Transportation Research Part B 182 (2024) 102922

Contents lists available at ScienceDirect

Transportation Research Part B

journal homepage: www.elsevier.com/locate/trb

Phase transitions of the price-of-anarchy function in multi-commodity routing games

Roberto Cominetti [a], Valerio Dose [b,*], Marco Scarsini [c]

[a] Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Diagonal las Torres 2640, Peñalolén, 7910000, Región Metropolitana, Chile
[b] Dipartimento di Ingegneria Informatica, Automatica e Gestionale, "Sapienza" Università di Roma, Via Ariosto 25, Roma, 00185, Italy
[c] Dipartimento di Economia e Finanza, Luiss University, Viale Romania 32, Roma, 00197, Italy

ARTICLE INFO

MSC:
91A14
91A07
91A43

Keywords:
Wardrop equilibrium
Network flows
Traffic demand

ABSTRACT

We consider the behavior of the price of anarchy and equilibrium flows in nonatomic multi-commodity routing games as a function of the traffic demand. We analyze their smoothness with a special attention to specific values of the demand at which the support of the Wardrop equilibrium exhibits a phase transition with an abrupt change in the set of optimal routes. Typically, when such a phase transition occurs, the price of anarchy function has a breakpoint, i.e., is not differentiable. We prove that, if the demand varies proportionally across all commodities, then, at a breakpoint, the largest left or right derivatives of the price of anarchy and of the social cost at equilibrium, are associated with the smaller equilibrium support. This proves – under the assumption of proportional demand – a conjecture of O'Hare et al. (2016), who observed this behavior in simulations. We also provide counterexamples showing that this monotonicity of the one-sided derivatives may fail when the demand does not vary proportionally, even if it moves along a straight line not passing through the origin.

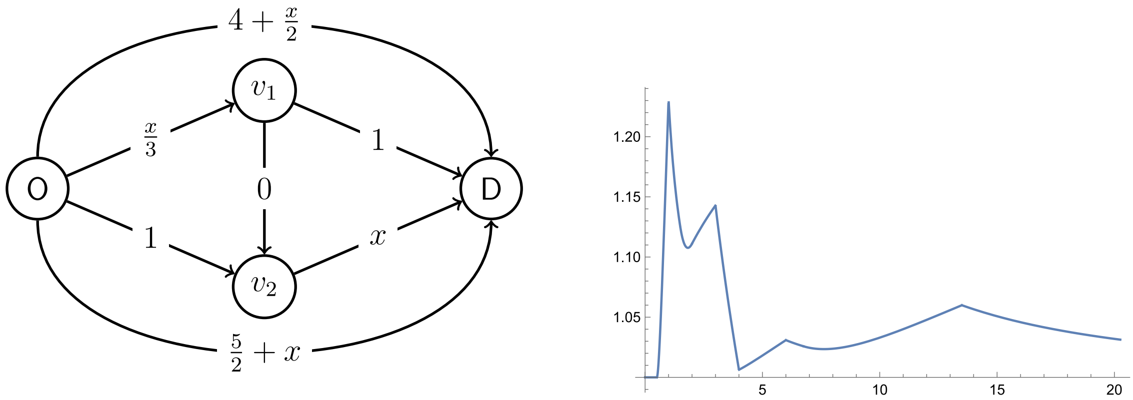
1. Introduction

Wardrop equilibria for nonatomic routing games provide a mathematical description of how traffic distributes over a network used by a large number of agents who do not coordinate or cooperate. Standard examples are road and telecommunication networks. This model is a special case of a nonatomic congestion game where the resources are the edges of a directed multigraph, and each edge e has an associated cost or delay $c_e(x_e)$ given by a non-decreasing and continuous function of the traffic load x_e on the edge. Different types of users have different origin–destination (OD) pairs, typically called commodities. A nonnegative traffic demand is associated to each OD pair, and users can choose different paths to go from their origin to their destination. In doing so, they generate a flow over the network. A Wardrop equilibrium is a distribution of traffic which satisfies the demand on each OD pair in a such a way that all the users of any given type only use paths of minimal cost.

Traffic demand is often difficult to predict as it is eminently a dynamic phenomenon, with variations between days of the week and across different periods within any given day. Moreover, traffic demand increases on a longer time scale as a result of the growth of the population and car-ownership. As a consequence, it is relevant to analyze the sensitivity of Wardrop equilibria under different demand scenarios to understand the extent to which this equilibrium notion approximates real traffic patterns. Using this analysis, a planner can anticipate the evolution of equilibria and better design policy interventions and infrastructure modifications that the increasing traffic level requires. These observations have motivated several authors to investigate how the Wardrop equilibrium is

* Corresponding author.
E-mail address: valerio.dose@uniroma1.it (V. Dose).

https://doi.org/10.1016/j.trb.2024.102922
Received 3 May 2023; Received in revised form 5 February 2024; Accepted 8 March 2024
Available online 15 March 2024
0191-2615/



(a) Some network (b) PoA for the network on the left

Fig. 1. A single-commodity example of the behavior of the PoA as a function of the demand. This function is nonsmooth at demands 1, 3, 4, 6 and 27/2. Between these breakpoints the set of optimum paths remains stable and the PoA behaves smoothly. At demand 4 the set of optimum paths exhibits a contraction and the derivative jumps up. At all the other breakpoints the set of optimum paths undergoes an expansion and the derivative jumps down.

affected by variations in the traffic demands, in terms of its continuity, smoothness, and monotonicity properties. In Section 1.2 we present a brief overview of previous works that have addressed these questions.

On the other hand, Wardrop equilibria are known to be inefficient in the sense that there may exist other traffic distributions that produce a lower total delay. Inefficiency of equilibria is usually measured via the *price of anarchy* (PoA), i.e., the ratio between the total equilibrium cost and the optimum total cost. This provides a benchmark for the maximal improvement that could be achieved in terms of social cost. While the initial studies on the PoA focused on finding tight bounds under worst case scenarios, recent work has turned to analyze the behavior of the PoA as a function of the traffic demand.

Fig. 1 shows a typical profile of the PoA as the demand varies: it is close to 1 when the demand is either very small or very large, and it oscillates in the middle range with occasional sharp kinks and smooth critical points at local minima. As reported by Youn et al. (2008), this type of profile is also observed in real networks such as Boston, London, and New York, where in addition the PoA values remain far from the worst case estimates. The fact that the PoA is close to 1 for both low and high demands was formally established under fairly general assumptions in Colini-Baldeschi et al. (2019, 2020) and Wu et al. (2021). For the intermediate range, O'Hare et al. (2016) observed that the kinks are associated with a sudden change in the set of optimum paths at equilibrium, and stated a conjecture that we analyze in this paper. A partial support for this was given by Cominetti et al. (2024b): in single-commodity routing games with affine cost functions, and in demand intervals where the set of optimum paths at equilibrium does not change, the PoA is differentiable and exhibits at most one critical point, which must be a local minimum. In this paper we further explore these questions for multi-commodity networks and more general cost functions.

1.1. Our results

We consider nonatomic routing games and analyze the behavior of the equilibrium flows, the social cost, and the PoA, as the traffic demands vary on potentially multiple OD pairs. The whole paper deals with *constrained* routing games in which every OD pair is allowed to use a restricted set of paths and not necessarily all the paths connecting its origin and destination. The main takeaway of our study is that when the demands vary with constant proportions across all OD pairs, the behavior of the PoA is analog to what is observed in the single-commodity case. However, this is not what happens if the demand varies without keeping constant proportions among the different OD pairs.

Specifically, we begin by generalizing to multiple OD pairs some results on the continuity and differentiability of equilibria as a function of the demands. We show that if the cost functions are C^1 with strictly positive derivative, and the demand varies along a smooth curve parameterized by a single variable t, then the equilibrium loads on each edge and the PoA are also C^1 as functions of t, as long as the demand curve stays in a region where the set of minimum-cost paths does not change.

Typically – although not always – this smoothness fails at points where the set of optimum paths changes and the equilibrium exhibits a phase transition. We call such points *breakpoints*. For networks with C^1 cost functions having strictly positive derivatives and proportionally varying demands, we prove a conjecture stated in O'Hare et al. (2016), showing that at a breakpoint the largest between the left and right derivatives of the PoA corresponds to the smaller set of optimum paths (see the example in Fig. 1). We also show with an example that the full conjecture, as originally stated in O'Hare et al. (2016), might fail when the demands do not vary proportionally, even if these demands move along an affine straight line in the space of demands.

1.2. Related work

The definition of equilibrium in nonatomic routing games that we adopt in this paper is due to Wardrop (1952). The characterization of Wardrop equilibria as solutions of a convex optimization problem was first described in Beckmann et al. (1956); and the first attempts to explicitly compute such equilibria were due to Tomlin (1966) for affine costs, and to Dafermos and Sparrow (1969) for general convex costs. For surveys on the topic the reader is referred to Florian and Hearn (1995) and Correa and Stier-Moses (2011).

