r_n\right)\right)(h) + \left((K' * L)(h) - \frac{K'(0)}{K(0)}\right)(r_n - x_0) \\
&\quad \leq \left(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\right)(x_0 - r_n), \quad h \geq 0.
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\left(\mathrm{d}\left(K' * L\right) * \left(x - x_0\right)\right)(h) \\
&\quad = \left(\mathrm{d}\left(K' * L\right) * \left(x - l_n\right)\right)(h) + \left((K' * L)(h) - \frac{K'(0)}{K(0)}\right)(l_n - x_0) \\
&\quad \geq \left(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\right)(x_0 - l_n), \quad h \geq 0.
\end{aligned}$$

Since $u_c'(\cdot; x_0 - l_n) \leq 0$ in (l, c) and $u_c'(\cdot; x_0 - r_n) \geq 0$ in $[c, r)$, the two previous equations show that $g^{(n)} \leq 0$, as desired.

Going back to (2.23) and recalling that $u_c(\cdot; x_0 - l_n, x_0 - r_n)$ is (strictly) positive on I, which entails that $\overline{Y}_t^{(n)} \geq 0$ by definition, we observe that the process $u_c(x_0; x_0 - l_n, x_0 - r_n) + \overline{\mathbf{II}}_t^{(n)}$, $t \geq 0$, is a nonnegative local martingale for $n \geq \bar{n}$. Indeed, (2.23) gives

$$u_c(x_0; x_0 - l_n, x_0 - r_n) + \overline{\mathbf{II}}_t^{(n)} = \overline{Y}_t^{(n)} - \bar{\mathbf{I}}_t^{(n)}, \quad t \geq 0, \, \mathbb{P}-\text{a.s.}$$

Hence it is a supermartingale, and so is $\overline{Y}^{(n)}$. By Fatou's lemma, we can claim that also the $[0, \infty]-$valued $\liminf-$process $\overline{Y} = (\overline{Y}_t)_{t \geq 0}$ given by

$$\overline{Y}_t = \liminf_{n \to \infty} \overline{Y}_t^{(n)} = e^{-(t \wedge S)} \liminf_{n \to \infty} u_c\big(X_t^{S_n}; x_0 - l_n, x_0 - r_n\big), \quad t \geq 0, \tag{2.24}$$

is a supermartingale. Note that, for every $t \geq 0$, the random variable $\overline{Y}_t$ is finite $\mathbb{P}-$a.s., because $t \mapsto \mathbb{E}[\overline{Y}_t]$ is nonincreasing and starts at a finite deterministic value $\overline{Y}_0$. Indeed, by (2.17) and (2.24),

$$\overline{Y}_0 = \liminf_{n \to \infty} u_c(x_0; x_0 - l_n, x_0 - r_n) \leq \liminf_{n \to \infty} e^{v_c(x_0; x_0 - l_n, x_0 - r_n)} \leq e^{v_c(x_0; x_0 - l_1, x_0 - r_1)} < \infty,$$

where we apply Lemma 2.11 for the second-to-last inequality.

We now prove the implication in Point (i) (the one in Point (ii) can be shown with analogous arguments). By contradiction, suppose that there exists $t > 0$ such that $\mathbb{P}(\{S = S_-\} \cap \{S_- < t\}) > 0$. Then, writing $\Omega' = \{S = S_-\} \cap \{S_- < t\}$,

$$X_t^{S_-}(\omega) = l, \quad \omega \in \Omega'.$$

Observe that, by the continuity of the process X, for every $\omega \in \Omega'$ there is an $N(\omega) \in \mathbb{N}$ so big that $l_n < c$ and that

$$u_c\big(X_t^{S_n}(\omega); x_0 - l_n, x_0 - r_n\big) = u_c(l_n; x_0 - l_n) \geq 1 + v_c(l_n; x_0 - l_n), \quad n \geq N(\omega),$$

where we use (2.17) for the inequality. Thus, from (2.24) and the hypotheses of this theorem,

$$\begin{aligned}\overline{Y}_t(\omega) &= e^{-S(\omega)} \liminf_{n\to\infty} u_c\big(X_t^{S_n}(\omega); x_0 - l_n, x_0 - r_n\big) \\ &\geq e^{-S(\omega)} \liminf_{n\to\infty} v_c(l_n; x_0 - l_n) = \infty, \quad \omega \in \Omega',\end{aligned}$$

which is absurd because we have argued that $\overline{Y}_t < \infty$ almost surely. Therefore,

$$\mathbb{P}(\{S = S_-\} \cap \{S_- < t\}) = 0 \quad \text{for every } t \geq 0.$$

Consequently,

$$\mathbb{P}(\{S = S_-\} \cap \{S_- < \infty\}) \leq \mathbb{P}\Big(\{S = S_-\} \cap \Big(\bigcup_{n\in\mathbb{N}} \{S_- < n\}\Big)\Big) = 0,$$

which completes the proof. □

Remark 2.14. Theorem 2.13 can be applied to general, possibly unbounded intervals. Roughly speaking, in the proof, the fact that the endpoints of I might be infinite is handled by adding terms depending on the (finite) quantities l_n and r_n. This enables to control the sign of the path-dependent term in the dynamics (2.4) of the stopped processes X^{S_n}. Such an argument requires a sufficient condition involving the sequences $(v_c(l_n; x_0 - l_n))_n$ and $(v_c(r_n; x_0 - r_n))_n$, contrary to the classical formulation of Feller's test for explosions, see [22] and [27].

However, when $l > -\infty$ or $r < \infty$, it is possible to impose stronger requirements which satisfy the hypotheses of Theorem 2.13 and resemble more the classical ones. More specifically, suppose that $l > -\infty$ and that $v_c(l+; x_0 - l) = \infty$. Then, by Lemma 2.11,

$$\liminf_{n\to\infty} v_c(l_n; x_0 - l_n) \geq \liminf_{n\to\infty} v_c(l_n; x_0 - l) = v_c(l+; x_0 - l) = \infty.$$

Analogously, if $r < \infty$, the condition $\lim_{n\to\infty} v_c(r_n; x_0 - r_n) = \infty$ is implied by $v_c(r-; x_0 - r) = \infty$.

Under stronger assumptions on the modified diffusion $\tilde{\sigma}$, it is possible to combine Theorems 2.12 and 2.13 to deduce a necessary and sufficient condition for $\mathbb{P}(S = \infty) = 1$. More precisely, by Lemma 2.11 and Remark 2.14, the next result holds.

Corollary 2.15. *Assume Hypotheses 1-2-3. Consider the stopping times* $S_- = \inf\{t \geq 0 : X_t = l\}$ *and* $S_+ = \inf\{t \geq 0 : X_t = r\}$, *so that* $S = \min\{S_-, S_+\}$. *Furthermore, suppose that* $I \subset \mathbb{R}$ *is a bounded interval and that* $\tilde{\sigma}^{-2}$ *is integrable in* I. *Then, given* $c \in I$ *and* $\beta, \gamma \in \mathbb{R}$,

(i) the condition $v_c(l+; \beta) = \infty$ *is necessary for* $\mathbb{P}(S_- = \infty) = 1$ *and sufficient for* $\mathbb{P}(\{S < S_-\} \cup \{S_- = \infty\}) = 1$;

(ii) the condition $v_c(r-; \gamma) = \infty$ *is necessary for* $\mathbb{P}(S_+ = \infty) = 1$ *and sufficient for* $\mathbb{P}(\{S < S_+\} \cup \{S_+ = \infty\}) = 1$.

In particular,

$$\mathbb{P}(S = \infty) = 1 \quad \textit{if and only if} \quad v_c(l+; \beta) = v_c(r-; \gamma) = \infty.$$

We notice that, by choosing $\beta = \gamma = 0$ in Corollary 2.15, we obtain a test where the auxiliary function $v_c(\cdot; 0, 0)$ depends solely on the modified drift $\tilde{b}$ (defined in the equation before (2.4)), without the need for any shifts β, γ.

Remark 2.16. In the classical case $K \equiv K(0)$, for some positive constant $K(0) > 0$, (2.2) reduces to a standard Itô diffusion without path-dependent component. In this context, since $K' \equiv 0$ on $\mathbb{R}_+$, it directly follows from the definition in (2.5) that $\tilde{b}_c(\cdot;\beta,\gamma)$ does not depend on β or γ. Consequently, by (2.6) and (2.8)-(2.9), $p_c(\cdot;\beta,\gamma)$ and $v_c(\cdot;\beta,\gamma)$ do not depend on β or γ, either.

The same conclusion holds in the case of a positive exponential kernel $K(t) = K(0)e^{-\Lambda t}$, see also (1.5) in Section 1. Here, indeed, since $\|K\|_{L^1(\mathbb{R}_+)} = \Lambda^{-1}K(0)$, according to (2.13) in Remark 2.7

$$\frac{K'(0)}{K(0)} = -\Lambda = (K' * L)(\infty),$$

implying that $\tilde{b}_c(\cdot;\beta,\gamma)$ in (2.5) does not depend on β or γ.

As a result, in these frameworks where the dynamics are path-independent, Corollary 2.15 continues to hold even removing the hypotheses of boundedness for the interval I and of integrability for the modified diffusion $\tilde{\sigma}$. Therefore, in these cases Corollary 2.15 coincides with the classical Feller's test for explosions in [22].

We also notice that $K(t) = K(0)e^{-\Lambda t}$ for some $\Lambda \in \mathbb{R}$ (including 0) and $K(0) > 0$ are the only kernels for which this argument applies, namely $\tilde{b}_c(\cdot;\beta,\gamma)$ in (2.5) is independent of β and γ. Indeed, if we consider a map $K \in C^1(\mathbb{R}_+)$ satisfying Hypotheses 1-3, then $(K' * L)(0) = (K' * L)(\infty)$ implies – by monotonicity – that $(K' * L)(t) = (K' * L)(0)$, for every $t \geq 0$. Thus, by the resolvent identity, convolving with K and computing the derivatives, the fundamental theorem of calculus implies that

$$K'(t) = (K' * L)(0)K(t) = \frac{K'(0)}{K(0)}K(t),$$

a linear first-order ODE whose unique solution, given the initial condition $K(0)$, is exactly $K(t) = K(0)\exp\left\{\frac{K'(0)}{K(0)}t\right\}$.

To conclude this section, we refer the reader to Appendix B, where we highlight the importance of our results and their contribution to the existing literature in the particular case of a kernel given by a sum of exponentials. In fact, in this case, the boundary attainment of Volterra processes can be reformulated as a problem of explosion to infinity of standard multidimensional Itô diffusions.

3 Examples

The purpose of this section is to apply the results of the previous analysis, especially the necessary conditions in Theorem 2.12 and the sufficient ones in Theorem 2.13, to three specifications of the dynamics (2.2) that are relevant in the literature. In particular, in Subsection 3.1 our focus lies on Volterra CIR processes, for which the state space ($\mathbb{R}_+$) boundary attainment coincides with the strict positivity of the trajectories. Subsection 3.2 deals instead with Volterra Jacobi processes, characterized by having a bounded interval as their state space. Finally, in Subsection 3.3, we consider Volterra power-type diffusion processes, for which the interest is in the possibility of having blow-ups to $\pm\infty$ in finite time.

In Subsections 3.1 and 3.2, in order to guarantee the existence of weak solutions to the equations under scrutiny (see (3.1) and (3.11) below), we suppose that the nonexplosive kernel K satisfies the following requirement in addition to Hypotheses 1-3.

Hypothesis 4. *The kernel K is nonnegative and nonincreasing. Moreover, its resolvent of the first kind L is nonincreasing, i.e., the map $s \mapsto L([s, s+t])$ is nonincreasing for every $t \geq 0$.*

Hypothesis 4 – a standard assumption in the literature on Volterra processes in both continuous (see [4, 6]) and jump (see [2, 15]) cases – is satisfied, for example, by completely monotone kernels.

3.1 The Volterra CIR process

Let $\kappa, \theta, \sigma > 0$ and consider the SVE

$$X_t = x_0 + \int_0^t K(t-s)\kappa(\theta - X_s)\,\mathrm{d}s + \sigma \int_0^t K(t-s)\sqrt{X_s}\,\mathrm{d}W_s, \quad t \geq 0,\ \mathbb{P}-\text{a.s.}, \tag{3.1}$$

where the starting point $x_0 > 0$. Since K belongs to $C^1(\mathbb{R}_+)$ and admits a resolvent of the first kind satisfying Hypothesis 4, the weak existence of a continuous nonnegative solution of (3.1) is guaranteed by [6, Theorem 3.6]. In fact, (3.1) admits a pathwise unique strong solution, as shown in [5, Proposition B.3] by adapting the original strategy in [33, Theorem 1]. We call such a solution $X = (X_t)_{t\in\mathbb{R}_+}$ a *Volterra CIR process*. Using the results of Section 2, we aim to determine conditions on the parameters κ, θ, σ and on the initial state x_0 under which the trajectories of X attain (or do not attain) the value 0.