Several authors have considered how the equilibrium flows and costs vary with the traffic demand. Hall (1978) proved that the equilibrium cost of any given OD pair is an increasing function of the amount of traffic on that OD pair. On the other hand Fisk (1979) presented an example showing that the total social cost can decrease even if the total demand increases. Further analysis of this paradoxical phenomenon can be found in Dafermos and Nagurney (1984). Concerning the smoothness of equilibrium flows, Patriksson (2004) gave necessary and sufficient conditions for the directional differentiability of equilibria, and Josefsson and Patriksson (2007) proved the directional differentiability of equilibrium costs, but observed that this might fail for the equilibrium edge loads. Concerning the set of optimum paths, Englert et al. (2010) showed examples of single OD routing games where an arbitrarily small increase in traffic demand generates a complete change of the set of paths used at equilibrium, although the change on the edge loads remains small. Moreover, for polynomial costs of degree at most d, if the demand increases by ε then the equilibrium costs increase at most by a multiplicative factor $(1+\varepsilon)^d$. The extension of this latter result to multiple OD pairs can be found in Takalloo and Kwon (2020). More recently, an analysis of instances where there exists a Wardrop equilibrium with loads that are nondecreasing for each edge was carried out in Cominetti et al. (2024a), whereas a study on conditions that allow Braess' paradox to happen at specific demands was performed by Verbree and Cherukuri (2023). In this last work the authors studied the slope of the equilibrium costs, in the setting of networks with affine cost functions; these results are similar to some theorems proved here.

Pigou (1920) was probably the first author who studied the inefficiency of equilibria in nonatomic routing games. The formal definition of PoA to measure this inefficiency is due to Koutsoupias and Papadimitriou (1999), and acquired its name in Papadimitriou (2001). Most of the early literature on the PoA concentrated on establishing sharp bounds for the PoA for different classes of games, such as congestion games and routing games. In a landmark paper, Roughgarden and Tardos (2002) proved that in every nonatomic congestion game with affine costs the PoA is bounded above by $4/3$ and showed that this bound is sharp. This result was generalized to polynomial cost functions of maximum degree d in Roughgarden (2003), showing that the PoA grows as $\Theta(d/\log d)$. Other results of this type can be found in Dumrauf and Gairing (2006), who focused on cost functions that are sums of monomials whose degrees are in a specified range, and in Roughgarden and Tardos (2004), who dealt with the class of differentiable cost functions $c(x)$ such that $x\,c(x)$ is convex. Less regular cost functions and different notions of social cost were studied in Correa et al. (2004, 2007, 2008).

Several papers addressed the computation of the PoA in real networks. Monnot et al. (2017) analyzed data from a large sample of Singaporean students who commute to go to school, observing a PoA which is much smaller than the theoretical worst case bounds. Also, Youn et al. (2008, 2009) studied the PoA in the networks of Boston, London, and New York when all the OD traffic demands are scaled by the same factor, observing again values of the PoA consistently smaller than the worst case bounds. The behavior of the PoA in these three cities shows a common pattern: it is close to 1 both for small and large demands, and it oscillates in the middle range with sharp kinks at the local maxima and smooth critical points at the local minima. O'Hare et al. (2016) noted that these kinks arise when the set of optimum paths at equilibrium undergoes an expansion or a contraction, and stated the conjecture that we analyze later in this paper.

Recent efforts attempted to mathematically explain the empirical behavior of the PoA observed in the studies mentioned above. An algorithm for computing Wardrop equilibria as a function of the traffic demand was given in Klimm and Warode (2019, 2022), in the case of piecewise linear cost functions. The efficiency of equilibria when the demand is close to zero or infinity (light and heavy traffic) was analyzed in Colini-Baldeschi et al. (2019, 2020) and Wu et al. (2021). Colini-Baldeschi et al. (2019) considered the case of single OD parallel networks showing that in heavy traffic the PoA converges to 1 when the cost functions are regularly varying. These results were extended to general networks in Colini-Baldeschi et al. (2020), considering also the behavior of PoA in the light traffic regime. The case of heavy traffic was treated with different techniques by Wu et al. (2021). The study of the intermediate range of demands, when the traffic is neither light nor heavy, was the main objective of Cominetti et al. (2024b), who proved that for affine cost functions the shape of the PoA function is the one observed in the empirical studies. Finally, we mention Wu and Möhring (2023), who established the continuity of the PoA in non atomic routing games, as a function of various parameters including not only the demands, but also the cost functions.

1.3. Organization of the paper

Section 2 recalls the model to be studied, and fixes the notations used thereafter. Section 3 presents some regularity and smoothness results for the equilibrium loads, the social cost at equilibrium, the optimum social cost, and the price of anarchy. Section 4 discusses the behavior of these quantities around breakpoints, where smoothness is typically lost. Section 5 contains the proof of the main result. Some other technical proofs are presented in Appendix A, whereas Appendix B contains a list of the symbols used in the paper.

2. Network games with variable demand

This section introduces the model and notations used throughout the paper. We consider routing games with multiple OD pairs, and we study their equilibria as functions of the multivariate demand, concentrating on the case where the demand vector varies along a smooth curve in the space of demands. In particular, we study the simplest and natural case where the demands are scaled by a common factor so that they change proportionally across OD pairs.

Let $G := (\mathcal{V}, \mathcal{E})$ be a directed multigraph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, where each edge $e \in \mathcal{E}$ has a continuous and nondecreasing *cost function* $c_e : [0, +\infty) \to [0, +\infty)$, with $c_e(x_e)$ representing the travel time of traversing the edge e when the load on that edge is x_e. An *origin–destination* (OD) h is defined by a triple $(\mathsf{O}^{(h)}, \mathsf{D}^{(h)}, \mathcal{P}^{(h)})$, where $\mathsf{O}^{(h)} \in \mathcal{V}$ is the origin vertex, $\mathsf{D}^{(h)} \in \mathcal{V}$ is the destination vertex, and $\mathcal{P}^{(h)}$ is a subset of simple paths from $\mathsf{O}^{(h)}$ to $\mathsf{D}^{(h)}$. The symbol $\mathcal{H}$ denotes the set of OD's. The sets $\mathcal{P}^{(h)}$ are assumed nonempty but could be smaller than the sets of all paths from origin to destination in the graph G. In order to simplify our notations and formulas, we assume that these sets are pairwise disjoint so that no path is common to two different OD's. In particular, when considering the flow of a path, we do not need to specify to which OD the traffic flow belongs. We observe that our results do not depend on this simplifying assumption (see Remark 4.7). The set of all feasible routes is the disjoint union of the sets $\mathcal{P}^{(h)}$ for $h \in \mathcal{H}$, denoted by

$$\mathcal{P} := \bigcup_{h\in\mathcal{H}} \mathcal{P}^{(h)}. \tag{2.1}$$

Definition 2.1. A *routing game structure* is a tuple $(G, \boldsymbol{c}, \mathcal{H})$, where G is a directed multigraph, $\boldsymbol{c}$ is the vector of nondecreasing and continuous edge costs, and $\mathcal{H}$ is the set of OD pairs.

A *traffic demand* for the routing game structure $(G, \boldsymbol{c}, \mathcal{H})$ is a vector $\boldsymbol{\mu} = (\mu^{(h)})_{h\in\mathcal{H}} \in \mathbb{R}_+^{\mathcal{H}}$. Every such $\boldsymbol{\mu}$ determines an associated set of *feasible flows* $\boldsymbol{f} = \left(f_p\right)_{p\in\mathcal{P}} \in \mathbb{R}_+^{\mathcal{P}}$ satisfying

$$\sum_{p\in\mathcal{P}^{(h)}} f_p = \mu^{(h)} \text{ for all } h \in \mathcal{H}. \tag{2.2}$$

These flows induce in turn a *load profile* $\boldsymbol{x} = (x_e)_{e\in\mathcal{E}}$, where x_e represents the aggregate traffic over the edge $e \in \mathcal{E}$ given by

$$x_e = \sum_{p\in\mathcal{P}} \delta_{ep} f_p \text{ for all } e \in \mathcal{E}, \tag{2.3}$$

with $\delta_{ep} = 1$ if $e \in p$ and $\delta_{ep} = 0$ otherwise. More concisely, (2.2) and (2.3) can be written in matrix form as

$$S\boldsymbol{f} = \boldsymbol{\mu}, \quad \boldsymbol{x} = \Delta \boldsymbol{f}. \tag{2.4}$$

We write $\mathcal{F}_{\boldsymbol{\mu}}$ for the set of pairs $(\boldsymbol{f}, \boldsymbol{x}) \in \mathbb{R}_+^{\mathcal{P}} \times \mathbb{R}_+^{\mathcal{E}}$ satisfying these flow/load conservation equations. We also write $\mathcal{X}_{\boldsymbol{\mu}}$ for the projection of $\mathcal{F}_{\boldsymbol{\mu}}$ onto the space of load variables $\boldsymbol{x}$.

With a slight abuse of notation, the total travel time on a path $p \in \mathcal{P}$ is denoted

$$c_p(\boldsymbol{x}) := \sum_{e\in p} c_e(x_e). \tag{2.5}$$

The total delay induced by $(\boldsymbol{f}, \boldsymbol{x})$ is called the *social cost* and is denoted by

$$\mathsf{SC}(\boldsymbol{f}, \boldsymbol{x}) := \sum_{p\in\mathcal{P}} f_p \cdot c_p(\boldsymbol{x}) = \sum_{e\in\mathcal{E}} x_e \cdot c_e(x_e). \tag{2.6}$$

2.1. Equilibria and price of anarchy

Let $(G, \boldsymbol{c}, \mathcal{H})$ be a routing game structure. For each demand vector $\boldsymbol{\mu} \in \mathbb{R}_+^{\mathcal{H}}$, we obtain a classical nonatomic routing game. We recall that a Wardrop equilibrium is a feasible flow-load pair $(\boldsymbol{f}^*, \boldsymbol{x}^*) \in \mathcal{F}_{\boldsymbol{\mu}}$ for which there exists $\lambda := (\lambda^{(h)})_{h\in\mathcal{H}} \in \mathbb{R}^{\mathcal{H}}$ such that

$$\begin{cases} c_p(\boldsymbol{x}^*) = \lambda^{(h)} & \text{for every } h \in \mathcal{H} \text{ and all } p \in \mathcal{P}^{(h)} \text{ with } f_p^* > 0, \\ c_p(\boldsymbol{x}^*) \geq \lambda^{(h)} & \text{for every } h \in \mathcal{H} \text{ and all } p \in \mathcal{P}^{(h)} \text{ with } f_p^* = 0. \end{cases} \tag{2.7}$$

The quantity $\lambda^{(h)}$ is called the *equilibrium cost* associated to the OD pair $h \in \mathcal{H}$.

From Beckmann et al. (1956) we know that the set of all equilibrium load profiles $\boldsymbol{x}^*$ coincides with the set of optimum solutions of the minimization problem

$$\min_{\boldsymbol{x}\in\mathcal{X}_{\boldsymbol{\mu}}} \Phi(\boldsymbol{x}), \tag{2.8}$$

where $\Phi(\boldsymbol{x}) := \sum_{e\in\mathcal{E}} C_e(x_e)$ with $C_e(x_e) := \int_0^{x_e} c_e(z)\,\mathrm{d}z$. Since the cost functions c_e are continuous and nondecreasing, the objective function $\Phi(\cdot)$ is C^1 and convex. Hence, the minimum is attained and therefore for every $\boldsymbol{\mu} \in \mathbb{R}_+^{\mathcal{H}}$ there exists at least one equilibrium.

An equilibrium $(\boldsymbol{f}^*, \boldsymbol{x}^*) \in \mathcal{F}_{\boldsymbol{\mu}}$ induces equilibrium edge costs $\tau_e := c_e(x_e^*)$. In Fukushima (1984), the equilibrium edge costs were shown to be optimum solutions of the strictly convex dual program

$$\min_{\boldsymbol{\tau}} \sum_{e\in\mathcal{E}} C_e^*(\tau_e) - \sum_{h\in\mathcal{H}} \left(\mu^{(h)} \min_{p\in\mathcal{P}^{(h)}} \sum_{e\in p} \tau_e \right), \tag{2.9}$$

where $C_e^*(\cdot)$ is the Fenchel conjugate of $C_e(\cdot)$, which is strictly convex. Thus, the edge costs τ_e at equilibrium are uniquely defined for each demand vector $\boldsymbol{\mu}$, and are the same in every equilibrium load profile $\boldsymbol{x}^*$. This defines maps $\boldsymbol{\mu} \mapsto \tau_e(\boldsymbol{\mu})$ that assign these equilibrium edge costs to each demand vector $\boldsymbol{\mu}$. It follows that, at equilibrium, the cost $c_p^*(\boldsymbol{\mu}) = \sum_{e \in p} \tau_e(\boldsymbol{\mu})$ of a path $p \in \mathcal{P}$, as well as the equilibrium costs $\lambda^{(h)}(\boldsymbol{\mu}) = \min_{p \in \mathcal{P}^{(h)}} c_p^*(\boldsymbol{\mu})$ for each $h \in \mathcal{H}$, are also uniquely defined for each $\boldsymbol{\mu}$ and do not depend on the specific equilibrium $(\boldsymbol{f}^*, \boldsymbol{x}^*)$ that we might consider.

Moreover, when the edge costs are strictly increasing, the equilibrium loads are also uniquely determined by $x_e^*(\boldsymbol{\mu}) = c_e^{-1}(\tau_e(\boldsymbol{\mu}))$. For later reference, we introduce the concept of active regime, which associates to each $\boldsymbol{\mu}$ the set of minimum-cost paths at equilibrium.

Definition 2.2. Consider the routing game structure $(G, \boldsymbol{c}, \mathcal{H})$.

(a) A subset $\mathcal{R} \subset \mathcal{P}$ is called a *regime* if $\mathcal{R} \cap \mathcal{P}^{(h)} \neq \emptyset$ for every $h \in \mathcal{H}$.
(b) The *active regime* at demand $\boldsymbol{\mu} \in \mathbb{R}_+^{\mathcal{H}}$ is the set

$$\widehat{\mathcal{P}}(\boldsymbol{\mu}) = \bigcup_{h \in \mathcal{H}} \left\{ p \in \mathcal{P}^{(h)} : c_p^*(\boldsymbol{\mu}) = \lambda^{(h)}(\boldsymbol{\mu}) \right\}. \tag{2.10}$$

All of this allows us to define the *equilibrium social cost* with demand $\boldsymbol{\mu} \in \mathbb{R}_+^{\mathcal{H}}$ as

$$\mathsf{SC}^*(\boldsymbol{\mu}) := \sum_{h \in \mathcal{H}} \mu^{(h)} \lambda^{(h)}(\boldsymbol{\mu}) = \sum_{p \in \mathcal{P}} f_p^* \, c_p(\boldsymbol{f}^*) = \sum_{e \in \mathcal{E}} x_e^* \, c_e(x_e^*), \tag{2.11}$$

where $(\boldsymbol{f}^*, \boldsymbol{x}^*) \in \mathcal{F}_{\boldsymbol{\mu}}$ is any equilibrium flow.

Also, the *optimum social cost* for a demand vector $\boldsymbol{\mu} \in \mathbb{R}_+^{\mathcal{H}}$ is defined as

$$\widetilde{\mathsf{SC}}(\boldsymbol{\mu}) := \min_{(\boldsymbol{f}, \boldsymbol{x}) \in \mathcal{F}_{\boldsymbol{\mu}}} \sum_{p \in \mathcal{P}} f_p \, c_p(\boldsymbol{x}) = \min_{\boldsymbol{x} \in \mathcal{X}_{\boldsymbol{\mu}}} \sum_{e \in \mathcal{E}} x_e \, c_e(x_e), \tag{2.12}$$

and the *price of anarchy* at demand $\boldsymbol{\mu} \in \mathbb{R}_+^{\mathcal{H}}$ is

$$\mathsf{PoA}(\boldsymbol{\mu}) := \frac{\mathsf{SC}^*(\boldsymbol{\mu})}{\widetilde{\mathsf{SC}}(\boldsymbol{\mu})} \geq 1. \tag{2.13}$$

3. Continuity and smoothness of equilibria and PoA

In what follows we study the behavior of the equilibrium, social cost, and price-of-anarchy, when the demand varies on a smooth curve parameterized by a scalar variable. For instance, a linear demand function $\boldsymbol{\mu}(t) = t\,\boldsymbol{r}$ for a fixed vector $\boldsymbol{r} \in \mathbb{R}_+^{H}$ represents a situation where the demands vary proportionally across different OD pairs.

Definition 3.1. A *nonatomic routing game with demand function* $\boldsymbol{\mu}(\cdot)$ is given by a tuple $(G, \boldsymbol{c}, \mathcal{H}, \boldsymbol{\mu}(\cdot))$, where the demand is a function $\boldsymbol{\mu} : [0, \infty) \to \mathbb{R}_+^{H}$.