We fix $l = 0$, $r = \infty$ and define the maps $\tilde{b}$, $\tilde{\sigma}\colon (0,\infty) \to \mathbb{R}$ by

$$\tilde{b}(x) = K(0)\kappa(\theta - x) + \frac{K'(0)}{K(0)}(x - x_0), \qquad \tilde{\sigma}(x) = K(0)\sigma\sqrt{x}, \quad x \in (0,\infty).$$

Since X is a global solution of (3.1) and $r = \infty$, we notice that

$$S = S_- = \inf\{t \geq 0 : X_t = 0\}.$$

Let $c = 1$ and, to simplify the notation, $C = 2(K(0)\sigma)^{-2}$ and $V = \frac{K'(0)}{K(0)} - (K' * L)(\infty)$. Given $\beta, \gamma \in \mathbb{R}$, the derivative of the scale function $p_1(\cdot;\beta,\gamma)\colon (0,\infty) \to \mathbb{R}$ associated with the dynamics (3.1) is (cf. the definitions in (2.5)-(2.6))

$$\begin{aligned}
p_1'(x;\beta) &= \exp\Bigg\{ -\frac{2}{K(0)^2\sigma^2}\int_1^x \frac{1}{y}\Bigg(K(0)\kappa(\theta - y) + \frac{K'(0)}{K(0)}(y - x_0) \\
&\qquad + \Big(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\Big)\beta\Bigg)\mathrm{d}y\Bigg\} \\
&= \exp\left\{-C\left(\left(\frac{K'(0)}{K(0)} - K(0)\kappa\right)(x-1) + \left(K(0)\kappa\theta - x_0\frac{K'(0)}{K(0)} + V\beta\right)\log x\right)\right\} \\
&= x^{-C\left(K(0)\kappa\theta - x_0K'(0)K^{-1}(0) + V\beta\right)} \exp\left\{-C\left(\frac{K'(0)}{K(0)} - K(0)\kappa\right)(x-1)\right\},
\end{aligned} \tag{3.2}$$

for $x \in (0,1)$. By analogous computations,

$$\begin{aligned}
p_1'(x;\gamma) &= x^{-C\left(K(0)\kappa\theta - x_0K'(0)K^{-1}(0) + V\gamma\right)} \exp\left\{-C\left(\frac{K'(0)}{K(0)} - K(0)\kappa\right)(x-1)\right\}, \\
&\quad x \in [1,\infty).
\end{aligned} \tag{3.3}$$

We notice that it is not possible to apply the results in Subsection 2.1 with this scale function p_1 for general kernels K satisfying Hypotheses 1-3-4 such that $V \neq 0$. More precisely, Point (i) in Proposition 2.9 does not apply because $r = \infty$, see Remark 2.10, while $p_1(\infty;x_0) = \lim_{x\to\infty} p_1(x;x_0) = \infty$ by (3.3), which violates the requirement of Point (ii). We also refer the reader to Remark 3.2 for the particular cases of K constant or exponential, when $V = 0$, cf. Remark 2.16.

As for the nonnegative function $v_1(\cdot;\beta,\gamma)\colon(0,\infty)\to\mathbb{R}_+$ defined in (2.9), we suppose that $K(0)\kappa\theta - x_0K'(0)K^{-1}(0)+V\beta\neq 0$. Then the following lower bound for $v_1(\cdot;\beta)$ holds in the interval $(0,1)$:

$$\begin{aligned} v_1(x;\beta) &= C\int_x^1 p_1'(y;\beta)\bigg(\int_y^1 z^{C\left(K(0)\kappa\theta-x_0K'(0)K^{-1}(0)+V\beta\right)-1} \\ &\qquad\qquad \times\exp\left\{C(z-1)\left(\frac{K'(0)}{K(0)}-K(0)\kappa\right)\right\}\mathrm{d}z\bigg)\mathrm{d}y \\ &\geq C_1\left(K(0)\kappa\theta - x_0\frac{K'(0)}{K(0)}+V\beta\right)^{-1}\int_x^1 p_1'(y;\beta)\left(1-y^{C\left(K(0)\kappa\theta-x_0\frac{K'(0)}{K(0)}+V\beta\right)}\right)\mathrm{d}y \\ &\geq C_1\int_x^1\left|y^{-C\left(K(0)\kappa\theta-x_0K'(0)K^{-1}(0)+V\beta\right)}-1\right|\mathrm{d}y,\quad x\in(0,1). \end{aligned} \tag{3.4}$$

We can also establish an analogous upper bound on the same interval, given by

$$v_1(x;\beta)\leq C_2\int_x^1\left|y^{-C\left(K(0)\kappa\theta-x_0K'(0)K^{-1}(0)+V\beta\right)}-1\right|\mathrm{d}y,\quad x\in(0,1). \tag{3.5}$$

Here C_1, $C_2>0$ are positive constants, possibly depending on K, κ, θ, σ, x_0 and β, which are allowed to change from line to line. Combining these estimates with the necessary and sufficient conditions in Subsection 2.2, we obtain the next theorem.

Theorem 3.1. *Assume Hypotheses 1-3-4.*

(i) The condition $2\kappa\theta\geq K(0)\sigma^2+2\frac{K'(0)}{K(0)^2}x_0$ *is necessary for* $\mathbb{P}(S=\infty)=1$.

(ii) The condition $2\kappa\theta\geq K(0)\sigma^2+2\frac{(K'*L)(\infty)}{K(0)}x_0$ *is sufficient for* $\mathbb{P}(S=\infty)=1$.

Proof. We start by proving the contrapositive implication of Point (i). Thus, we suppose

$$2|K'(0)|x_0<K(0)^2(K(0)\sigma^2-2\kappa\theta)\quad\Longleftrightarrow\quad C\left[K(0)\kappa\theta+x_0\frac{|K'(0)|}{K(0)}\right]<1.$$

Considering also that $K(0)\kappa\theta+x_0|K'(0)|K(0)^{-1}>0$, we can find a $\bar\beta<0$ sufficiently close to 0 such that

$$C\left[K(0)\kappa\theta+x_0\frac{|K'(0)|}{K(0)}+V\bar\beta\right]\in(0,1).$$

As a result, recalling that $|K'(0)|=-K'(0)$, by (3.5) we have, for a positive constant $C_{\bar\beta}$,

$$v_1(0+;\bar\beta)\leq C_{\bar\beta}\int_0^1\left|y^{-C\left(K(0)\kappa\theta+x_0|K'(0)|K^{-1}(0)+V\bar\beta\right)}-1\right|\mathrm{d}y<\infty.$$

Hence the necessary condition in Theorem 2.12 (i) is violated, which yields $\mathbb{P}(S<\infty)>0$. This demonstrates Point (i).

As for Point (ii), we assume $2\kappa\theta-2\frac{(K'*L)(\infty)}{K(0)}x_0\geq K(0)\sigma^2$ and observe that, by (3.4),

$$v_1(0+;x_0)\geq\widetilde{C}\int_0^1\left(y^{-C\left(K(0)\kappa\theta-(K'*L)(\infty)x_0\right)}-1\right)\mathrm{d}y=\infty,\quad\text{for some }\widetilde{C}>0.$$

Therefore, thanks to Remark 2.14, we invoke Theorem 2.13 (i) to conclude that $\mathbb{P}(S=\infty)=1$. The proof is then complete. □

Remark 3.2. When $K\equiv K(0)e^{-\Lambda t}$, for some constant $K(0)>0$ and $\Lambda\geq 0$, the necessary and sufficient conditions for $\mathbb{P}(S=\infty)=1$ in Theorem 3.1 coincide, see Remark 2.16. Therefore,

$$\mathbb{P}(X_t>0,\,t\in\mathbb{R}_+)=1\quad\text{ if and only if }\quad 2\kappa\theta\geq K(0)\sigma^2-2\frac{\Lambda}{K(0)}x_0. \tag{3.6}$$

Notice that, choosing $K \equiv 1$, (3.6) reduces to the well–known Feller condition for CIR processes, see for instance [26, Section 3.1, Chapter 6].

In addition, if $2\kappa\theta < K(0)\sigma^2 - 2\frac{\Lambda}{K(0)}x_0$, then not only X attains the value 0 with positive probability, but also $\mathbb{P}(\sup_{t\in[0,S)} X_t < \infty) = 1$, meaning that its trajectories are bounded almost surely in $[0, S)$. This follows from Proposition 2.9 (i), whose hypotheses are swiftly verified by (3.2), (3.3) and recalling that $K' * L$ is constant on $\mathbb{R}_+$, hence $p_1(\cdot;\beta,\gamma)$ does not depend on β or γ, see also Remark 2.16. By contrast, Proposition 2.9 (ii) does not apply by (3.3), which gives $p_1(\infty; x_0) = \infty$.

Remark 3.3. Fix a nonintegrable kernel $K \in C^1(\mathbb{R}_+)$ that is completely monotone on $(0,\infty)$ and such that $V \neq 0$; by Remark 2.7, $V = \frac{K'(0)}{K(0)}$. Consider an SVE as in (3.1), but replace the drift $x \mapsto \kappa(\theta - x)$ with $x \mapsto \kappa(\theta + x)$, namely

$$Y_t = x_0 + \int_0^t K(t-s)\kappa(\theta + Y_s)\,\mathrm{d}s + \sigma\int_0^t K(t-s)\sqrt{Y_s}\,\mathrm{d}W_s, \quad t \geq 0,\ \mathbb{P}-\text{a.s.}$$

As in the case of (3.1), strong existence and pathwise uniqueness of a global solution $Y = (Y_t)_{t\geq 0}$ taking values in $\mathbb{R}_+$ follow from [6, Theorem 3.6] and [5, Proposition B.3]. For this specification of the dynamics, computations analogous to (3.2)–(3.3) yield

$$p_1'(x;\beta) = x^{-C\left(K(0)\kappa\theta + (\beta - x_0)K'(0)K^{-1}(0)\right)} \exp\left\{-C\left(\frac{K'(0)}{K(0)} + K(0)\kappa\right)(x-1)\right\}, \ \ x \in (0,1).$$

The same expression holds for $p_1'(x;\gamma)$, $x > 1$, with γ replacing β. Taking κ sufficiently large so that

$$CK(0)\kappa\theta > 1 \qquad \text{and} \qquad K(0)\kappa > \frac{|K'(0)|}{K(0)},$$

we obtain

$$p_1(0+;x_0) = -\infty, \qquad p_1(\infty;x_0) < \infty.$$

From (2.10) in Lemma 2.4 (see also Corollary 2.5), we infer that $v_1(0+;x_0) = \infty$. Hence, by Remark 2.14, we can apply Point (i) of Theorem 2.13 to deduce that $\mathbb{P}(S = \infty) = 1$. Moreover, Point (ii) of Proposition 2.9 ensures that

$$\mathbb{P}\left(\inf_{0\leq t<\infty} X_t > 0\right) = 1.$$

Approximating the fractional kernel In the mathematical finance literature on stochastic volatility models, particular attention has been recently devoted to the dynamics (3.1) where K is the fractional kernel, namely $K(t) = K_\alpha(t) = \Gamma(\alpha)^{-1}t^{\alpha-1}$, $t > 0$, with $\alpha \in (0,1)$. In fact, considering $K = K_\alpha$, (3.1) describes the spot variance in the so-called rough Heston model, see [20, 21]. Note that, when $\alpha \in (0, \frac{1}{2}]$, K_α is only locally integrable (and not locally square-integrable), hence (3.1) has to be understood in an integral form, see [2]. As already mentioned in Section 1, our approach does not cover such kernels, as $K_\alpha(0+) = \infty$. Moreover, weak convergence results approximating the solution of (3.1) for $K = K_\alpha$ with a sequence of solutions corresponding to nonexplosive kernels (see, e.g., [2, Theorem 2.8] and [5, Theorem 3.5]) do not allow to extend the conditions in Theorem 3.1 to the fractional limiting case. Indeed, the limit of a sequence of nonexplosive processes may very well be explosive, and viceversa.

Nevertheless, Theorem 3.1 can be applied to approximations of K_α which are commonly used in applications. In particular, we here focus on

$$K_{\alpha,T}(t) = \int_0^T e^{-xt}\mu(\mathrm{d}x),\ t \geq 0, \quad \text{where} \quad \mu(\mathrm{d}x) = \frac{x^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}\mathrm{d}x =: w(x)\,\mathrm{d}x, \tag{3.7}$$

for some $T > 0$. Since $K_{\alpha,T}$ belongs to $C^1(\mathbb{R}_+)$ and is completely monotone on $(0,\infty)$, it satisfies Hypotheses 1-3-4. We denote by $X_{\alpha,T}$ the pathwise unique strong solution of (3.1) associated with this kernel. Note that $K_{\alpha,T}$ – considered, for instance, in [10] – is obtained by truncating the integral in the Bernstein-Widder representation of K_α, that is, $K_\alpha(t) = \int_0^\infty e^{-xt}\mu(\mathrm{d}x)$, $t > 0$. A direct computation entails that

$$K_{\alpha,T}(0) = \frac{T^{1-\alpha}}{\Gamma(\alpha)\Gamma(2-\alpha)}, \qquad K'_{\alpha,T}(0) = -\frac{T^{2-\alpha}}{(2-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}.$$

Given an input $(x_0,\kappa,\theta,\sigma)$ for (3.1), since $K_{\alpha,T}(0) \to \infty$ and, by the monotone convergence theorem, $\|K_{\alpha,T}\|_{L^1(\mathbb{R}_+)} \to \infty$ as $T \to \infty$, also using (2.13) in Remark 2.7, the sufficient condition in Theorem 3.1 (ii) is not verified for T sufficiently large. This might suggest that, for T large enough, $X_{\alpha,T}$ attains the value 0. To rigorously prove this intuition, it suffices to show that the necessary condition in Theorem 3.1 (i) is violated. We then compute, denoting by C_1, C_2 two positive constants possibly depending on α,κ,θ and σ,

$$\sigma^2\frac{K_{\alpha,T}(0)^3}{2|K'_{\alpha,T}(0)|} - \kappa\theta\frac{K_{\alpha,T}(0)^2}{|K'_{\alpha,T}(0)|} = C_1\frac{1}{T^{2\alpha-1}} - C_2\frac{1}{T^\alpha} \underset{T\to\infty}{\longrightarrow} \infty \quad \text{if and only if} \quad \alpha \in \Big(0,\frac{1}{2}\Big).$$

As a result, the previous conjecture is only proven for $\alpha \in (0,\frac{1}{2})$, namely, for these values of α, the process $X_{\alpha,T}$ reaches 0 with positive probability for every initial condition $x_0 > 0$, when T is sufficiently large.