Theorem 3.2. *Let $(G, \boldsymbol{c}, \mathcal{H}, \boldsymbol{\mu}(\cdot))$ be a nonatomic routing game with a continuously differentiable demand function $\boldsymbol{\mu}(\cdot)$. Let the cost functions c_e be C^1 with strictly positive derivative, and suppose that the active regime $\widehat{\mathcal{P}}(\boldsymbol{\mu}(t))$ is constant on a neighborhood of $t^0 \in \mathbb{R}_+$. Then:*

(a) *The equilibrium load map $t \mapsto \boldsymbol{x}^*(\boldsymbol{\mu}(t))$ is continuously differentiable on a neighborhood of t^0. In particular the equilibrium costs $\left(\lambda^{(h)}(\boldsymbol{\mu}(t))\right)_{h \in \mathcal{H}}$ and the social cost at equilibrium $\mathsf{SC}^*(\boldsymbol{\mu}(t))$ are of class C^1 on a neighborhood of t^0.*
(b) *If in addition the maps $x_e \mapsto x_e c_e(x_e)$ are convex, then minimum social cost $\widetilde{\mathsf{SC}}(\boldsymbol{\mu}(t))$ and the price of anarchy $\mathsf{PoA}(\boldsymbol{\mu}(t))$ are of class C^1 on a neighborhood of t^0.*

The simplest situation where the active regime $\widehat{\mathcal{P}}(\mu(t))$ is constant on a neighborhood of t^0 is when every path of minimal cost at demand $\mu(t^0)$ carries a strictly positive flow. In this case, by continuity of the equilibrium costs, we are assured that the active regime will not change after small changes in the demands.

The proof of Theorem 3.2 exploits the regularity properties of the equilibrium costs and optimum social cost stated below in Lemmas 3.3 and 3.4. Lemma 3.3 extends Cominetti et al. (2024b, proposition 3.1) to the multi-OD setting, as well as Hall (1978), who considered the case of strictly increasing costs. The proof of Lemma 3.3 can be found in Cominetti et al. (2024a, proposition 1), while Lemma 3.4 is proved in Appendix A.

Lemma 3.3. *Let $(G, \boldsymbol{c}, \mathcal{H})$ be a routing game structure. Then, the equilibrium edge costs $\boldsymbol{\mu} \mapsto \tau_e(\boldsymbol{\mu})$ are uniquely defined and continuous. Moreover, the equilibrium costs $\boldsymbol{\mu} \mapsto \lambda^{(h)}(\boldsymbol{\mu})$ are continuous.*

Lemma 3.4. *Let $(G, \boldsymbol{c}, \mathcal{H})$ be a routing game structure. If the cost functions c_e are C^1 and the functions $x_e \mapsto x_e c_e(x_e)$ are convex, then the optimum social cost function $\boldsymbol{\mu} \mapsto \widetilde{\mathsf{SC}}(\boldsymbol{\mu})$ is convex and C^1 everywhere.*

In addition to these lemmas, our proof of Theorem 3.2 involves the analysis of some auxiliary minimization problems without sign constraints on the variables. In order to properly formulate these problems, we extend the edge cost functions to the whole $\mathbb{R}$ in such a way that they remain C^1 with strictly positive derivatives and with $\lim_{x\to-\infty} c_e(x) < 0$ for all edges $e \in \mathcal{E}$. Then, for each fixed regime $\mathcal{R}$, we consider the auxiliary problems:

$$\min_{x\in\mathcal{X}_\mu^{\mathcal{R}}} \Phi(x) \tag{$\mathrm{P}_\mu^{\mathcal{R}}$}$$

where the feasible set $\mathcal{X}_\mu^{\mathcal{R}}$ comprises all *signed* load vectors $x \in \mathbb{R}^{\mathcal{E}}$ induced by some flow $f \in \mathbb{R}^{\mathcal{P}}$ with support in $\mathcal{R}$, that is

$$\mathcal{X}_\mu^{\mathcal{R}} = \left\{ x \in \mathbb{R}^{\mathcal{E}} : (\exists f \in \mathbb{R}^{\mathcal{P}}) \text{ s.t. } Sf = \mu,\ x = \Delta f,\ \text{and } f_p = 0 \text{ for all } p \notin \mathcal{R} \right\}. \tag{3.1}$$

Lemma A.1(b) in Appendix A shows that when the costs c_e are C^1 with strictly positive derivative, then ($\mathrm{P}_\mu^{\mathcal{R}}$) has a unique optimal solution $x_{\mathcal{R}}(\mu)$ and the map $\mu \to x_{\mathcal{R}}(\mu)$ is of class C^1. Moreover, Lemma A.1(a) provides explicit expressions for the Lagrange multipliers, whose uniqueness is used in the proof of Theorem 4.4(ii).

Let us insist that in ($\mathrm{P}_\mu^{\mathcal{R}}$) the regime $\mathcal{R}$ is fixed and the variables have no sign constraints, so this problem is a relaxation of (2.8) and we might have $f_p < 0$ for some paths $p \in \mathcal{P}^{(h)} \cap \mathcal{R}$. Moreover, although for every optimal solution x of ($\mathrm{P}_\mu^{\mathcal{R}}$) and each OD pair h the costs of all paths $p \in \mathcal{P}^{(h)} \cap \mathcal{R}$ turn out to be equal to some common value $m^{(h)}$, it can happen that some paths $p \in \mathcal{P}^{(h)} \setminus \mathcal{R}$ may have a smaller cost $c_p(x) < m^{(h)}$. However, if for some demand vector μ we have an optimal solution with $f_p \geq 0$ for all $p \in \mathcal{P}^{(h)} \cap \mathcal{R}$ and $c_p(x) \geq m^{(h)}$ for all $p \in \mathcal{P}^{(h)} \setminus \mathcal{R}$, then this is also an optimal solution of the original problem (2.8) and therefore it corresponds to a Wardrop equilibrium.

Proof of Theorem 3.2. (a) A sufficient condition for a point $x \in \mathcal{X}_\mu^{\mathcal{R}}$ to be an optimal solution of ($\mathrm{P}_\mu^{\mathcal{R}}$), is the existence of multipliers (m, η, ν) such that

$$\begin{aligned} c_e(x_e) &= \eta_e && \forall e \in \mathcal{E},\\ m^{(h)} &= \sum_{e\in p} \eta_e && \forall h \in \mathcal{H}\ \forall p \in \mathcal{P}^{(h)} \cap \mathcal{R},\\ m^{(h)} &= \sum_{e\in p} \eta_e - \nu_p && \forall h \in \mathcal{H}\ \forall p \in \mathcal{P}^{(h)} \setminus \mathcal{R}. \end{aligned}$$

For any demand μ the equilibrium load vector $x^*(\mu)$ satisfies these conditions for $\mathcal{R} = \widehat{\mathcal{P}}(\mu)$, with $\eta_e = \tau_e = c_e(x_e^*)$, $m^{(h)} = \lambda^{(h)}(\mu)$, and $\nu_p = \sum_{e\in p} \tau_e - \lambda^{(h)}(\mu)$ for $p \notin \mathcal{R}$. This shows that the equilibrium load $x^*(\mu)$ coincides with the unique optimum solution $x_{\mathcal{R}}(\mu)$ of ($\mathrm{P}_\mu^{\mathcal{R}}$) for $\mathcal{R} = \widehat{\mathcal{P}}(\mu)$. Now, by assumption $\widehat{\mathcal{P}}(\mu(t)) \equiv \mathcal{R}$ is constant for t near t_0, so that $x^*(\mu(t)) = x_{\mathcal{R}}(\mu(t))$, which is continuously differentiable as a composition of the C^1 demand function $t \mapsto \mu(t)$ and the map $\mu \mapsto x_{\mathcal{R}}(\mu)$, which is also C^1 in view of Lemma A.1(b).

(b) The smoothness of $t \mapsto \widetilde{\mathsf{SC}}(\mu(t))$ is a direct consequence of Lemma 3.4. This, combined with part (a), implies the smoothness of $t \mapsto \mathsf{PoA}(\mu(t)) = \mathsf{SC}^*(\mu(t))/\widetilde{\mathsf{SC}}(\mu(t))$. □

Remark 3.5.

(a) Observe that Theorem 3.2 requires the active regime $\widehat{\mathcal{P}}(\mu(t))$ to be constant near $t^0 \in \mathbb{R}_+$ but only along the demand curve, and not necessarily in a neighborhood of $\mu(t^0)$ in the ambient space $\mathbb{R}_+^{\mathcal{H}}$. This covers some special situations in which the curve $\mu(\cdot)$ may slide along the boundary between two regions with different active regimes.

(b) Invoking Remark A.2 in Appendix A, the strict positivity of the cost derivatives in Theorem 3.2 can be slightly weakened by assuming that the edge costs $c_e(\cdot)$ are just strictly increasing and the set of edges e with $c_e'(x_e(\mu(t))) = 0$ do not contain undirected cycles.

The following examples show that the assumptions in Theorem 3.2 are close to minimal: the presence of a single edge with a non-convex cost whose derivative vanishes at a single point, may give rise to non-smooth equilibrium loads, even if the set of optimum paths is constant around the reference point. In Examples 3.6 and 3.7 below, the point where the cost functions have zero derivative is not the origin. Whether Theorem 3.2 can be extended to cost functions that exhibit a zero derivative only at the origin remains an open problem. This would be of major interest since this feature is shared by Bureau of Public Roads (BPR) cost functions.

Example 3.6. In this example with strictly increasing C^1 cost functions, the equilibrium flow is nondifferentiable at some point which is not a breakpoint. Consider a two link parallel network where the cost functions (plotted in Fig. 2) are

$$c_1(x) = \begin{cases} -(x-1)^2 + 1 & \text{if } x \leq 1,\\ (x-1)^2 + 1 & \text{if } x > 1, \end{cases} \tag{3.2}$$

$$c_2(x) = \begin{cases} -(x-1)^2 + 1 & \text{if } x \leq 1,\\ 2(x-1)^2 + 1 & \text{if } x > 1. \end{cases} \tag{3.3}$$

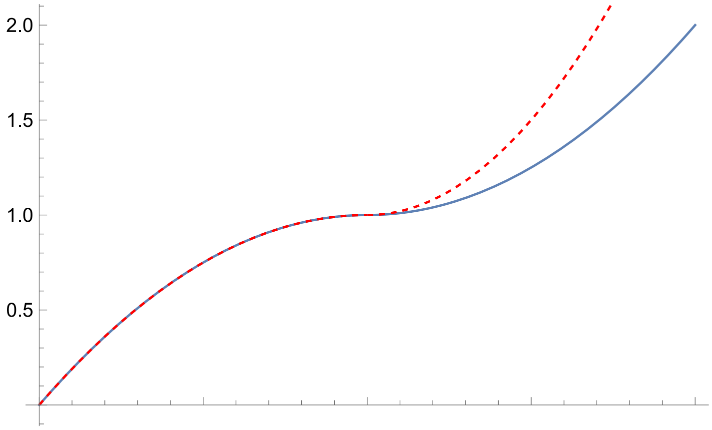


Fig. 2. Plot of the cost functions in (3.2) (in blue) and (3.3) (in dotted red). The two functions are C^1 but not convex.

These functions are C^1 and their derivatives vanish at $x = 1$. The equilibrium flows are:

Interval	x_1	x_2
$\mu \in [0, 2)$	$\mu/2$	$\mu/2$
$\mu \in [2, +\infty)$	$\frac{\sqrt{2}\cdot\mu-\sqrt{2}+1}{\sqrt{2}+1}$	$\frac{\mu+\sqrt{2}-1}{\sqrt{2}+1}$

Both flows are nondifferentiable at $\mu = 2$.

Example 3.7. Nonsmoothness can also be observed at a point where a flow vanishes. Consider the Wheatstone network in Fig. 4(a) with

$$
\begin{aligned}
c_1(x) &= x,\\
c_2(x) &= x,\\
c_3(x) &= \begin{cases} -\frac{1}{10}(x-1)^2+1 & x \le 1,\\ 10(x-1)^2+1 & x > 1,\end{cases}\\
c_4(x) &= \begin{cases} -(x-1)^2+1 & x \le 1,\\ (x-1)^2+1 & x > 1,\end{cases}\\
c_5(x) &= x^2.
\end{aligned}
$$

The plots of c_3 and c_4 can be found in Fig. 3. All three paths are active in a neighborhood of $\mu = 2$. When the demand approaches 2 from below, the flow on the zig-zag path decreases, and it vanishes when $\mu = 2$, whereas it increases with positive derivative after 2. A plot of the flow on the zig-zag path for $\mu \in [1, 3]$ is shown in Fig. 4(b).

4. Behavior around breakpoints

In this section we describe the behavior of the functions SC^* and PoA around breakpoints. In particular we study their left and right derivatives when the demand vector depends on a single real parameter. To formally describe this situation we make use of the following concept:

Definition 4.1. We say that $\bar{t} \in \mathbb{R}_+$ is a $\widehat{\mathcal{P}}$*-breakpoint* for $(G, \boldsymbol{c}, \mathcal{H}, \boldsymbol{\mu}(\cdot))$ if there exist $\varepsilon > 0$ such that $\widehat{\mathcal{P}}(\boldsymbol{\mu}(t))$ is constant as a function of t over each of the intervals $[\bar{t}-\varepsilon, \bar{t})$ and $(\bar{t}, \bar{t}+\varepsilon]$ with $\widehat{\mathcal{P}}(\boldsymbol{\mu}(\bar{t}-\varepsilon)) \neq \widehat{\mathcal{P}}(\boldsymbol{\mu}(\bar{t}+\varepsilon))$. In this case, we use the symbols

$$\widehat{\mathcal{P}}(\bar{t}^-) := \widehat{\mathcal{P}}(\boldsymbol{\mu}(\bar{t}-\varepsilon)), \quad \widehat{\mathcal{P}}(\bar{t}^+) := \widehat{\mathcal{P}}(\boldsymbol{\mu}(\bar{t}+\varepsilon)). \tag{4.1}$$

O'Hare et al. (2016), considered two regimes around a breakpoint and observed that, empirically, if one of these two regimes contains the other, then the derivatives of SC^* and PoA associated to the smaller regime are larger than the derivatives in the larger regime. They formalized this idea in their conjectures 4.5, 4.6, and 4.9. Using our language and notation, we can summarize these conjectures in the following statement, where we define $g(\bar{t}^-) := \lim_{t\to\bar{t}^-} g(t)$ and $g(\bar{t}^+) := \lim_{t\to\bar{t}^+} g(t)$.

Conjecture 4.2 (*O'Hare et al., 2016*)**.** *Let $(G, \boldsymbol{c}, \mathcal{H}, \boldsymbol{\mu}(\cdot))$ be a routing game where the cost functions are continuous, differentiable, strictly increasing with positive second derivative, and the demand function $t \to \boldsymbol{\mu}(t)$ is piecewise affine and componentwise nondecreasing with t. If $\bar{t} \in \mathbb{R}_+$ is a $\widehat{\mathcal{P}}$-breakpoint, then the following hold:*

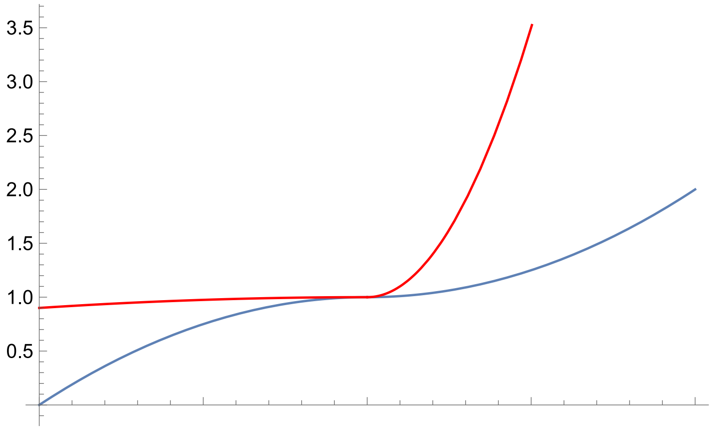


Fig. 3. Plot of the cost functions c_3 (in red) and c_4 (in blue) in Example 3.7. The two functions are C^1 but not convex. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

(a) *if* $\widehat{\mathcal{P}}(\bar{t}^-) \subset \widehat{\mathcal{P}}(\bar{t}^+)$, *then* $(\mathsf{SC}^* \circ \boldsymbol{\mu})'(\bar{t}^-) > (\mathsf{SC}^* \circ \boldsymbol{\mu})'(\bar{t}^+)$ *and* $(\mathsf{PoA} \circ \boldsymbol{\mu})'(\bar{t}^-) > (\mathsf{PoA} \circ \boldsymbol{\mu})'(\bar{t}^+)$*;*
(b) *if* $\widehat{\mathcal{P}}(\bar{t}^-) \supset \widehat{\mathcal{P}}(\bar{t}^+)$, *then* $(\mathsf{SC}^* \circ \boldsymbol{\mu})'(\bar{t}^-) < (\mathsf{SC}^* \circ \boldsymbol{\mu})'(\bar{t}^+)$ *and* $(\mathsf{PoA} \circ \boldsymbol{\mu})'(\bar{t}^-) < (\mathsf{PoA} \circ \boldsymbol{\mu})'(\bar{t}^+)$*,*

As illustrated by the next example with affine costs and an affine demand function, this conjecture does not hold in its most general form as stated above. However, we will show later that a restricted form of the conjecture is indeed valid.