The integral expression of $K_{\alpha,T} = \int_0^T e^{-x\cdot}\mu(\mathrm{d}x)$ is at the basis of multi-factor Markovian approximations of rough volatility models in the spirit of [1, 5]. We also refer to [11, 12] for more efficient implementations of the same method. In these approaches, the idea is to consider an increasing (finite) sequence $(\xi_{n,N})_{n=0,\dots,N} \subset \mathbb{R}_+$, for some $N \in \mathbb{N}$, and then to approximate – on the resulting intervals $[\xi_{n-1,N},\xi_{n,N}]$, $n = 1,\dots,N$ – $\mu(\mathrm{d}x)$ in (3.7) with a weighted sum of q Dirac measures, with $q \in \mathbb{N}$. Consequently, the approximated kernels $K_{\alpha,Nq}$ have the form

$$K_{\alpha,Nq}(t) = \sum_{n=1}^{Nq} m_{n,Nq}\, e^{-x_{n,Nq}t}, \quad t \geq 0, \text{ for some } (m_{n,Nq})_n, (x_{n,Nq})_n \in (0,\infty)^{Nq}.$$

We denote by $X_{\alpha,Nq}$ the solution process of (3.1) corresponding to $K_{\alpha,Nq} \in C^1(\mathbb{R}_+)$, which is completely monotone on $(0,\infty)$ because it is a weighted sum of exponentials. If, as is the case for [1, 5, 11], the masses $(m_{n,Nq})_n$ and the centers of mass $(x_{n,Nq})_n$ are selected according to Gaussian quadrature rules of order q with the weight function $w(x)$ in (3.7), then (see e.g. [23])

$$\begin{aligned} K_{\alpha,Nq}(0) &= \sum_{n=1}^{Nq} m_{n,Nq} = \mu([\xi_{0,N},\xi_{N,N}]) = \frac{\xi_{N,N}^{1-\alpha} - \xi_{0,N}^{1-\alpha}}{\Gamma(\alpha)\Gamma(2-\alpha)}, \\ K'_{\alpha,Nq}(0) &= -\sum_{n=1}^{Nq} m_{n,Nq}x_{n,Nq} = -\int_{\xi_{0,N}}^{\xi_{N,N}} x\mu(\mathrm{d}x) = -\frac{\xi_{N,N}^{2-\alpha} - \xi_{0,N}^{2-\alpha}}{(2-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}. \end{aligned} \tag{3.8}$$

It is customarily supposed that the sequence $(\xi_{N,N})_N$ [resp., $(\xi_{0,N})_N$] diverges to ∞ [resp., is bounded], which implies that $K_{\alpha,Nq}(0) \to \infty$ as $N \to \infty$. Moreover, the condition $\liminf_{N\to\infty} \|K_{\alpha,Nq}\|_{L^1(\mathbb{R}_+)} > 0$ is typically satisfied.[3] Therefore, also recalling (2.13), Theorem 3.1 (ii) is not satisfied for N big enough. As for the necessary condition in Theorem 3.1 (i), we note that, for a constant $C_3 = C_3(\alpha,\sigma) > 0$,

$$\sigma^2\frac{K_{\alpha,Nq}(0)^3}{2|K'_{\alpha,Nq}(0)|} - \kappa\theta\frac{K_{\alpha,Nq}(0)^2}{|K'_{\alpha,Nq}(0)|} \sim C_3\frac{1}{\xi_{N,N}^{2\alpha-1}} \quad \text{as} \quad N \to \infty. \tag{3.9}$$

[3] A sufficient condition – fulfilled in most multi-factor schemes – for $\liminf_{N\to\infty} \|K_{\alpha,Nq}\|_{L^1(\mathbb{R}_+)} > 0$ is the

Thus, when $\alpha \in (0, \frac{1}{2})$, analogously to $X_{\alpha,T}$, the process $X_{\alpha,Nq}$ attains the value 0 for every initial condition with positive probability when N is sufficiently large. This case corresponds to the multi-factor approximation of hyper-rough Volterra Heston models, see [2, Section 7].

The same general conclusion cannot be drawn for $\alpha \in (\frac{1}{2}, 1)$, when $\lim_{N\to\infty} \xi_{N,N}^{1-2\alpha} = 0$ and all initial states $x_0 > 0$ satisfy Theorem 3.1 (i) for N large enough. For this range of α, there might be parameters sets for which Feller's test for explosions in Theorem 3.1 is inconclusive for every N, that is, the necessary condition (i) is satisfied and the sufficient condition (ii) is violated. This is the case, for instance, for the test parameters in [5, Subsection 4.2] (which have also been considered in [12, Subsection 4.2]) when we approximate the fractional kernel $K_{0.6}$ with the kernel $K_{0.6,Nq}$ constructed as follows: $N = 20$, $q = 1$ and $\xi_{n,20} = r^{n-10}$, $n = 1, \dots, 20$. Here, $r > 1$ is obtained by minimizing the L^2-distance between $K_{0.6}$ and $K_{0.6,Nq}$ in the interval $(0, 5\mathrm{e}+05)$ according to [18, Equation (5.2)]. In this setting, we perform numerical experiments varying $\sigma > 0$ (the so-called *vol-of-vol*) in the range of values where Feller's test does not apply, namely

$$\left(\frac{2\kappa\theta}{K_{0.6,Nq}(0)} + 2\frac{|(K'_{0.6,Nq} * L_{0.6,Nq})(\infty)|}{K^2_{0.6,Nq}(0)}x_0\right)^{\frac{1}{2}} < \sigma \leq \left(\frac{2\kappa\theta}{K_{0.6,Nq}(0)} + 2\frac{|K'_{0.6,Nq}(0)|}{K_{0.6,Nq}(0)^3}x_0\right)^{\frac{1}{2}}.$$

Recalling (2.13), the previous estimates can be rewritten as follows:

$$\left(\frac{2}{K_{0.6,Nq}(0)}\right)^{\frac{1}{2}}\left(\kappa\theta + \frac{x_0}{\|K_{0.6,Nq}\|_{L^1(\mathbb{R}_+)}}\right)^{\frac{1}{2}} < \sigma \leq \left(\frac{2\kappa\theta}{K_{0.6,Nq}(0)} + 2\frac{|K'_{0.6,Nq}(0)|}{K_{0.6,Nq}(0)^3}x_0\right)^{\frac{1}{2}}. \quad (3.10)$$

In particular, in (3.10), the lower bound corresponds to a violation of the sufficient condition in Theorem 3.1 (ii), whereas the upper bound guarantees that the necessary condition in Theorem 3.1 (i) holds. More specifically, we simulate $M = 5\mathrm{e}{+}04$ trajectories of the process (3.1) (with $K_{0.6,Nq}$ instead of K) up to time $T = 5\mathrm{e}+05$ using a variation of constants and an explicit Euler scheme with discretization step $\delta = 1\mathrm{e}-01$. We refer the reader to [16, Section 5], where the same method is used to approximate numerically the solution of deterministic Riccati-Volterra equations driven by sum-of-exponentials kernels. The results are displayed in Table 1, where we report the empirical probability of the event $\{S \leq T\}$, i.e., of touching 0 before T, together with the empirical mean of S conditional on the same event. These outcomes show that, as σ increases, i.e., as σ approaches the upper bound in (3.10), the probability of the event $\{S \leq T\}$ increases, while the conditional empirical mean of S decreases. On the other hand, the smaller σ is, the smaller the empirical probability of $\{S \leq T\}$ and the higher the conditional empirical mean of S. Based on these results – which, in any case, are affected by an Euler approximation error – we conjecture that the necessary [resp., sufficient] condition in Theorem 3.1 (i) [resp., Theorem 3.1 (ii)] is not optimal, as the numerical evidence suggests that it could be more stringent [resp., weaker]. We plan to investigate theoretically this issue in a future research project, which aims to study the distribution

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following:

there exists $(j_N)_N \subset \mathbb{N}$, with $j_N \leq N$, such that $(\xi_{j_N,N})_N$ is bounded above and $\limsup_{N\to\infty} \frac{\xi_{0,N}}{\xi_{j_N,N}} < 1$.

Indeed, under this assumption, denoting by $C_\alpha > 0$ a constant that depends on α, by (3.8),

$$\begin{aligned}\|K_{\alpha,Nq}\|_{L^1(\mathbb{R}_+)} &= \sum_{n=1}^{Nq} \frac{m_{n,Nq}}{x_{n,Nq}} \geq \frac{1}{\xi_{j_N,N}} \sum_{n=1}^{j_N q} m_{n,Nq} = \frac{C_\alpha}{\xi_{j_N,N}}(\xi_{j_N,N}^{1-\alpha} - \xi_{0,N}^{1-\alpha}) \\ &\geq C_\alpha \xi_{j_N,N}^{-\alpha}\big(1 - (\xi_{0,N}/\xi_{j_N,N})^{1-\alpha}\big).\end{aligned}$$

Table 1: Results of the numerical experiment on $K_{0.6,Nq}$, with $N = 20$ and $q = 1$, using $M = 5\text{e} + 04$ trajectories of (3.1) with $\kappa = 0.3$, $\theta = 0.0667$ and $x_0 = 0.02$ as in [5, Subsection 3.2] (where $\sigma = 0.3$). The lower [resp., upper] bound in (3.10) is 0.0038 [resp., 1.09]

σ	$\mathbb{P}(S \leq 5\text{e} + 05)$	$\mathbb{E}[S\|\{S \leq 5\text{e} + 05\}]$
0.9	100%	0.41
0.3	100%	1.64
0.2	100%	6.22
0.15	100%	4.68e+01
0.12	100%	9.08e+02
0.11	100%	4.23e+03
0.1	100%	2.97e+04
0.09	73.8%	1.93e+05
0.08	4.16%	2.38e+05
0.07	0.03%	2.17e+05
0.06	0.00%	Not defined

of S using different techniques from those of the present paper. In fact, describing the $0-$attainment of Volterra CIR processes when σ varies on the range (3.10) is challenging and might require relationships that depend on more quantities than just $(K' * L)(0)$ and $(K' * L)(\infty)$. For instance, already in the case of the sum of two exponentials, namely $K_2(t) = m_1 e^{-x_1 t} + m_2 e^{-x_2 t}$, it is possible to determine different pairs of masses (m_1, m_2) and centers of mass (x_1, x_2) for which the lower and upper bounds in (3.10) coincide, while the boundary attainment of the corresponding Volterra CIR process X_2 is completely different, for a fixed σ satisfying (3.10). In our experiments, we consider the following two specifications, which correspond to a lower [resp., upper] bound in (3.10) of 0.002 [resp., 0.2]:

Case	Masses (m_1, m_2)	Centers of mass (x_1, x_2)
I	(1,1e4)	(1e-3, 1e8)
II	(8.87e+03, 1.13e+03)	(8.87, 8.87e+08)

Here, we fix $\sigma = 0.06$ and keep all the other test parameters as in [5, Subsection 4.2]. We then simulate $M = 5\text{e}{+}04$ trajectories of X_2 up to $T = 1\text{e}{+}05$ with discretization step $\delta = 1\text{e}{-}2$. In Case **I**, no trajectory hits 0, hence $\mathbb{P}(S \leq 1\text{e} + 05) = 0$, whereas in Case **II**, every trajectory attains 0, i.e., $\mathbb{P}(S \leq 1\text{e} + 05) = 1$, and $\mathbb{E}[S|\{S \leq 1\text{e} + 05\}] = 1.67\text{e}{-}2$.

Going back to the multi-factor approximations $K_{\alpha,Nq}$ of the fractional kernel K_α, it is also worth noting that the closer α is to its lower bound $\frac{1}{2}$, the slower the decay of $\xi_{N,N}^{1-2\alpha}$ and the more restrictive the necessary condition in Theorem 3.1 (i), see (3.9). As a result, for a small α, one might still expect such necessary condition to be violated across a wide range of starting points. This intuition is particularly relevant when the number of factors Nq is low, as is often the case in applications for improved efficiency. For instance, the test parameters in [5, Subsection 4.2] do not satisfy Theorem 3.1 (i) when we consider $\alpha = 0.51$ and approximate the fractional kernel $K_{0.51}$ by the kernel $K_{0.51,Nq}$ constructed in the same way as in the numerical experiment of Table 1.

We finally comment on the geometric and non-geometric Gaussian schemes in the recent work [12]. These methods are preferable because they offer faster numerical performance, which is useful, for example, when calibrating models involving $X_{\alpha,Nq}$ to real market data, see [16, Section 6]. In these approaches, $\xi_{0,N} = 0$ for all N, $(\xi_{1,N})_N$

is bounded, $(\xi_{N,N})_N$ diverges and the Gaussian quadrature rules of order q used to determine $(m_{n,Nq})_n$ and $(x_{n,Nq})_n$ employ the weight function $w(x) \equiv 1$ (apart from $[0, \xi_{1,N}]$, where $w(x)$ is as in (3.7)). As a result, similarly to (3.8),

$$K_{\alpha,Nq}(0) = \frac{\xi_{1,N}^{1-\alpha}}{\Gamma(\alpha)\Gamma(2-\alpha)} + (\xi_{N,N} - \xi_{1,N}),$$
$$K'_{\alpha,Nq}(0) = -\frac{\xi_{1,N}^{2-\alpha}}{(2-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)} - \frac{1}{2}(\xi_{N,N}^2 - \xi_{1,N}^2).$$

In particular,

$$K_{\alpha,Nq}(0) \to \infty \quad \text{and} \quad \sigma^2 \frac{K_{\alpha,Nq}(0)^3}{2|K'_{\alpha,Nq}(0)|} - \kappa\theta \frac{K_{\alpha,Nq}(0)^2}{|K'_{\alpha,Nq}(0)|} \to \infty \quad \text{as} \quad N \to \infty.$$

As regards Theorem 3.1, these computations entail that, for all $\alpha \in (0,1)$ and $x_0 > 0$, both the sufficient condition in Point (ii) and the necessary condition in Point (i) are violated for N large enough. In fact, we have verified that $X_{\alpha,Nq}$ attains 0 in a nonnegligible event in the two following literature cases, where Nq is low for numerical efficiency.

- [12], both using the geometric Gaussian scheme as specified in [12, Theorem 3.9] and the non-geometric Gaussian scheme as specified in [12, Theorem 3.16], with $a = 6.4$. Here $\alpha \in \{0.4, 0.5, \dots, 0.9\}$, $Nq = 4$ and the test parameters are in [12, Subsection 4.2];
- [16, Subsection 6.1], where an L^1-optimized version of the geometric Gaussian scheme in [12] is considered. Here $Nq = 20$ and the parameters set comes from a joint calibration of the rough Heston model to SPX and VIX market data.