Example 4.3. Consider the network in Fig. 5(a) studied in Fisk (1979), with affine cost functions and three OD pairs whose demands and routes as shown in the next table:

OD	Demand	Feasible routes
(a,b)	1	$p_1 = a \to b$
(a,c)	t	$p_2 = a \to c$ and $p_3 = a \to b \to c$
(b,c)	100	$p_4 = b \to c$

where the map $t \mapsto \boldsymbol{\mu}(t)$ is affine and only the demand of (a,c) increases with t.

The following table shows the path flows and social cost at equilibrium for all $t \geq 0$:

Interval	f_1^*	f_2^*	f_3^*	f_4^*	$\mathsf{SC}^*(\mu(t))$
$t \in [0,11)$	1	t	0	100	$10001 + 90t + t^2$
$t \in [11,+\infty)$	1	$\frac{1}{3}(11+2t)$	$\frac{1}{3}(t-11)$	100	$\frac{1}{3}(28892 + 382t + 2t^2)$

whereas the optimum flow and minimum social cost are given by

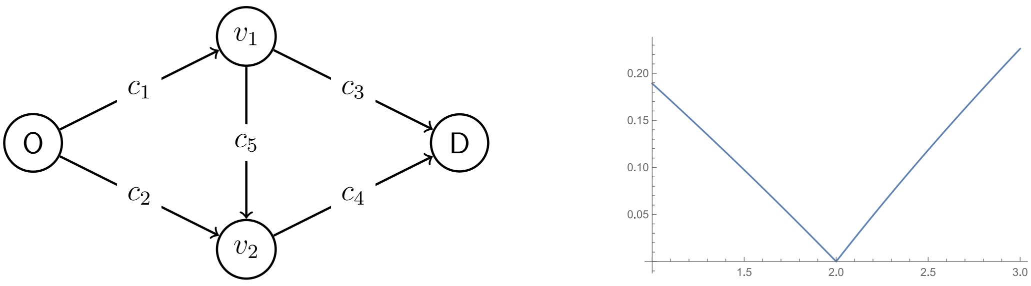
Interval	$\widetilde{f_1}$	$\widetilde{f_2}$	$\widetilde{f_3}$	$\widetilde{f_4}$	$\widetilde{\mathsf{SC}}(\mu(t))$
$t \in [0,56)$	1	t	0	100	$10001 + 90t + t^2$
$t \in [56,+\infty)$	1	$\frac{1}{3}(56+2t)$	$\frac{1}{3}(t-56)$	100	$\frac{1}{3}(26867 + 382t + 2t^2)$.

Fig. 5(b) shows the plot of the PoA, which is

$$\mathsf{PoA}(\mu(t)) = \begin{cases} 1 & \text{if } t \in [0,11), \\ \dfrac{28892 + 382t + 2t^2}{3(10001 + 90t + t^2)} & \text{if } t \in [11,56), \\ \dfrac{28892 + 382t + 2t^2}{26867 + 382t + 2t^2} & \text{if } t \geq 56, \end{cases} \tag{4.2}$$

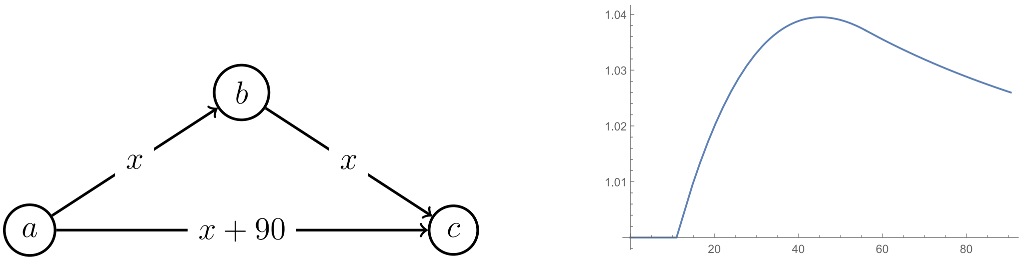
with derivative

$$(\mathsf{PoA} \circ \mu)'(t) = \begin{cases} 0 & \text{if } t \in [0,11), \\ -\dfrac{2(-610051 + 8890t + 101t^2)}{3(10001 + 90t + t^2)^2} & \text{if } t \in [11,56), \\ -\dfrac{4050(191 + 2t)}{(26867 + 382t + 2t^2)^2} & \text{if } t \geq 56. \end{cases} \tag{4.3}$$



(a) (b)

Fig. 4. On the left we have the classical Wheatstone network with the cost functions defined in Example 3.7. The plot on the right shows the load on the vertical link with demand varying in the interval $[1, 3]$. Notice that at demand 2 the load is zero and the load function is not differentiable.



(a) Fisk's Network (b) PoA for Fisk's Network with affine demand

Fig. 5. An example where Conjecture 4.2 does not hold.

At $t = 11$ the active regime experiences an expansion, where the route p_3 becomes optimal for the pair (a, c) at equilibrium. However, the right derivatives of $\mathsf{SC}^* \circ \boldsymbol{\mu}$ and $\mathsf{PoA} \circ \boldsymbol{\mu}$ (respectively 142 and $5/1852$) are strictly larger than the respective left derivatives (112 and 0). This provides a counter-example for Conjecture 4.2(a) as originally stated in O'Hare et al. (2016). Note also that, as predicted by Lemma 3.4, both $\widetilde{\mathsf{SC}} \circ \boldsymbol{\mu}$ and $\mathsf{PoA} \circ \boldsymbol{\mu}$ are smooth at $t = 56$, despite the presence of a phase transition in the social optimum. For simplicity, in the example the demand was chosen to move along the direction $(0, 1, 0)$, however the counterexample also works for strictly positive directions close to $(0, 1, 0)$.

We now proceed to prove that a restricted version of Conjecture 4.2 is true, where the demand function is assumed to be linear, so that the proportion among the demands in each OD pair is maintained constant as the total demand increases.

Theorem 4.4. *Let $(G, \boldsymbol{c}, \mathcal{H}, \boldsymbol{\mu}(\cdot))$ be a nonatomic routing game with a linear demand function $\boldsymbol{\mu}(t) = t\,\boldsymbol{r}$ for a fixed $\boldsymbol{r} \in \mathbb{R}_+^{\mathcal{H}}$. Let $\bar{t} \in \mathbb{R}_+$ be a $\widehat{\mathcal{P}}$-breakpoint.*

(i) *If the edge cost functions are C^1 with strictly positive derivatives, then properties (a) and (b) in Conjecture 4.2 hold with weak inequalities.*
(ii) *If the edge costs are nondecreasing affine functions, then properties (a) and (b) in Conjecture 4.2 hold with strict inequalities.*

Remark 4.5. Theorem 4.4(i) can be generalized to networks with just strictly increasing cost functions, assuming that the set of edges e where $c_e'(x_e(\boldsymbol{\mu}(\bar{t}))) = 0$ form a graph without undirected cycles.

Remark 4.6. The hypothesis of strictly positive derivatives in Theorem 4.4(i) excludes from the result the case of BPR cost functions. The result would hold also in the BPR case if we had a generalization of Theorem 3.2(a) to strictly increasing smooth cost functions with zero derivative only possibly at zero.

Remark 4.7. Violating the model assumption of having the sets $\mathcal{P}^{(h)}$ pairwise disjoint for every $h \in \mathcal{H}$, does not affect the results of this paper. Indeed, one can always add a dummy origin $\tilde{\mathsf{O}}^{(h)}$ and attach a zero cost edge $(\tilde{\mathsf{O}}^{(h)}, \mathsf{O}^{(h)})$ to each of the original paths $p \in \mathcal{P}^{(h)}$. This operation does not change the equilibrium; hence the equilibrium loads on the original edges remain unchanged. Furthermore, this transformation does not create any new undirected cycle, thus, as noted in Remark 4.5 the results are still valid.

Example 4.8. At a $\widehat{\mathcal{P}}$-breakpoint $\bar{t}$ where neither $\widehat{\mathcal{P}}(\bar{t}^-) \subset \widehat{\mathcal{P}}(\bar{t}^+)$ nor $\widehat{\mathcal{P}}(\bar{t}^-) \supset \widehat{\mathcal{P}}(\bar{t}^+)$ are true, all inequalities are possible between the left and the right derivatives of the social cost and the price of anarchy. In order to illustrate this phenomenon, consider the single

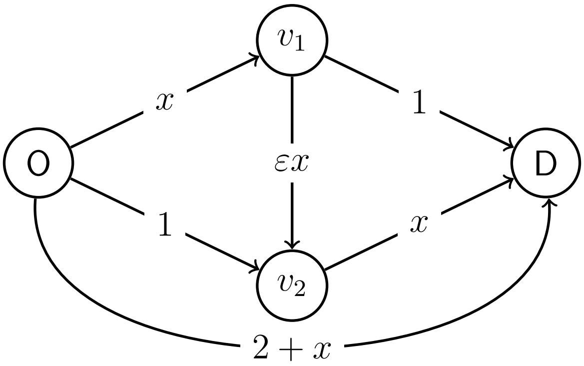


Fig. 6. In this network, varying the parameter ϵ we can observe any type of inequality between the left and right derivative of the equilibrium cost at a breakpoint.

OD network in Fig. 6 where one cost function depends on a parameter $\epsilon > 0$. For every ϵ, there is a breakpoint at $\bar{\mu} = 2$ with neither $\widehat{\mathcal{P}}^-(\bar{\mu}) \subset \widehat{\mathcal{P}}^+(\bar{\mu})$ nor $\widehat{\mathcal{P}}^-(\bar{\mu}) \supset \widehat{\mathcal{P}}^+(\bar{\mu})$, and the left and right derivatives at $\bar{\mu}$ of SC^* and PoA are ranked in different order depending on the value of ϵ. There are four paths

$$\begin{aligned} p_1 &= \mathsf{O} \to v_1 \to \mathsf{D} \\ p_2 &= \mathsf{O} \to v_2 \to \mathsf{D} \\ p_3 &= \mathsf{O} \to v_1 \to v_2 \to \mathsf{D} \\ p_4 &= \mathsf{O} \to \mathsf{D} \end{aligned}$$

and the equilibrium flow vector $\boldsymbol{f}$ is given in the following table:

Interval	$\lambda(\mu)$	f_1	f_2	f_3	f_4
$\mu \in [0, \frac{1}{1+\varepsilon})$	$(2+\varepsilon)\mu$	0	0	μ	0
$\mu \in [\frac{1}{1+\varepsilon}, 2)$	$\frac{\varepsilon\mu+2+2\varepsilon}{1+2\varepsilon}$	$\frac{(1+\varepsilon)\mu-1}{1+2\varepsilon}$	$\frac{(1+\varepsilon)\mu-1}{1+2\varepsilon}$	$\frac{2-\mu}{1+2\varepsilon}$	0
$\mu \in [2, +\infty)$	$\frac{\mu+4}{3}$	$\frac{\mu+1}{3}$	$\frac{\mu+1}{3}$	0	$\frac{\mu-2}{3}$

where we can observe that

$$\lim_{\mu \to 2^-} \lambda'(\mu) = \frac{\varepsilon}{1+2\varepsilon}, \quad \lim_{\mu \to 2^+} \lambda'(\mu) = \frac{1}{3}.$$

Therefore,

$$\begin{aligned} \lim_{\mu \to 2^-} \lambda'(\mu) &> \lim_{\mu \to 2^+} \lambda'(\mu) \quad \text{if } \varepsilon < 1, \\ \lim_{\mu \to 2^-} \lambda'(\mu) &= \lim_{\mu \to 2^+} \lambda'(\mu) \quad \text{if } \varepsilon = 1, \\ \lim_{\mu \to 2^-} \lambda'(\mu) &< \lim_{\mu \to 2^+} \lambda'(\mu) \quad \text{if } \varepsilon > 1, \end{aligned}$$

which implies the same relations at $\bar{\mu} = 2$ for the left and right derivatives of SC^* and PoA.

As shown in Theorem 4.4(ii), with affine cost functions the inequalities of Conjecture 4.2 can never hold with equality at a breakpoint. It is an open question whether this is also the case for convex costs or BPR costs. On the other hand, the following example shows that the inequalities of Conjecture 4.2 can indeed hold with equality if we allow for non-convex costs.

Example 4.9. Consider a single-commodity parallel network with two paths and cost functions:

$$c_1(x) = \begin{cases} -(x-1)^2 + 1 & \text{if } x \le 1, \\ (x-1)^2 + 1 & \text{if } x > 1, \end{cases} \qquad c_2(x) = 1 + x^2. \tag{4.4}$$

For $\mu \le 1$ only the first link is used, and the equilibrium cost $\lambda(\mu)$ is equal to $-(\mu-1)^2 + 1$, whose derivative at 1 is zero. For $\mu \in [1, +\infty)$ the equilibrium flow routes $(\mu+1)/2$ through the first link and $(\mu-1)/2$ through the second link. Hence, the equilibrium cost is

$$\lambda(\mu) = 1 + \left(\frac{\mu-1}{2}\right)^2$$

for every $\mu \in [1, +\infty)$, whose derivative at 1 also equals zero.

5. Proof of Theorem 4.4

The proof of Theorem 4.4 will be split into a number of preliminary intermediate steps. The general principle is described in Lemma 5.1 which assumes *a priori* the existence of a selection of equilibrium flows that is smooth both to the left and to the right of a breakpoint. This assumption is then shown to hold both when the cost functions have strictly positive derivatives (Theorem 4.4(i)), and also when the costs are affine (Theorem 4.4(ii)).

Lemma 5.1. *Let $(G, \boldsymbol{c}, \mathcal{H}, \boldsymbol{\mu}(\cdot))$ be a nonatomic routing game with C^1 nondecreasing costs and a linear demand function $\boldsymbol{\mu}(t) = t\,\boldsymbol{r}$ with $\boldsymbol{r} \in \mathbb{R}_+^{\mathcal{H}}$. Let $\bar{t} \in \mathbb{R}_+$ be a $\widehat{\mathcal{P}}$-breakpoint with active regimes $\mathcal{R}^- := \widehat{\mathcal{P}}(\bar{t}^-)$ over $I^- := (\bar{t}-\varepsilon, \bar{t})$ and $\mathcal{R}^+ := \widehat{\mathcal{P}}(\bar{t}^+)$ over $I^+ := (\bar{t}, \bar{t}+\varepsilon)$. Suppose that there exist maps $\boldsymbol{f}^-, \boldsymbol{f}^+ : (\bar{t}-\varepsilon, \bar{t}+\varepsilon) \to \mathbb{R}_+^{\mathcal{P}}$ of class C^1 such that*

- *$\boldsymbol{f}^-(t)$ is an equilibrium flow of demand $\boldsymbol{\mu}(t)$ for all $t \in I^-$,*
- *$\boldsymbol{f}^+(t)$ is an equilibrium flow of demand $\boldsymbol{\mu}(t)$ for all $t \in I^+$,*
- *$\boldsymbol{f}^-(\bar{t}) = \boldsymbol{f}^+(\bar{t})$.*

Then Conjecture 4.2 holds with weak inequalities.

Proof. Let $\boldsymbol{x}^-(t)$ and $\boldsymbol{x}^+(t)$ be the loads induced by $\boldsymbol{f}^-(t)$ and $\boldsymbol{f}^+(t)$, and let $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{f}}$ be their corresponding common values at $\bar{t}$. For notational simplicity, for each $t \in (\bar{t}-\varepsilon, \bar{t}+\varepsilon)$ we let $\lambda^{(h)}(t) = \lambda^{(h)}(\boldsymbol{\mu}(t))$ denote the equilibrium costs and $\mathsf{SC}^*(t) = \mathsf{SC}^*(\boldsymbol{\mu}(t))$ the social cost at equilibrium with demand $\boldsymbol{\mu}(t)$, so that

$$\mathsf{SC}^*(t) = \sum_{h\in\mathcal{H}} \mu^{(h)}(t) \cdot \lambda^{(h)}(t). \tag{5.1}$$

By Lemma 3.3, the maps $t \mapsto \lambda^{(h)}(t)$ are continuous. They are also C^1 over both I^- and I^+, with well defined unilateral limits $(\lambda^{(h)})'(\bar{t}^-)$ and $(\lambda^{(h)})'(\bar{t}^+)$ at $\bar{t}$. Indeed, for each $h \in \mathcal{H}$ we may fix an active path $p \in \mathcal{R}^+$ so that, for all $t \in I^+$, we have $\lambda^{(h)}(t) = c_p(\boldsymbol{x}^+(t))$, which is smooth with a well defined limit for its derivative at $\bar{t}^+$. A similar argument using $\boldsymbol{x}^-(\cdot)$ applies for $t \in I^-$.