Remark 3.4. Another nonexplosive kernel commonly used in financial applications, see e.g. [7], is the shifted power-law $K_{\epsilon,H}(t) = (t+\epsilon)^{H-\frac{1}{2}}$, for fixed $H \in (-\infty, \frac{1}{2})$ and $\epsilon > 0$. Since $K_{\epsilon,H}$ is completely monotone, it falls within our framework. By Theorem 3.1, a necessary condition for $\mathbb{P}(S = \infty) = 1$ is

$$2\kappa\theta \geq \epsilon^{H-\frac{1}{2}}\sigma^2 + 2\Big(H - \frac{1}{2}\Big)\epsilon^{-H-\frac{1}{2}}x_0,$$

while, recalling also Remark 2.7 and using the convention $\frac{1}{\infty} = 0$, a sufficient condition for $\mathbb{P}(S = \infty) = 1$ is

$$2\kappa\theta \geq \epsilon^{H-\frac{1}{2}}\sigma^2 - \frac{2x_0}{\|K_{\epsilon,H}\|_{L^1(\mathbb{R}_+)}}.$$

This shows that the sufficient condition is never satisfied when $\epsilon \downarrow 0$, as its lower bound diverges to ∞. As regards the necessary condition, its right-hand side diverges to ∞ as $\epsilon \downarrow 0$ when $H < 0$. This proves that, for every choice of parameters and $H < 0$, the Volterra CIR process corresponding to $K_{\epsilon,H}$ hits 0 with positive probability for ϵ sufficiently small. We observe that $H < 0$ corresponds to the hyper-rough regime $\alpha \in (0, \frac{1}{2})$ discussed in the above paragraph for multi-factor exponential kernels, for which we reach the same conclusion regarding the process hitting 0 with positive probability.

3.2 The Volterra Jacobi process

Fix $a, b \in \mathbb{R}$ such that $a < b$, $\kappa > 0$, $\sigma > 0$, and let $\theta \in (a, b)$. Define the nonnegative quadratic function $Q_{a,b}\colon [a,b] \to \mathbb{R}_+$ by $Q_{a,b}(x) = (x-a)(b-x)$, $x \in [a,b]$. Consider the SVE

$$X_t = x_0 + \int_0^t K(t-s)\kappa(\theta - X_s)\,\mathrm{d}s + \sigma\int_0^t K(t-s)\sqrt{Q_{a,b}(X_s)}\,\mathrm{d}W_s, \quad t \geq 0,\ \mathbb{P}-\text{a.s.}, \tag{3.11}$$

where $x_0 \in (a,b)$. By Hypotheses 1-4, for every finite time horizon $N \in \mathbb{N}$, the existence of a continuous weak solution to (3.11) in $[0,N]$ is established in [4, Corollary 2.8]. Note that, for every $x, y \in [a,b]$, by the subadditivity of the square-root function,

$$\begin{aligned}\left|\sqrt{Q_{a,b}(x)} - \sqrt{Q_{a,b}(y)}\right| &\leq \sqrt{|Q_{a,b}(x) - Q_{a,b}(y)|}\frac{\sqrt{Q_{a,b}(x) + Q_{a,b}(y)}}{\sqrt{Q_{a,b}(x)} + \sqrt{Q_{a,b}(y)}} \\ &\leq \sqrt{3(|a| + |b|)}\sqrt{|x-y|},\end{aligned}$$

which gives the $\frac{1}{2}-$Hölder's continuity of $\sqrt{Q_{a,b}}$ in $[a,b]$. Therefore, by the Yamada-Watanabe type argument in [5, Proposition B.3], we infer that there exists a pathwise unique continuous strong solution $X^N = (X_t^N)_{t\in[0,N]}$ of (3.11) in $[0,N]$. If we define the process $X = (X_t)_{t\geq 0}$ by

$$X_t = X_t^{[t]+1}, \quad t \geq 0,$$

then X is the pathwise unique, $[a,b]-$valued continuous strong solution of (3.11) in $\mathbb{R}_+$. We call such X a *Volterra Jacobi process*. As in Subsection 3.1, we now want to apply the results in Section 2 to determine conditions on the parameters κ, θ, σ and the initial point x_0 such that the paths of X touch (or do not touch) the boundary of $[a,b]$.

Fix $l = a$, $r = b$ and define the modified drift and diffusion mappings $\tilde{b}$, $\tilde{\sigma}\colon (a,b) \to \mathbb{R}$ by

$$\tilde{b}(x) = K(0)\kappa(\theta - x) + \frac{K'(0)}{K(0)}(x - x_0), \qquad \tilde{\sigma}(x) = K(0)\sigma\sqrt{Q_{a,b}(x)}, \quad x \in (a,b).$$

For an arbitrary $c \in (a,b)$, given $\beta, \gamma \in \mathbb{R}$ and defining $V = \frac{K'(0)}{K(0)} - (K' * L)(\infty)$ as in Subsection 3.1, by (2.5)-(2.6), the derivative of the scale function $p_c(\cdot;\beta,\gamma)\colon (a,b) \to \mathbb{R}$ associated with the dynamics (3.11) in the interval (a,c) is

$$\begin{aligned}p_c'(x;\beta) &= \exp\left\{-\frac{2}{K(0)^2\sigma^2}\int_c^x \frac{1}{Q_{a,b}(y)}\left(K(0)\kappa(\theta - y) + \frac{K'(0)}{K(0)}(y - x_0) + V\beta\right)\mathrm{d}y\right\} \\ &= \exp\left\{-\frac{2}{K(0)^2\sigma^2}\int_c^x \left(\frac{A(\beta)}{y-a} + \frac{B(\beta)}{b-y}\right)\mathrm{d}y\right\} \\ &= \left[\left(\frac{x-a}{c-a}\right)^{A(\beta)}\left(\frac{b-x}{b-c}\right)^{-B(\beta)}\right]^{-2(K(0)\sigma)^{-2}}, \quad x \in (a,c),\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}A(\beta) &= \frac{1}{b-a}\left(K(0)\kappa(\theta - a) + \frac{K'(0)}{K(0)}(a - x_0) + V\beta\right), \\ B(\beta) &= \frac{1}{b-a}\left(K(0)\kappa(\theta - b) + \frac{K'(0)}{K(0)}(b - x_0) + V\beta\right).\end{aligned} \tag{3.13}$$

Analogously, for every $x \in [c,b)$,

$$p_c'(x;\gamma) = \left[\left(\frac{x-a}{c-a}\right)^{A(\gamma)}\left(\frac{b-x}{b-c}\right)^{-B(\gamma)}\right]^{-2(K(0)\sigma)^{-2}}, \tag{3.14}$$

with $A(\gamma)$ and $B(\gamma)$ defined as in (3.13) replacing β by γ.

We now focus on the nonnegative map $v_c(\cdot;\beta,\gamma)\colon (a,b) \to \mathbb{R}_+$ in (2.9). Defining $C = 2(K(0)\sigma)^{-2}$ and supposing that $A(\beta) \neq 0$, the following lower bound for $v_c(\cdot;\beta)$

holds in the interval (a,c):

$$\begin{aligned} v_c(x;\beta) &= C(c-a)^{-A(\beta)C}(b-c)^{B(\beta)C}\int_x^c p_c'(y;\beta)\left(\int_y^c (z-a)^{A(\beta)C-1}(b-z)^{-B(\beta)C-1}\mathrm{d}z\right)\mathrm{d}y \\ &\geq C_1\left(A(\beta)C\right)^{-1}\int_x^c p_c'(y;\beta)\left((c-a)^{A(\beta)C}-(y-a)^{A(\beta)C}\right)\mathrm{d}y \\ &\geq C_1\int_x^c \left|\left(\frac{y-a}{c-a}\right)^{-A(\beta)C}-1\right|\mathrm{d}y, \quad x\in(a,c). \end{aligned} \tag{3.15}$$

We can also establish an analogous upper bound on the same interval, given by

$$v_c(x;\beta)\leq C_2\int_x^c \left|\left(\frac{y-a}{c-a}\right)^{-A(\beta)C}-1\right|\mathrm{d}y, \quad x\in(a,c). \tag{3.16}$$

Supposing that also $B(\gamma)\neq 0$, similar computations enable us to estimate $v_c(\cdot;\gamma)$ in the interval $[c,b)$, namely

$$(-1)^i v_c(x;\gamma)\leq (-1)^i C_i\int_c^x \left|\left(\frac{b-y}{b-c}\right)^{B(\gamma)C}-1\right|\mathrm{d}y, \quad x\in[c,b),\, i=3,4. \tag{3.17}$$

Here C_1, C_2, C_3, $C_4>0$ are positive constants, possibly depending on $K,\kappa,\theta,\sigma,\beta,\gamma,a,b$, x_0 and c, which are allowed to change from line to line. Combining the previous estimates with the results in Subsection 2.2, we obtain the next theorem.

Theorem 3.5. *Assume Hypotheses 1-3-4. Consider the stopping times* $S_-=\inf\{t\geq 0: X_t=a\}$, $S_+=\inf\{t\geq 0: X_t=b\}$ *and* $S=S_-\wedge S_+=\inf\{t\geq 0: X_t\in\{a,b\}\}$.

(i) The condition

$$2\kappa(\theta-a)\geq K(0)\sigma^2(b-a)+2\frac{K'(0)}{K^2(0)}(x_0-a)$$

is necessary for $\mathbb{P}(S_-=\infty)=1$. *Additionally, the condition*

$$2\kappa(b-\theta)\geq K(0)\sigma^2(b-a)+2\frac{K'(0)}{K(0)^2}(b-x_0)$$

is necessary for $\mathbb{P}(S_+=\infty)=1$.

(ii) The condition $2\kappa(\theta-a)\geq K(0)\sigma^2(b-a)+2\frac{(K'*L)(\infty)}{K(0)}(x_0-a)$ *is sufficient for* $\mathbb{P}(\{S<S_-\}\cup\{S_-=\infty\})=1$. *Additionally, the condition* $2\kappa(b-\theta)\geq K(0)\sigma^2(b-a)+$ $+2\frac{(K'*L)(\infty)}{K(0)}(b-x_0)$ *is sufficient for* $\mathbb{P}(\{S<S_+\}\cup\{S_+=\infty\})=1$. *Consequently,*

$$\begin{aligned} &2\min\left\{\kappa(\theta-a)+\frac{|(K'*L)(\infty)|}{K(0)}(x_0-a),\kappa(b-\theta)+\frac{|(K'*L)(\infty)|}{K(0)}(b-x_0)\right\} \\ &\quad\geq K(0)\sigma^2(b-a) \end{aligned}$$

implies that $\mathbb{P}(S=\infty)=1$.

Proof. Let $c\in(a,b)$. We will only prove the assertions concerning S_-, as those about S_+ can be obtained by analogous arguments relying on the estimate (3.17). We first demonstrate the necessary condition in Point (i) by working with its contrapositive statement. Consider $2|K'(0)|x_0<2|K'(0)|a+K(0)^2\left(K(0)\sigma^2(b-a)-2\kappa(\theta-a)\right)$, which amounts to supposing that

$$2\big(K(0)\sigma\big)^{-2}\frac{1}{b-a}\left(K(0)\kappa(\theta-a)+\frac{K'(0)}{K(0)}(a-x_0)\right)<1,$$

where we also use that $K'(0) \leq 0$. Then, recalling (3.13), for a $\bar{\beta} < 0$ sufficiently close to 0 we have $CA(\bar{\beta}) < 1$. In this way, by (3.16), for a positive constant $C_{\bar{\beta}}$,

$$v_c(a+;\bar{\beta}) \leq C_{\bar{\beta}} \int_a^c \left| \left(\frac{y-a}{c-a} \right)^{-A(\bar{\beta})C} - 1 \right| \mathrm{d}y < \infty.$$

Thus, the necessary condition in Theorem 2.12 (i) is violated, which yields $\mathbb{P}(S_- < \infty) > 0$. This proves Point (i).

As for Point (ii), we assume $2\kappa(\theta-a) \geq K(0)\sigma^2(b-a) + 2\frac{(K'*L)(\infty)}{K(0)}(x_0-a)$ and observe that, by (3.15) and the definition of $A(x_0-a)$ in (3.13),

$$v_c(a+;x_0-a) \geq \widetilde{C} \int_a^c \left(\left(\frac{c-a}{y-a} \right)^{2\left(\kappa(\theta-a) + \frac{(K'*L)(\infty)}{K(0)}(a-x_0)\right)(K(0)\sigma^2(b-a))^{-1}} - 1 \right) \mathrm{d}y = \infty,$$

for some $\widetilde{C} > 0$. Therefore, thanks to Remark 2.14, we invoke Theorem 2.13 (i) to conclude that $\mathbb{P}(\{S < S_-\} \cup \{S_- = \infty\}) = 1$. The theorem is now completely proved. □

In the Volterra Jacobi framework the interval $I = (a,b)$ is bounded. Thus, we can employ Proposition 2.9 to obtain the following result, see also Remark 2.10.

Proposition 3.6. *Assume Hypotheses 1-3-4. Define S_+, S_- and S as in Theorem 3.5.*

(i) If $2\kappa(b-\theta) \geq K(0)\sigma^2(b-a) + 2\frac{(K'*L)(\infty)}{K(0)}(b-x_0)$ and $2\kappa(\theta-a) + 2\frac{(K'*L)(\infty)}{K(0)}(b-x_0) < \left(K(0)\sigma^2 - 2|K'(0)|K(0)^{-2}\right)(b-a)$, then

$$\mathbb{P}\left(\sup_{0 \leq t < S} X_t < b \right) = 1;$$

(ii) If $2\kappa(\theta-a) \geq K(0)\sigma^2(b-a) + 2\frac{(K'*L)(\infty)}{K(0)}(x_0-a)$ and $2\kappa(b-\theta) + 2\frac{(K'*L)(\infty)}{K(0)}(x_0-a) < \left(K(0)\sigma^2 - 2|K'(0)|K(0)^{-2}\right)(b-a)$, then

$$\mathbb{P}\left(\inf_{0 \leq t < S} X_t > a \right) = 1.$$

Proof. We focus only on Point (i), as Point (ii) can be deduced by similar arguments in a straightforward way. By (3.13), the condition $2\kappa(\theta-a) + 2\frac{(K'*L)(\infty)}{K(0)}(b-x_0) < \left(K(0)\sigma^2 - 2|K'(0)|K(0)^{-2}\right)(b-a)$ is equivalent to

$$A(x_0-b)C = \frac{2}{K(0)\sigma^2(b-a)} \left(\kappa(\theta-a) + \frac{K'(0)}{K(0)^2}(a-b) + \frac{(K'*L)(\infty)}{K(0)}(b-x_0) \right) < 1.$$

Furthermore, $2\kappa(b-\theta) \geq K(0)\sigma^2(b-a) + 2\frac{(K'*L)(\infty)}{K(0)}(b-x_0)$ is equivalent to

$$B(x_0-b)C = 2\frac{\kappa(\theta-b) + (K'*L)(\infty)(K(0))^{-1}(b-x_0)}{K(0)\sigma^2(b-a)} \leq -1.$$

By (3.12)-(3.14), the previous estimates show that $p_c(a+;x_0-b) > -\infty$ and $p_c(b-;x_0-b) = \infty$, for some $c \in (a,b)$. Hence an application of Proposition 2.9 (i) yields the desired conclusion, and the proof is complete. □

Remark 3.7. When $K \equiv K(0)e^{-\Lambda t}$, for some constant $K(0) > 0$ and $\Lambda \leq 0$, the necessary and sufficient conditions for $\mathbb{P}(S = \infty) = 1$ in Theorem 3.5 coincide. Consequently, the following equivalence holds: $\mathbb{P}(X_t \in (a,b),\, t \in \mathbb{R}_+) = 1$ if and only if

$$\begin{aligned} &2\min\left\{ \kappa(\theta-a) + \frac{\Lambda}{K(0)}(x_0-a), \kappa(b-\theta) + \frac{\Lambda}{K(0)}(b-x_0) \right\} \\ &\quad \geq K(0)\sigma^2(b-a). \end{aligned} \tag{3.18}$$

Notice that, choosing $K \equiv 1$, (3.18) reduces to boundary attainment results for (classical) Jacobi processes already established in the literature, see for instance [8, Theorem 2.1].

In fact, when the condition in (3.18) is not met, we can infer more properties on the trajectories of X using Proposition 3.6. Specifically, if $2\kappa(\theta - a) < K(0)\sigma^2(b - a) - 2\frac{\Lambda}{K(0)}(x_0 - a)$, then by Theorem 3.5(i) the paths of X attain the value a with positive probability. If we further suppose that $2\kappa(b-\theta) \geq K(0)\sigma^2(b-a) - 2\frac{\Lambda}{K(0)}(b - x_0)$, then $\mathbb{P}(\sup_{t\in[0,S)} X_t < b) = 1$ by Proposition 3.6(i), which can be applied recalling that, for this particular choice of kernels, $(K' * L)(\infty) = K'(0)(K(0))^{-1}$. On the other hand, if $2\kappa(b-\theta) < K(0)\sigma^2(b-a) - 2\frac{\Lambda}{K(0)}(b - x_0)$, then the trajectories of X attain the value b with positive probability again by Theorem 3.5(i). Furthermore, if also $2\kappa(\theta - a) \geq K(0)\sigma^2(b-a) - 2\frac{\Lambda}{K(0)}(x_0 - a)$, then $\mathbb{P}(\inf_{t\in[0,S)} X_t > a) = 1$ by Proposition 3.6(ii).

3.3 The Volterra power-type diffusion process

Given $\sigma > 0$ describing the strength of the random perturbation and coefficients $\alpha > 1$ and $\delta \in [0, 1)$ characterizing the power-type drift and diffusion, respectively, we consider the SVE with multiplicative noise

$$X_t = x_0 + \int_0^t K(t-s)|X_s|^\alpha \mathrm{d}s + \sigma \int_0^t K(t-s)|X_s|^{\frac{\delta}{2}}\,\mathrm{d}W_s, \quad t \geq 0,\ \mathbb{P}-\text{a.s.}, \tag{3.19}$$

where $x_0 \in \mathbb{R}$. We suppose that there exists a continuous $\overline{\mathbb{R}}-$valued process X satisfying (3.19) locally in the sense of (2.2). In this framework, boundary attainment implies *blow-ups* (also called *explosions*) to $\pm\infty$ of the trajectories. Our aim in this subsection is to determine a relation between the exponents α and δ ensuring that X explodes in finite time, see Theorem 3.8 below. Note that, in the limiting case $\alpha = 1$, the drift reduces to a map with linear growth, hence the existence of a global weak solution to (3.19) is given by [6, Theorem 3.4], for every $\delta \in [0, 2]$. We also refer to [34, Theorem 3.7], which establishes the existence and uniqueness of a maximal (local) solution to (3.19). Such a result requires locally Lipschitz continuous coefficients, hence it can be applied to our setting only when $\delta = 0$.

Fix $l = -\infty$, $r = \infty$ and define the mappings $\tilde{b}$, $\tilde{\sigma}\colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{b}(x) = K(0)|x|^\alpha + \frac{K'(0)}{K(0)}(x - x_0), \qquad \tilde{\sigma}(x) = \sigma K(0)|x|^{\frac{\delta}{2}}, \quad x \in \mathbb{R}.$$

We observe that the modified drift $\tilde{b}$ and diffusion $\tilde{\sigma}$ satisfy Hypothesis 2 because $\delta < 1$.

To ease the notation, as in the two previous subsections we let $C = 2(K(0)\sigma)^{-2}$ and $V = \frac{K'(0)}{K(0)} - (K' * L)(\infty)$. For β, $\gamma \in \mathbb{R}$ and $c = 0$, the derivative of the scale function $p_0(\cdot;\beta,\gamma)\colon \mathbb{R} \to \mathbb{R}$ is, by (2.5)-(2.6),

$$\begin{aligned} p_0'(x;\beta) = \exp\Bigg\{ -C\Bigg(&-\frac{K(0)}{\alpha-\delta+1}(-x)^{\alpha-\delta+1} + \frac{K'(0)}{(2-\delta)K(0)}(-x)^{2-\delta} \\ &+ \Big(x_0\frac{K'(0)}{K(0)} - V\beta\Big)\frac{(-x)^{1-\delta}}{1-\delta}\Bigg)\Bigg\}, \quad x < 0, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} p_0'(x;\gamma) = \exp\Bigg\{ -C\Bigg(&\frac{K(0)}{\alpha-\delta+1}x^{\alpha-\delta+1} + \frac{K'(0)}{(2-\delta)K(0)}x^{2-\delta} \\ &- \Big(x_0\frac{K'(0)}{K(0)} - V\gamma\Big)\frac{x^{1-\delta}}{1-\delta}\Bigg)\Bigg\}, \quad x \geq 0. \end{aligned} \tag{3.21}$$

We notice that, since $l = -\infty$ and $r = \infty$, Proposition 2.9 cannot be applied for general kernels K such that $V \neq 0$, as explained in Remark 2.10. In the particular case when K is exponential or constant (see Remark 2.16), Proposition 2.9 (i) does not apply either, as $p_0(r_n; x_0 - r_n)$ is bounded above by (3.21). On the other hand, considering also that $p_0(-\infty; \beta) = -\infty$ for every $\beta \in \mathbb{R}$ by (3.20), Proposition 2.9 (ii) ensures that $\mathbb{P}\left(\inf_{0 \leq t < S} X_t > -\infty\right) = 1$.

Observe that we can estimate from below the map $v_0(\cdot; \beta) \colon (-\infty, -1) \to \mathbb{R}_+$, defined in (2.8)-(2.9), as follows:

$$\begin{aligned} v_0(x;\beta) &\geq C \int_x^{-1} p_0'(y;\beta) \left(\int_y^0 \frac{\mathrm{d}z}{(-z)^\delta p_0'(z;\beta)} \right) \mathrm{d}y \\ &\geq C \left(\int_{-1}^0 \frac{\mathrm{d}z}{(-z)^\delta p_0'(z;\beta)} \right) \left(\int_x^{-1} p_0'(y;\beta) \mathrm{d}y \right), \quad x < -1. \end{aligned}$$

In particular, when $\beta = x_0 + \tilde{\beta}$ for some $\tilde{\beta} \in \mathbb{R}$ such that $V\tilde{\beta} \leq 0$, by (3.20),

$$\begin{aligned} v_0(x; x_0 + \tilde{\beta}) \geq C \left(\int_{-1}^0 \frac{\mathrm{d}z}{(-z)^\delta p_0'(z;\beta)} \right) \exp\left\{ CV\tilde{\beta} \frac{(-x)^{1-\delta}}{1-\delta} \right\} \int_x^{-1} \mathrm{d}y \Bigg[\exp\Bigg\{ C \\ \left(\frac{K(0)}{\alpha - \delta + 1} (-y)^{\alpha - \delta + 1} + \frac{|K'(0)|}{(2-\delta)K(0)} (-y)^{2-\delta} - (K' * L)(\infty) x_0 \frac{(-y)^{1-\delta}}{1-\delta} \right) \Bigg\} \Bigg], \end{aligned} \tag{3.22}$$

for every $x < -1$. Combining this estimate with further assumptions on α and δ which enable us to obtain un upper bound for $v_0(\cdot; \gamma)$ in a positive half-line, we infer that the solution X of (3.19) blows up in finite time with positive probability. More specifically, the next result holds.

Theorem 3.8. *Assume Hypotheses 1-3. Consider a continuous $\overline{\mathbb{R}}-$valued process X solving* (3.19) *in the sense of* (2.2). *Define the corresponding stopping times $S_- = \inf\{t \geq 0 : X_t = -\infty\}$, $S_+ = \inf\{t \geq 0 : X_t = \infty\}$ and $S = S_- \wedge S_+ = \inf\{t \geq 0 : X_t \in \{-\infty, \infty\}\}$. Then*

$$\mathbb{P}(\{S < S_-\} \cup \{S_- = \infty\}) = 1.$$

Furthermore, if $\alpha > 1 + \delta$, then $\mathbb{P}(S_+ < \infty) > 0$.

Proof. We first verify the sufficient condition in Theorem 2.13 (i), which guarantees that $\mathbb{P}(\{S < S_-\} \cup \{S_- = \infty\}) = 1$, as desired. We consider $l_n = -n$, $n \in \mathbb{N}$, and aim to show that

$$\liminf_{n \to \infty} v_0(-n; x_0 + n) = \infty. \tag{3.23}$$

Hypothesis 3 guarantees that $V \leq 0$. Then for all $x < -1 \wedge x_0$, thanks to the lower bound in (3.22) with $\tilde{\beta} = -x$,

$$\begin{aligned} v_0(x; x_0 - x) &\geq C \left(\int_{-1}^0 \frac{\mathrm{d}z}{(-z)^\delta p_0'(z; x_0 - x)} \right) \times \exp\left\{ -C|V| \frac{(-x)^{2-\delta}}{1-\delta} \right\} \times \int_x^{-1} \mathrm{d}y \Bigg[\exp\Bigg\{ C \\ &\quad \left(\frac{K(0)}{\alpha - \delta + 1} (-y)^{\alpha - \delta + 1} + \frac{|K'(0)|}{(2-\delta)K(0)} (-y)^{2-\delta} - (K' * L)(\infty) x_0 \frac{(-y)^{1-\delta}}{1-\delta} \right) \Bigg\} \Bigg] \\ &=: C(\mathbf{I} \times \mathbf{II}^{-1} \times \mathbf{III})(x). \end{aligned}$$

Note that, by (3.20), since $x_0 - x > 0$, $p_0'(z; x_0 - x) \leq p_0'(z; 0)$ for all $z < 0$. Thus,

$$\mathbf{I}(x) \geq \int_{-1}^0 \frac{\mathrm{d}z}{(-z)^\delta p_0'(z;0)}, \quad x < -1 \wedge x_0.$$

Next, denoting by C_i, $i = 1, \dots, 5$, constants possibly depending on C, K, V, x_0, α and δ, with $C_i > 0$ for $i = 1, \dots, 4$, since $\alpha > 1$ we have

$$\lim_{x \to -\infty} \frac{\mathbf{III}'(x)}{\mathbf{II}'(x)} = \lim_{x \to -\infty} \frac{\exp\left\{C_1(-x)^{\alpha-\delta+1} + C_2(-x)^{2-\delta} + C_5(-x)^{1-\delta}\right\}}{C_3(-x)^{1-\delta}\exp\left\{C_4(-x)^{2-\delta}\right\}} = \infty.$$

Therefore, $\lim_{x\to-\infty} \mathbf{II}^{-1}(x) \times \mathbf{III}(x) = \infty$ by De l'Hôpital's rule, whence (3.23).

Secondly, we assume $\alpha > 1 + \delta$ and argue that the necessary condition in Theorem 2.12 (ii) is violated. More precisely, we are going to show that $v_0(\infty-;\gamma) < \infty$ for all $\gamma \in \mathbb{R}$. We denote by $g(x)$, $x \geq 0$, the exponent of $p_0'(\cdot;\gamma)$ in (3.21) divided by $-C$, so that we can write

$$p_0'(x;\gamma) = \exp\left\{-Cg(x)\right\}, \quad x \geq 0. \tag{3.24}$$

Since $g \sim \frac{K(0)}{\alpha-\delta+1}x^{\alpha-\delta+1}$, $g' \sim K(0)x^{\alpha-\delta}$ and $g'' \sim K(0)(\alpha-\delta)x^{\alpha-\delta-1}$ as $x \to \infty$, there exists a threshold $\tilde{c} = \tilde{c}(\alpha, \gamma, \delta, K, x_0, V) > 1$ such that g and g' are increasing and strictly positive in the half-line $[\tilde{c}, \infty)$. As a result, we can deduce an upper bound for $v_0(\cdot;\gamma)$ in $(\tilde{c}, \infty)$ using the following decomposition:

$$\begin{aligned} v_0(x;\gamma) \leq{}& C\left(\int_0^{\tilde{c}} p_0'(y;\gamma)\mathrm{d}y\right)\left(\int_0^{\tilde{c}} \frac{\mathrm{d}z}{z^\delta p_0'(z;\gamma)}\right) + C\left(\int_0^{\tilde{c}} \frac{\mathrm{d}z}{z^\delta p_0'(z;\gamma)}\right)\left(\int_{\tilde{c}}^x p_0'(y;\gamma)\mathrm{d}y\right) \\ &+ C\int_{\tilde{c}}^x p_0'(y;\gamma)\left(\int_{\tilde{c}}^y \frac{\mathrm{d}z}{z^\delta p_0'(z;\gamma)}\right)\mathrm{d}y, \quad x > \tilde{c}. \end{aligned} \tag{3.25}$$

In particular, if we focus on the last addend in the right-hand side of (3.25), then, by Tonelli's theorem, (3.24) and the fact that $\tilde{c} > 1$,

$$\begin{aligned} \int_{\tilde{c}}^x p_0'(y;\gamma)\left(\int_{\tilde{c}}^y \frac{\mathrm{d}z}{z^\delta p_0'(z;\gamma)}\right)\mathrm{d}y &\leq \int_{\tilde{c}}^\infty \frac{1}{p_0'(z;\gamma)}\left(\int_z^\infty \frac{g'(y)}{g'(y)}p_0'(y;\gamma)\mathrm{d}y\right)\mathrm{d}z \\ &\leq \frac{1}{C}\int_{\tilde{c}}^\infty \frac{1}{g'(z)}\mathrm{d}z, \quad x > \tilde{c}. \end{aligned}$$

Plugging this estimate in (3.25), we obtain, for every $x > \tilde{c}$,

$$v_0(x;\gamma) \leq C\left(\int_0^{\tilde{c}} \frac{\mathrm{d}z}{z^\delta p_0'(z;\gamma)}\right)\left(\int_0^\infty p_0'(y;\gamma)\mathrm{d}y\right) + \int_{\tilde{c}}^\infty \frac{1}{g'(z)}\mathrm{d}z < \infty.$$

Here, the two addends on the right-hand side do not depend on x and are finite. Indeed, $\delta < 1$ and the maps $p_0'(\cdot;\gamma)$ and $(g')^{-1}$ are integrable at ∞ because $\alpha > 1 + \delta$. Therefore, $v_0(\infty-;\gamma) < \infty$ and the proof is complete. □

A Proofs of some technical results

In this appendix, we collect the proofs of Lemma 2.4, Corollary 2.5 and Lemma 2.11 in Section 2.

Proof of Lemma 2.4. Fix β, $\gamma \in \mathbb{R}$ and c_1, $c_2 \in I$ such that $c_1 < c_2$. All the equalities in (2.10) and (2.11) directly follow from the definitions of $p_c(\cdot;\beta,\gamma)$, see (2.6), and $v_c(\cdot;\beta,\gamma)$, see (2.8)-(2.9). Here, we only show how to obtain the second equation in (2.11) as an example. To do this, we take $x \in (c_2, r)$ and compute, according to (2.6)-(2.9) and omitting γ from the notation,

$$\begin{aligned} v_{c_1}(x) &= v_{c_1}(c_2) + 2\int_{c_2}^x \exp\left\{-2\int_{c_1}^y \frac{\tilde{b}_{c_1}(t)}{\tilde{\sigma}^2(t)}\mathrm{d}t\right\}\left(\int_{c_1}^y \exp\left\{2\int_{c_1}^z \frac{\tilde{b}_{c_1}(t)}{\tilde{\sigma}^2(t)}\mathrm{d}t\right\}\frac{1}{\tilde{\sigma}^2(z)}\mathrm{d}z\right)\mathrm{d}y \\ &= v_{c_1}(c_2) + 2p_{c_1}'(c_2)\int_{c_2}^x p_{c_2}'(y)\left(\int_{c_1}^{c_2} \frac{1}{p_{c_1}'(z)\tilde{\sigma}^2(z)}\mathrm{d}z + \frac{1}{p_{c_1}'(c_2)}\int_{c_2}^y \frac{1}{p_{c_2}'(z)\tilde{\sigma}^2(z)}\mathrm{d}z\right)\mathrm{d}y \end{aligned}$$

$$= v_{c_1}(c_2) + 2p'_{c_1}(c_2)p_{c_2}(x)\int_{c_1}^{c_2}\frac{1}{p'_{c_1}(z)\tilde{\sigma}^2(z)}\mathrm{d}z + v_{c_2}(x)$$
$$= v_{c_1}(c_2) + p_{c_2}(x)v'_{c_1}(c_2) + v_{c_2}(x).$$

Here, for the second equality we use that, by (2.5), $\tilde{b}_{c_1}(t;\gamma) = \tilde{b}_{c_2}(t;\gamma)$ when $t \in [c_2, r)$. The previous expression is the one for v_{c_1} in (c_2, r) that we are looking for. □

Proof of Corollary 2.5. Consider β, $\gamma \in \mathbb{R}$ and c_1, $c_2 \in I$ such that $c_1 < c_2$. We first focus on demonstrating that $v_{c_2}(r-;\gamma) = \infty$ is equivalent to $v_{c_1}(r-;\gamma) = \infty$. The implication

$$v_{c_2}(r-;\gamma) = \infty \quad \Longrightarrow \quad v_{c_1}(r-;\gamma) = \infty$$

follows directly from the second equality in (2.11), which gives $v_{c_1}(\cdot;\gamma) \geq v_{c_2}(\cdot;\gamma)$ in (c_2, r). For the reverse one, namely

$$v_{c_1}(r-;\gamma) = \infty \quad \Longrightarrow \quad v_{c_2}(r-;\gamma) = \infty,$$

we suppose that $v_{c_1}(r-;\gamma) = \infty$ and argue by cases, corresponding to $p_{c_2}(r-;\gamma) < \infty$ or $p_{c_2}(r-;\gamma) = \infty$. In the former, i.e., $p_{c_2}(r-;\gamma) < \infty$, we rewrite the second equation in (2.11) as

$$v_{c_2}(x;\gamma) = v_{c_1}(x;\gamma) - v_{c_1}(c_2;\gamma) - v'_{c_1}(c_2;\gamma)p_{c_2}(x;\gamma), \quad x \in (c_2, r).$$

Taking the limit as $x \uparrow r$ we obtain $v_{c_2}(r-;\gamma) = \infty$. In the latter case, i.e., $p_{c_2}(r-;\gamma) = \infty$, the fact that $v_{c_2}(r-;\gamma)$ is infinite is independent from the hypothesis $v_{c_1}(r-;\gamma) = \infty$. Indeed, fixing a generic $\bar{c} \in (c_2, r)$, by (2.9) we compute, for $x \in (\bar{c}, r)$,

$$\begin{aligned} v_{c_2}(x;\gamma) &= v_{c_2}(\bar{c};\gamma) + 2\int_{\bar{c}}^{x} p'_{c_2}(y;\gamma)\left(\int_{c_2}^{y}\frac{\mathrm{d}z}{p'_{c_2}(z;\gamma)\tilde{\sigma}^2(z)}\right)\mathrm{d}y \\ &\geq 2\left(\int_{c_2}^{\bar{c}}\frac{\mathrm{d}z}{p'_{c_2}(z;\gamma)\tilde{\sigma}^2(z)}\right)\left(p_{c_2}(x;\gamma) - p_{c_2}(\bar{c};\gamma)\right). \end{aligned}$$

Thus, we deduce that $v_{c_2}(r-;\gamma) = \infty$ by letting $x \uparrow r$.

The equivalence

$$v_{c_1}(l+;\beta) = \infty \quad \Longleftrightarrow \quad v_{c_2}(l+;\beta) = \infty$$

can be argued analogously, using the second equality in (2.10) (which entails that $v_{c_2}(\cdot;\beta) \geq v_{c_1}(\cdot;\beta)$ in (l, c_1)) and showing the implication $p_{c_1}(l+;\beta) = -\infty \Longrightarrow v_{c_1}(l+;\beta) = \infty$. Therefore the proof is complete. □

Proof of Lemma 2.11. Fix $c \in I$ and parameters β_i, γ_i, $i = 1, 2$, according to the statement of the lemma, namely, $\beta_1 \geq \beta_2$ and $\gamma_1 \leq \gamma_2$. Differentiating (2.6) and recalling (2.5), for every $c < z < y < r$

$$\begin{aligned} \frac{p'_c(y;\gamma_1)}{p'_c(z;\gamma_1)} &= \exp\left\{-2\int_z^y \frac{\tilde{b}(h)}{\tilde{\sigma}^2(h)}\mathrm{d}h - 2\gamma_1\left(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\right)\int_z^y \frac{1}{\tilde{\sigma}^2(h)}\mathrm{d}h\right\} \\ &\leq \exp\left\{-2\int_z^y \frac{\tilde{b}(h)}{\tilde{\sigma}^2(h)}\mathrm{d}h - 2\gamma_2\left(\frac{K'(0)}{K(0)} - (K' * L)(\infty)\right)\int_z^y \frac{1}{\tilde{\sigma}^2(h)}\mathrm{d}h\right\} = \frac{p'_c(y;\gamma_2)}{p'_c(z;\gamma_2)}, \end{aligned}$$

where the inequality holds by Hypothesis 3, which in particular guarantees that

$$(K' * L)(0) = \frac{K'(0)}{K(0)} \leq (K' * L)(\infty).$$

Hence, by (2.8)-(2.9), $v_c(\cdot;\gamma_1) \leq v_c(\cdot;\gamma_2)$ in (c, r). Analogously, we demonstrate that the inequality $v_c(\cdot;\beta_1) \leq v_c(\cdot;\beta_2)$ holds in (l, c).

Suppose now that $\int_l^r \tilde{\sigma}^{-2}(z)\mathrm{d}z < \infty$. Arguing as in the above computation yields, for all $y,\, z \in I$ such that $l < y < z < c$ or $c < z < y < r$,

$$\frac{p_c'(y;\beta_2,\gamma_2)}{p_c'(z;\beta_2,\gamma_2)} \leq \exp\left\{2\max\{\beta_1-\beta_2,\gamma_2-\gamma_1\}\left((K'*L)(\infty) - \frac{K'(0)}{K(0)}\right)\int_l^r \frac{1}{\tilde{\sigma}^2(h)}\mathrm{d}h\right\} \times \frac{p_c'(y;\beta_1,\gamma_1)}{p_c'(z;\beta_1,\gamma_1)} =: C\frac{p_c'(y;\beta_1,\gamma_1)}{p_c'(z;\beta_1,\gamma_1)},$$

whence the second conclusion of the lemma. This completes the proof. □

B A different view on the sum-of-exponentials case

As we observe in a paragraph in Section 1, when the kernel K driving the dynamics (2.2) is a sum of exponentials, that is, $K(t) = K_N(t) = \sum_{n=1}^N m_n e^{-\Lambda_n t}$, for $m_n,\, \Lambda_n > 0$, $n = 1,\dots,N$, one can construct a local $\mathbb{R}-$valued solution X of (2.2) from an $N-$dimensional stochastic process $Y = [Y_1,\dots,Y_N]^\top$ starting from 0 that solves, locally, a system of SDEs, see (1.6). To do this, it suffices to take $X = x_0 + \sum_{n=1}^N m_n Y_n$. Thus, to study blow-ups to $\pm\infty$ of solutions of SVEs in finite time, one can analyze necessary and sufficient conditions such that an $N-$dimensional Itô diffusion is defined globally on $\mathbb{R}_+$. This procedure can also be extended to determine conditions under which solutions to SVEs take values in an open interval $I = (l,r)$, with possibly finite endpoints l and r, on the nonnegative half-line $\mathbb{R}_+$. Assuming that $l > -\infty$, this can be achieved by increasing the dimension of the system of SDEs in (1.6), including the dynamics of a real-valued transform $Y^{N+1} = f\big(x_0 + \sum_{n=1}^N m_n Y_n\big)$ such that $|f(l)| = \infty$. An additional SDE corresponding to another transform of $x_0 + \sum_{n=1}^N m_n Y_n$ must be added to the system if r is finite. Following this machinery, the problem of global $I-$valued solutions to SVEs driven by the kernel K_N translates into the study of necessary and sufficient conditions such that an $(N+2)-$dimensional system of SDEs admits a global $\mathbb{R}^{N+2}-$valued solution that does not explode in finite time. On a related note, we refer to [3] for a recent study of invariant state spaces for degenerate systems similar to (1.6), arising from and motivated by K_N-driven SVEs.

Motivated by these arguments, we consider a continuous $N-$dimensional $(\overline{\mathbb{R}})^N-$valued stochastic process $X = [X_1,\dots,X_N]^\top = (X_t)_{t\geq 0}$ and define $\tau_n = \inf\{t \geq 0 : |X_t| \geq n\}$, so that $S = \lim_{n\to\infty}\tau_n = \inf\{t \geq 0 : |X_{i,t}| = \infty, \text{ for some } i\}$. Suppose that, for every $n \in \mathbb{N}$, X satisfies the path-independent $N-$dimensional SDE

$$\mathrm{d}X_t = B(t,X_t)\mathrm{d}t + \Sigma(t,X_t)\mathrm{d}\bar{W}_t, \quad X_0 = x_0 \in \mathbb{R}^N, \qquad t \in [0,\tau_n],\, \mathbb{P}-\text{a.s.}, \tag{B.1}$$

where $B\colon \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, $\Sigma\colon \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^{N\times N}$, and $\bar{W} = [W_1,\dots,W_N]^\top$ is a vector of one-dimensional Brownian motions with positive semi-definite covariance matrix C, namely

$$C_{i,j}t = \langle W_i, W_j\rangle_t, \quad t \geq 0,\, i,j = 1,\dots,N.$$

In this appendix, we give necessary and sufficient conditions for the non-explosion of X, that is, for $\mathbb{P}(S = \infty) = 1$. This is an extension of the classical Khasminskii's criterion in [29, Theorem I], see also [28] for related topics, that applies to systems with degenerate noise and time-dependent coefficients. Such an extension is necessary to apply the results to SVEs governed by a sum-of-exponentials kernel, as we detail in Example B.2 below. Since our result is technical and non-trivial and [29] does not include a proof, we present one for the sake of completeness, relying on Itô's formula in a way similar to Feller's test. We notice that another sufficient Khasminskii-type criterion for non-explosion appears in [32, Theorem 52.12, Chapter V], but the proof of this result exploits a convexity argument which is not justified and seems difficult to establish in general.

Theorem B.1. *Assume $B\colon \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ and $\Sigma\colon \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^{N\times N}$ locally bounded, namely*

$$\sup_{t\in[0,M],\,|x|\le M} |B(t,x)| < \infty \text{ and } \sup_{t\in[0,M],\,|x|\le M} |\Sigma(t,x)| < \infty, \quad M \in \mathbb{N}. \tag{B.2}$$

Define $a(t,x) = \Sigma(t,x)C\Sigma(t,x)^\top$ for $t \ge 0$ and $x \in \mathbb{R}^N$.

- Sufficient condition. *Fix $T > 0$. Suppose that there exists an $\bar\epsilon_T > 0$ such that*

$$\bar C_T = 1 \vee \sup_{t\le T,\, x^\top a(t,x)x \le \bar\epsilon_T} \Big(2\langle x, B(t,x)\rangle + \mathrm{Tr}(a(t,x))\Big) < \infty \tag{B.3}$$

and consider the maps

$$\begin{aligned} \bar A_T(r) &= \max\Big\{1, 2 \sup_{t\in[0,T],|x|^2=r} x^\top a(t,x)x\Big\}, \\ \bar D_T(r) &= \max\left\{1, \sup_{t\in[0,T],|x|^2=r} \frac{2\langle x, B(t,x)\rangle + \mathrm{Tr}(a(t,x))}{2(x^\top a(t,x)x \vee \bar\epsilon_T)}\right\}, \end{aligned} \tag{B.4}$$

where $r \ge 0$. Define on $\mathbb{R}_+$ the auxiliary functions

$$\begin{aligned} \bar p_T(r) &= \exp\Big\{-\int_0^r \bar D_T(x)\mathrm{d}x\Big\}, \\ \bar v_T(r) &= \int_0^r \bar p_T(x)\Big(\int_0^x [\bar p_T(y)\bar A_T(y)]^{-1}\mathrm{d}y\Big)\mathrm{d}x. \end{aligned} \tag{B.5}$$

If

$$\lim_{r\to\infty} \bar v_T(r) = \infty, \tag{B.6}$$

then $\mathbb{P}(S > T) = 1$.

- Necessary condition. *Suppose that there exists an $\epsilon > 0$ such that*

$$\inf_{t\ge 0,\, x^\top a(t,x)x \le \epsilon} \Big(2\langle x, B(t,x)\rangle + \mathrm{Tr}(a(t,x)) - 2x^\top a(t,x)x\Big) > 0. \tag{B.7}$$

For every $T > 0$, fix an $\underline\epsilon_T \in (0,\epsilon)$ sufficiently small such that $x^\top a(t,x)x \le \epsilon$ for all $(t,x) \in [0,T]\times\mathbb{R}^N$ with $|x|^2 \le \underline\epsilon_T$, and consider the maps

$$\begin{aligned} \underline A_T(r) &= \min\Big\{1, 2 \inf_{t\in[0,T],|x|^2=(r-\underline\epsilon_T)\vee 0} (x^\top a(t,x)x \vee \underline\epsilon_T)\Big\}, \\ \underline D_T(r) &= \min\left\{1, \inf_{t\in[0,T],|x|^2=(r-\underline\epsilon_T)\vee 0} \frac{2\langle x, B(t,x)\rangle + \mathrm{Tr}(a(t,x))}{2(x^\top a(t,x)x \vee \underline\epsilon_T)}\right\}, \end{aligned} \tag{B.8}$$

where $r \ge 0$. Define for $r \ge 0$ the auxiliary functions

$$\begin{aligned} \underline p_T(r) &= \exp\Big\{-\int_0^r \underline D_T(x)\mathrm{d}x\Big\}, \\ \underline v_T(r) &= \int_0^r \underline p_T(x)\Big(\int_0^x [\underline p_T(y)\underline A_T(y)]^{-1}\mathrm{d}y\Big)\mathrm{d}x. \end{aligned} \tag{B.9}$$

If $\mathbb{P}(S = \infty) = 1$, then $\sup_{T>0}\{\lim_{r\to\infty} \underline v_T(r)\} = \infty$.

Proof. We first prove the sufficient part. Fix $T > 0$, an arbitrary $n \in \mathbb{N}$ and consider the process $R = (R_t)_{t\ge 0}$ defined by $R_t = |X_t|^2$. Recalling (B.1), an application of Itô's formula yields

$$\begin{aligned} \mathrm{d}R_t &= 2\langle X_t, B(t,X_t)\,\mathrm{d}t + \Sigma(t,X_t)\,\mathrm{d}\bar W_t\rangle + \mathrm{Tr}(\Sigma(t,X_t)C\Sigma(t,X_t)^\top)\mathrm{d}t \\ &= 2\langle X_t, B(t,X_t)\,\mathrm{d}t + \Sigma(t,X_t)\,\mathrm{d}\bar W_t\rangle + \mathrm{Tr}(a(t,X_t))\mathrm{d}t, \quad t \in [0, \tau_n \wedge T],\ \mathbb{P}-\text{a.s.} \end{aligned}$$

By the generalized Itô's formula in [27, Problem 7.3, Chapter 3] and the stochastic product rule, using the constant $\bar{C}_T$ defined in (B.3) and the map $\bar{v}_T$ in (B.5), the dynamics of the process $\bar{Z} = (\bar{Z}_t)_{t\geq 0}$ given by $\bar{Z}_t = e^{-(\bar{C}_T+2\bar{\epsilon}_T)t}(1+\bar{v}_T(R_t))$ read

$$\begin{aligned}
\mathrm{d}\bar{Z}_t &= -(\bar{C}_T+2\bar{\epsilon}_T)\bar{Z}_t\mathrm{d}t + 2e^{-(\bar{C}_T+2\bar{\epsilon}_T)t}[\bar{A}_T(R_t)]^{-1}X_t^\top a(t,X_t)X_t\mathrm{d}t \\
&\quad + 2\bar{v}_T'(R_t)e^{-(\bar{C}_T+2\bar{\epsilon}_T)t}\langle X_t, \Sigma(t,X_t)\mathrm{d}\bar{W}_t\rangle \\
&\quad + e^{-(\bar{C}_T+2\bar{\epsilon}_T)t}\Big(2\langle X_t, B(t,X_t)\rangle + \mathrm{Tr}(a(t,X_t)) - 2\bar{D}_T(R_t)X_t^\top a(t,X_t)X_t\Big)\bar{v}_T'(R_t)\mathrm{d}t \\
&=: (\bar{\mathbf{I}}_t + \overline{\mathbf{II}}_t)\mathrm{d}t + \mathrm{d}\,\overline{\mathbf{III}}_t + \overline{\mathbf{IV}}_t\mathrm{d}t,
\end{aligned} \tag{B.10}$$

which holds for every $t \in [0, \tau_n \wedge T]$, $\mathbb{P}-$a.s. Observe that the process $t \mapsto 1_{\{t\leq\tau_n\wedge T\}}\overline{\mathbf{III}}_t$ is a martingale, as by (B.2), $1_{\{t\leq\tau_n\wedge T\}}\bar{v}_T'(R_t)e^{-(\bar{C}_T+2\bar{\epsilon}_T)t}|X_t||\Sigma(t,X_t)|$ is bounded on $\Omega\times\mathbb{R}_+$.

We now consider the case $X_t^\top a(t,X_t)X_t \leq \bar{\epsilon}_T$; since $\bar{A}_T(r), \bar{D}_T(r) \geq 1$ (see (B.4)), from (B.5) we estimate

$$\bar{v}_T'(r) \leq \bar{p}_T(r)\int_0^r \frac{1}{\bar{D}_T(y)}\frac{\bar{D}_T(y)}{\bar{p}_T(y)}\mathrm{d}y \leq 1 - e^{-\int_0^r \bar{D}_T(x)\mathrm{d}x} \leq 1, \quad r \geq 0. \tag{B.11}$$

Hence, also noticing that $e^{(\bar{C}_T+2\bar{\epsilon}_T)t}\bar{Z}_t \geq 1$,

$$\begin{aligned}
&1_{\{X_t^\top a(t,X_t)X_t\leq\bar{\epsilon}_T\}}(\bar{\mathbf{I}}_t + \overline{\mathbf{II}}_t + \overline{\mathbf{IV}}_t) \\
&\quad \leq 1_{\{X_t^\top a(t,X_t)X_t\leq\bar{\epsilon}_T\}}e^{-(\bar{C}_T+2\bar{\epsilon}_T)t}\Big[-(\bar{C}_T+2\bar{\epsilon}_T) + 2\bar{\epsilon}_T + \bar{C}_T\bar{v}_T'(R_t)\Big] \leq 0,
\end{aligned}$$

which holds on $[0, \tau_n \wedge T]$, $\mathbb{P}-$a.s. As for the case $X_t^\top a(t,X_t)X_t > \bar{\epsilon}_T$, note that

$$t \mapsto 1_{\{t\leq\tau_n\wedge T\}}1_{\{X_t^\top a(t,X_t)X_t>\bar{\epsilon}_T\}}(\bar{\mathbf{I}}_t + \overline{\mathbf{II}}_t)$$

is nonpositive, because $\bar{C}_T \geq 1$ and, by (B.4), $2X_t^\top a(t,X_t)X_t \leq \bar{A}_T(R_t)$ for $t \leq T$. Regarding $\overline{\mathbf{IV}}_t$, by the definition in (B.5) $\bar{v}_T'(r) \geq 0$ and, by (B.4),

$$1_{\{X_t^\top a(t,X_t)X_t>\bar{\epsilon}_T\}}\bar{D}_T(R_t) \geq 1_{\{X_t^\top a(t,X_t)X_t>\bar{\epsilon}_T\}}\frac{2\langle X_t, B(t,X_t)\rangle + \mathrm{Tr}(a(t,X_t))}{2X_t^\top a(t,X_t)X_t}, \quad t \in [0,T].$$

We then infer that

$$1_{\{X_t^\top a(t,X_t)X_t>\bar{\epsilon}_T\}}\overline{\mathbf{IV}}_t \leq 0, \quad t \in [0, \tau_n \wedge T],\ \mathbb{P}-\text{a.s.}$$

Therefore the stopped process $\bar{Z}^{\tau_n\wedge T}$ is a supermartingale and from (B.10) we deduce that

$$\mathbb{E}\Big[e^{-(\bar{C}_T+2\bar{\epsilon}_T)(t\wedge\tau_n\wedge T)}\bar{v}_T(|X_{t\wedge\tau_n\wedge T}|^2)\Big] \leq 1 + \bar{v}_T(|x_0|^2), \quad t \geq 0;$$

taking the limit as $t \to \infty$, by the continuity of the process X and the map $\bar{v}_T$ on $\mathbb{R}_+$, the dominated convergence theorem yields

$$\mathbb{E}\Big[1_{\{\tau_n\leq T\}}e^{-(\bar{C}_T+2\bar{\epsilon}_T)\tau_n}\Big] \leq \frac{1+\bar{v}_T(|x_0|^2)}{\bar{v}_T(n^2)}.$$

The limit as $n \to \infty$ gives, recalling the hypothesis (B.6),

$$\mathbb{E}\Big[1_{\{S\leq T\}}e^{-(\bar{C}_T+2\bar{\epsilon}_T)S}\Big] = 0,$$

whence $\mathbb{P}(S \leq T) = 0$. This proves the sufficient condition in the statement.

As regards the necessary condition, we focus on its contrapositive statement, letting

$$\underline{s} \coloneqq \sup_{T>0}\underline{v}_T(\infty) = \sup_{T>0}\big\{\lim_{r\to\infty}\underline{v}_T(r)\big\} < \infty. \tag{B.12}$$

First of all, we notice that, for every $T > 0$, there exists an $\underline{\epsilon}_T \in (0, \epsilon)$ such that $x^\top a(t,x)x \le \epsilon$ for every $(t,x) \in [0,T] \times \mathbb{R}^N$ with $|x|^2 \le \underline{\epsilon}_T$, as $\Sigma(t,x)$ is locally bounded in t and x, see (B.2). By (B.7), we then define the constant

$$\underline{C} \coloneqq \min\left\{1, \inf_{\substack{t \le T \\ x^\top a(t,x)x \wedge |x|^2 \le \underline{\epsilon}_T}} \Big(2\langle x, B(t,x)\rangle + \mathrm{Tr}(a(t,x)) - 2x^\top a(t,x)x\Big)\right\} > 0. \tag{B.13}$$

By (B.8), $\underline{A}_T(r), \underline{D}^+_T(r) \le 1$, where we denote by $(\cdot)^+$ [resp., $(\cdot)^-$] the positive part [resp., negative part] of a real number. Thus, similarly to (B.11), for every $r \ge \underline{\epsilon}_T$

$$\begin{aligned}
\underline{v}'_T(r) &\ge \underline{p}_T(r) \int_0^r \frac{1}{\underline{D}^+_T(x)} \frac{\underline{D}^+_T(x)}{\underline{p}_T(x)} \mathrm{d}x \\
&\ge e^{-\int_0^r \underline{D}^+_T(x)\mathrm{d}x} \int_0^r \underline{D}^+_T(x) e^{\int_0^x \underline{D}^+_T(s)\mathrm{d}s} e^{\int_x^r \underline{D}^-_T(s)\mathrm{d}s} \mathrm{d}x \\
&\ge 1 - e^{-\int_0^r \underline{D}^+_T(x)\mathrm{d}x} \ge 1 - e^{-\int_0^{\underline{\epsilon}_T} \underline{D}^+_T(x)\mathrm{d}x} =: \underline{l} > 0.
\end{aligned} \tag{B.14}$$

Here, to claim that $\underline{l} > 0$, we use the fact that $\underline{D}^+_T = \underline{D}_T > 0$ on $[0, \underline{\epsilon}_T]$ by (B.8)-(B.13). By reasoning in a similar manner as above, we work with the process $\underline{Z} = (\underline{Z}_t)_{t \ge 0}$ defined by $\underline{Z}_t = \exp\Big\{-\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}t\Big\}\big(1 + \underline{v}_T(R_t + \underline{\epsilon}_T)\big)$, whose dynamics read

$$\begin{aligned}
\mathrm{d}\underline{Z}_t &= -\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}\,\underline{Z}_t \mathrm{d}t + 2e^{-\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}t}[\underline{A}_T(R_t + \underline{\epsilon}_T)]^{-1} X_t^\top a(t, X_t) X_t \,\mathrm{d}t \\
&\quad + 2\underline{v}'_T(R_t + \underline{\epsilon}_T) e^{-\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}t} \langle X_t, \Sigma(t, X_t)\mathrm{d}\bar{W}_t\rangle \\
&\quad + e^{-\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}t}\Big(2\langle X_t, B(t,X_t)\rangle + \mathrm{Tr}(a(t,X_t)) - 2\underline{D}_T(R_t + \underline{\epsilon}_T) X_t^\top a(t,X_t) X_t\Big) \\
&\quad \times \underline{v}'_T(R_t + \underline{\epsilon}_T)\mathrm{d}t \\
&=: (\underline{\mathbf{I}}_t + \underline{\mathbf{II}}_t)\mathrm{d}t + \mathrm{d}\,\underline{\mathbf{III}}_t + \underline{\mathbf{IV}}_t \mathrm{d}t, \quad t \in [0, \tau_n \wedge T],\ \mathbb{P}-\text{a.s.}
\end{aligned} \tag{B.15}$$

As before, the process $t \mapsto 1_{\{t \le \tau_n \wedge T\}} \underline{\mathbf{III}}_t$ is a martingale. When $X_t^\top a(t, X_t) X_t \le \underline{\epsilon}_T$ or $R_t \le \underline{\epsilon}_T$, considering that $\exp\Big\{\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}t\Big\}\underline{Z}_t \le 1 + \underline{v}_T(\infty)$ and $\underline{C}, \underline{l} > 0$, by (B.13)-(B.14)

$$\begin{aligned}
1_{\{X_t^\top a(t,X_t)X_t \wedge R_t \le \underline{\epsilon}_T\}}(\underline{\mathbf{I}}_t + \underline{\mathbf{II}}_t + \underline{\mathbf{IV}}_t) &\ge 1_{\{X_t^\top a(t,X_t)X_t \wedge R_t \le \underline{\epsilon}_T\}}(\underline{\mathbf{I}}_t + \underline{\mathbf{IV}}_t) \\
&\ge 1_{\{X_t^\top a(t,X_t)X_t \wedge R_t \le \underline{\epsilon}_T\}} e^{-\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}t}\Big[-\underline{l}\,\underline{C} + \underline{C}\bar{v}'_T(R_t + \underline{\epsilon})\Big] \ge 0,
\end{aligned}$$

which holds for $t \in [0, \tau_n \wedge T]$, $\mathbb{P}-$a.s., where in the second inequality we also use that $\underline{D}_T \le 1$.

On the other hand, when $X_t^\top a(t, X_t) X_t > \underline{\epsilon}_T$ and $R_t > \underline{\epsilon}_T$, the process

$$t \mapsto 1_{\{t \le \tau_n \wedge T\}} 1_{\{X_t^\top a(t,X_t)X_t \wedge R_t > \underline{\epsilon}_T\}}(\underline{\mathbf{I}}_t + \underline{\mathbf{II}}_t) \text{ is nonnegative on } \mathbb{R}_+,$$

because $\underline{C} \le 1$, $\underline{l} \le 1$ and, by (B.8), $2X_t^\top a(t,X_t)X_t \ge \underline{A}(R_t + \underline{\epsilon}_T)$ for $t \le T$. Moreover, again by (B.8)

$$1_{\{X_t^\top a(t,X_t)X_t \wedge R_t > \underline{\epsilon}_T\}} \underline{D}(R_t + \underline{\epsilon}_T) \le 1_{\{X_t^\top a(t,X_t)X_t \wedge R_t > \underline{\epsilon}_T\}} \frac{2\langle X_t, B(t,X_t)\rangle + \mathrm{Tr}(a(t,X_t))}{2X_t^\top a(t,X_t)X_t},$$

hence $t \mapsto 1_{\{X_t^\top a(t,X_t)X_t \wedge R_t > \underline{\epsilon}_T\}} \underline{\mathbf{IV}}_t$ is nonnegative on $[0, \tau_n \wedge T]$, as well. It follows that the process $\underline{Z}^{\tau_n}$ is a submartingale. Taking expectations in (B.15) then yields

$$1 + \underline{v}_T(|x_0|^2 + \underline{\epsilon}_T) \le \mathbb{E}\Big[e^{-\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}(t \wedge \tau_n \wedge T)}\big(1 + \underline{v}_T(|X_{t\wedge\tau_n\wedge T}|^2 + \underline{\epsilon}_T)\big)\Big], \quad t \ge 0;$$

sending $t \to \infty$, by the continuity of the process X and the map $\underline{v}_T$ on $\mathbb{R}_+$, we apply the dominated convergence theorem to infer that

$$\begin{aligned}\big(1+\underline{v}_T(n^2+\underline{\epsilon}_T)\big)\mathbb{E}\Big[1_{\{\tau_n\le T\}}e^{-\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}\tau_n}\Big]\\ \ge 1-e^{-\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}T}\mathbb{E}\Big[1_{\{\tau_n>T\}}\big(1+\underline{v}_T(|X_T|^2+\underline{\epsilon}_T)\big)\Big],\end{aligned}$$

whence

$$\mathbb{E}\Big[1_{\{\tau_n\le T\}}e^{-\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}\tau_n}\Big] \ge \frac{1}{1+\underline{v}_T(\infty)} - e^{-\frac{\underline{l}}{1+\underline{v}_T(\infty)}\underline{C}T}.$$

Considering that $\underline{C}$, defined in (B.13), depends on T, i.e., $\underline{C} = \underline{C}_T$, with $\underline{I} \coloneqq \inf_{T>0}\underline{C}_T > 0$ by (B.7), recalling (B.12) it follows that

$$\mathbb{E}\Big[1_{\{S\le T\}}e^{-\frac{\underline{l}}{1+\underline{s}}\underline{I}S}\Big] = \lim_{n\to\infty}\mathbb{E}\Big[1_{\{\tau_n\le T\}}e^{-\frac{\underline{l}}{1+\underline{s}}\underline{I}\tau_n}\Big] \ge \frac{1}{1+\underline{s}} - e^{-\frac{\underline{l}}{1+\underline{s}}\underline{I}T}.$$

The limit as $T\to\infty$ gives

$$\mathbb{E}\Big[1_{\{S<\infty\}}e^{-\frac{\underline{l}}{1+\underline{s}}\underline{I}S}\Big] \ge \frac{1}{1+\underline{s}} > 0,$$

whence $\mathbb{P}(S<\infty)>0$. This demonstrates that the condition $\sup_{T>0}\underline{v}_T(\infty)=\infty$ is necessary for $\mathbb{P}(S=\infty)=1$. The proof is now complete. □

We conclude this appendix with an example showing how the results of Theorem B.1 can be applied to determine necessary and sufficient conditions for the non-explosion of a SVE with a sum-of-exponentials kernel.

Example B.2. Consider the path-independent system with common noise (1.6); recall that this $N-$dimensional SDE enables us to construct (local) $\mathbb{R}-$valued solutions of one-dimensional SVEs driven by $K_N = \sum_{n=1}^N m_n e^{-\Lambda_n t}$. For simplicity of exposition, we suppose that the initial condition of the SVE is $x_0 = 0$ and the exponential weights $m_n = 1$, $n = 1,\dots,N$. We assume that b is locally bounded and $\sigma^2(\cdot) = c^{-2} + f^2(\cdot)$, for some constant $c>0$ and locally bounded function f. Denote by $\mathbf{1}$ the vector $\mathbf{1} = [1,\dots,1]^\top \in \mathbb{R}^N$. The covariance matrix $C\in\mathbb{R}^{N\times N}$ of the noise in (1.6) is $C = \mathbf{1}\,\mathbf{1}^\top$, i.e., C has all entries equal to 1, since we have a degenerate system driven only by a one-dimensional Brownian motion W. By the expression of $\Sigma(\cdot;x_0)$ in (1.7) and using the notation in Theorem B.1, given $\bar\epsilon>0$

$$y^\top a(y)y = \sigma^2(\langle\mathbf{1},y\rangle)(\langle\mathbf{1},y\rangle)^2 \le \bar\epsilon \quad\Longrightarrow\quad |\langle\mathbf{1},y\rangle| \le c\sqrt{\bar\epsilon} \quad\text{and}\quad f^2(\langle\mathbf{1},y\rangle)(\langle\mathbf{1},y\rangle)^2 \le \bar\epsilon.$$

Therefore the hypothesis in (B.3) is satisfied for every $\bar\epsilon>0$. Indeed, recalling $B(\cdot;x_0)$ defined in (1.7), for every $y\in\mathbb{R}^N$ with $y^\top a(y)y \le \bar\epsilon$, the following inequality holds, also using that $\Lambda_n>0$, $n=1,\dots,N$:

$$\begin{aligned}2\langle y, B(y;x_0)\rangle + \mathrm{Tr}(a(y)) &= -2\sum_{n=1}^N \Lambda_n y_n^2 + 2\langle\mathbf{1},y\rangle b(\langle\mathbf{1},y\rangle) + N\sigma^2(\langle\mathbf{1},y\rangle)\\ &\le 2c\sqrt{\bar\epsilon}\sup_{|x|\le c\sqrt{\bar\epsilon}}|b(x)| + N\sup_{|x|\le c\sqrt{\bar\epsilon}}|\sigma^2(x)| < \infty.\end{aligned}$$

Consequently, Theorem B.1 provides a sufficient condition for the non-explosion of a solution to (1.6) (that is, for $\mathbb{P}(S=\infty)=1$), and hence, by extension, for the non-explosion of a solution to the corresponding SVE (2.2) driven by K_N.

As regards a necessary condition for $\mathbb{P}(S=\infty)=1$, we notice that the requirement in (B.7) is not satisfied for the coefficients in (1.6). For this reason, we focus on the process $\tilde Y = [\tilde Y_1,\dots,\tilde Y_N]^\top$ constructed from Y as follows:

$$\tilde Y_{n,t} = e^{\Lambda_n t}Y_{n,t},\quad n=1,\dots,N.$$

By the stochastic product rule and (1.6), $\tilde{Y}$ is a local solution of the $N-$dimensional system of SDEs with time-dependent coefficients

$$\mathrm{d}\tilde{Y}_{n,t} = e^{\Lambda_n t} b\Big(\sum_{i=1}^{N} e^{-\Lambda_i t}\tilde{Y}_{i,t}\Big)\mathrm{d}t + e^{\Lambda_n t}\sigma\Big(\sum_{i=1}^{N} e^{-\Lambda_i t}\tilde{Y}_{i,t}\Big)\mathrm{d}W_t, \quad \tilde{Y}_{n,0} = 0. \tag{B.16}$$

Thus, denoting by $\tilde{S}$ the explosion time of $\tilde{Y}$, i.e., $\tilde{S} = \inf\{t \geq 0 : |[\tilde{Y}_{1,t}, \dots, \tilde{Y}_{N,t}]^\top| = \infty\}$, we want to establish a necessary condition for $\mathbb{P}(\tilde{S} = \infty) = 1$, as this is equivalent to $\mathbb{P}(S = \infty) = 1$. To do this, we further assume that there exists an $R > 0$ such that $|b(x)| \lesssim |f(x)|$ when $|x| > R$. Defining, for every $t \geq 0$ and $y = [y_1, \dots, y_N]^\top \in \mathbb{R}^N$, the locally bounded functions

$$\begin{aligned}
\tilde{B}(t,y) &= b\Big(\sum_{n=1}^{N} e^{-\Lambda_n t} y_n\Big)[e^{\Lambda_1 t}, \dots, e^{\Lambda_N t}]^\top \in \mathbb{R}^N, \\
\tilde{\Sigma}(t,y) &= \sigma\Big(\sum_{n=1}^{N} e^{-\Lambda_n t} y_n\Big)\mathrm{diag}[e^{\Lambda_1 t}, \dots, e^{\Lambda_N t}] \in \mathbb{R}^{N\times N},
\end{aligned}$$

we then compute, for every $\epsilon > 0$ and $(t,y) \in \mathbb{R}_+ \times \mathbb{R}^N$ such that $y^\top \tilde{a}(t,y) y \leq \epsilon$, where $\tilde{a}(t,y) = \tilde{\Sigma}(t,y) C \tilde{\Sigma}(t,y)^\top$ according to the notation of Theorem B.1,

$$\begin{aligned}
&2\langle y, \tilde{B}(t,y)\rangle + \mathrm{Tr}(\tilde{a}(t,y)) - 2y^\top \tilde{a}(t,y) y \\
&\quad \geq 2\Big(\sum_{n=1}^{N} e^{\Lambda_n t} y_n\Big) b\Big(\sum_{n=1}^{N} e^{-\Lambda_n t} y_n\Big) + \Big(\sum_{n=1}^{N} e^{2\Lambda_n t}\Big)\sigma^2\Big(\sum_{n=1}^{N} e^{-\Lambda_n t} y_n\Big) - 2\epsilon \\
&\quad \geq c^{-2}\Big(\sum_{n=1}^{N} e^{2\Lambda_n t}\Big) - 2\epsilon + 2\Big(\sum_{n=1}^{N} e^{\Lambda_n t} y_n\Big) b\Big(\sum_{n=1}^{N} e^{-\Lambda_n t} y_n\Big) \\
&\quad \geq c^{-2} - 2\epsilon + 2\Big(\sum_{n=1}^{N} e^{\Lambda_n t} y_n\Big) b\Big(\sum_{n=1}^{N} e^{-\Lambda_n t} y_n\Big).
\end{aligned} \tag{B.17}$$

Note that in the last inequality in (B.17) we use that $\Lambda_n > 0$, $n = 1, \dots, N$. Since, recalling also the definition of σ^2,

$$y^\top \tilde{a}(t,y) y \leq \epsilon \quad \Longleftrightarrow \quad \Big(\sum_{n=1}^{N} e^{\Lambda_n t} y_n\Big)^2 \Big[\frac{1}{c^2} + f^2\Big(\sum_{n=1}^{N} e^{-\Lambda_n t} y_n\Big)\Big] \leq \epsilon,$$

(B.17) and the fact that, by hypothesis, $|b(x)|$ is bounded above by $|f(x)|$ for x sufficiently large guarantee that we can choose an $\epsilon > 0$ so small that (B.7) holds. Hence Theorem B.1 provides the desired necessary condition for $\mathbb{P}(\tilde{S} = \infty) = 1$.

Remark B.3. Despite the fact that we can use Theorem B.1 to determine necessary and sufficient conditions for the non-explosion of SVEs with a sum-of-exponentials kernel (as discussed in Example B.2), the coefficients involved in this Khasminskii-type test, defined in (B.4) and (B.8), are evidently difficult to compute in practice. As a result, Theorem B.1 does not appear to be a viable option for establishing usable boundary attainment criteria for SVEs. By contrast, the Feller-type test in Theorems 2.12-2.13 is considerably more practical, as it relies on a shifted drift (see (2.5)) that can be obtained by evaluating the convolution $(K' * L)$ at 0 and computing its limit at ∞ (for the latter, see also Remark 2.7). This shows that our approach, despite being more general, provides significant contributions to the literature even in this particular sum-of-exponentials case.

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