Since $\mu^{(h)}(t) = t\,r^{(h)}$, we can express the left and the right derivatives at $\bar{t}$ of the social cost as

$$(\mathsf{SC}^*)'(t^-) = \sum_{h\in\mathcal{H}} r^{(h)} \cdot \lambda^{(h)}(\bar{t}) + \bar{t}\,r^{(h)} \cdot (\lambda^{(h)})'(\bar{t}^-),$$
$$(\mathsf{SC}^*)'(t^+) = \sum_{h\in\mathcal{H}} r^{(h)} \cdot \lambda^{(h)}(\bar{t}) + \bar{t}\,r^{(h)} \cdot (\lambda^{(h)})'(\bar{t}^+).$$

It follows that $(\mathsf{SC}^*)'(t^-) \geq (\mathsf{SC}^*)'(t^+)$ if and only if $\theta(\bar{t}^-) \geq \theta(\bar{t}^+)$, where for $t \neq \bar{t}$ we define

$$\theta(t) := \sum_{h\in\mathcal{H}} r^{(h)} \cdot (\lambda^{(h)})'(t). \tag{5.2}$$

We now derive an alternative characterization for $\theta(\bar{t}^-)$ and $\theta(\bar{t}^+)$. Let us consider the latter. For $t \in I^+$ we have that $\boldsymbol{f}(t) := \boldsymbol{f}^+(t)$ is an equilibrium flow with associated induced equilibrium load $\boldsymbol{x}(t) := \boldsymbol{x}^+(t)$. In particular,

$$\sum_{p\in\mathcal{P}^{(h)}} f_p(t) = \mu^{(h)}(t) = t\,r^{(h)}, \tag{5.3}$$

so that differentiating we get $\sum_{p\in\mathcal{P}^{(h)}} f_p'(t) = r^{(h)}$. Therefore for $t > \bar{t}$ we have

$$\theta(t) = \sum_{h\in\mathcal{H}} \sum_{p\in\mathcal{P}^{(h)}} f_p'(t) \cdot (\lambda^{(h)})'(t). \tag{5.4}$$

Now, for any inactive path $p \notin \mathcal{R}^+$ we have $f_p'(t) = 0$, so the inner sum in (5.4) can be restricted to the active paths $p \in \mathcal{R}^+$. For these paths $p \in \mathcal{P}^{(h)} \cap \mathcal{R}^+$ we have $\lambda^{(h)}(t) = c_p(\boldsymbol{x}(t))$ and therefore

$$(\lambda^{(h)})'(t) = \sum_{e\in p} c_e'(x_e(t)) \cdot x_e'(t) = \sum_{e\in p} c_e'(x_e(t)) \cdot \sum_{q\in\mathcal{P}} \delta_{eq} f_q'(t). \tag{5.5}$$

Plugging this into (5.4) and using Fubini's rule to exchange the order of sums, we can write $\theta(t)$ in the equivalent form

$$\theta(t) = \sum_{e\in\mathcal{E}} c_e'(x_e(t)) \left(\sum_{p\in\mathcal{P}} \delta_{ep} f_p'(t) \right)^2. \tag{5.6}$$

Let us recall that $\bar{\boldsymbol{x}}$ denotes the equilibrium load induced by $\bar{\boldsymbol{f}} = \boldsymbol{f}(\bar{t})$, and consider the positive semidefinite quadratic form

$$Q_{\bar{\boldsymbol{x}}}(\boldsymbol{y}) := \frac{1}{2} \sum_{e\in\mathcal{E}} c_e'(\bar{x}_e) \left(\sum_{p\in\mathcal{P}} \delta_{ep} y_p \right)^2 \geq 0. \tag{5.7}$$

We claim that $\theta(\bar{t}^+)$ coincides with the optimal value of the convex quadratic program

$$\theta(\bar{t}^+) = \min_{\boldsymbol{y}} \; Q_{\bar{\boldsymbol{x}}}(\boldsymbol{y}) \quad \text{s.t.} \sum_{p\in\mathcal{P}^{(h)}} y_p = r^{(h)} \text{ for all } h \in \mathcal{H}, \text{ and } y_p = 0 \text{ for } p \notin \mathcal{R}^+, \tag{5.8}$$

with optimal solution $\bar{\boldsymbol{y}} = \boldsymbol{f}'(\bar{t})$. Indeed, we already observed that $\boldsymbol{f}'(t)$ satisfies the constraints in (5.8) for all $t > \bar{t}$, so that letting $t \downarrow \bar{t}$ the same holds for $\bar{y} = \boldsymbol{f}'(\bar{t})$. On the other hand, letting $z_e = \sum_{p\in\mathcal{P}} \delta_{ep} y_p$, we have $\partial Q_{\bar{\boldsymbol{x}}}/\partial y_p = \sum_{e\in p} c_e'(\bar{\boldsymbol{x}})\, z_e$, so that the first-order optimality conditions for (5.8) are

$$\forall h \in \mathcal{H}, \forall p \in \mathcal{P}^{(h)}, \quad \sum_{e\in p} c_e'(\bar{\boldsymbol{x}})\, z_e = \begin{cases} m^{(h)} & \text{if } p \in \mathcal{R}^+, \\ m^{(h)} + v_p & \text{if } p \notin \mathcal{R}^+, \end{cases} \tag{5.9}$$

where the $m^{(h)}$ are Lagrange multipliers for the constraints $\sum_{p\in\mathcal{P}^{(h)}} y_p = r^{(h)}$ for all $h \in \mathcal{H}$, and v_p are multipliers for the constraints $y_p = 0$ for $p \notin \mathcal{R}^+$. Now, since for all $t \in I^+$ the pair $(\boldsymbol{f}(t), \boldsymbol{x}(t))$ is an equilibrium flow with $\widehat{\mathcal{P}}(\boldsymbol{\mu}(t)) = \mathcal{R}^+$, we have

$$\forall h \in \mathcal{H}, \forall p \in \mathcal{P}^{(h)} \quad \sum_{e\in p} c_e(x_e(t)) = \begin{cases} \lambda^{(h)}(t) & \text{if } p \in \mathcal{R}^+, \\ \lambda^{(h)}(t) + (c_p(\boldsymbol{x}(t)) - \lambda^{(h)}(t)) & \text{if } p \notin \mathcal{R}^+. \end{cases} \tag{5.10}$$

Differentiating in t and letting $t \downarrow \bar{t}$, we deduce that (5.9) holds with $m^{(h)} = (\lambda^{(h)})'(\bar{t}^+)$ for all $h \in \mathcal{H}$ and $v_p = (c_p\circ\boldsymbol{x})'(\bar{t}) - (\lambda^{(h)})'(\bar{t}^+)$ for $p \in \mathcal{P}^{(h)} \setminus \mathcal{R}^+$. Since the quadratic program is convex, these optimality conditions imply optimality. Hence $\bar{\boldsymbol{y}} = \boldsymbol{f}'(\bar{t})$ is an optimal solution and then the equality (5.8) follows by letting $t \to \bar{t}^+$ in (5.6).

Repeating the argument for $t \in I^-$, this time with $\boldsymbol{f}(t) = \boldsymbol{f}^-(t)$ and $\boldsymbol{x}(t) = \boldsymbol{x}^-(t)$, and noting that $\boldsymbol{x}^-(\bar{t}) = \bar{\boldsymbol{x}}$, we obtain that $\theta(\bar{t}^-)$ is similarly characterized as

$$\theta(\bar{t}^-) = \min_{\boldsymbol{y}} \quad Q_{\bar{\boldsymbol{x}}}(\boldsymbol{y}) \quad \text{s.t.} \sum_{p\in\mathcal{P}^{(h)}} y_p = r^{(h)} \text{ for all } h \in \mathcal{H}, \text{ and } y_p = 0 \text{ for } p \notin \mathcal{R}^-. \tag{5.11}$$

The objective function in both (5.8) and (5.11) is the same, and the only difference between these quadratic programs are the constraints $y_p = 0$ for all $p \notin \mathcal{R}^-$ or $p \notin \mathcal{R}^+$: a larger regime implies fewer constraints and thus a smaller optimal value. Explicitly, if $\mathcal{R}^- \subset \mathcal{R}^+$, then $\theta(\bar{t}^-) \geq \theta(\bar{t}^+)$; and therefore $(\mathsf{SC}^*)'(\bar{t}^-) \geq (\mathsf{SC}^*)'(\bar{t}^+)$. Similarly, if $\mathcal{R}^- \supset \mathcal{R}^+$, then $\theta(\bar{t}^-) \leq \theta(\bar{t}^+)$ and $(\mathsf{SC}^*)'(\bar{t}^-) \leq (\mathsf{SC}^*)'(\bar{t}^+)$. This establishes half of Conjecture 4.2 with weak inequalities.

Concerning the derivative of $\mathsf{PoA}\circ\boldsymbol{\mu}$, when it exists, it is

$$(\mathsf{PoA}\circ\boldsymbol{\mu})'(t) = \frac{(\mathsf{SC}^*\circ\boldsymbol{\mu})'(t) \cdot \widetilde{\mathsf{SC}}(\boldsymbol{\mu}(t)) - \mathsf{SC}^*(\boldsymbol{\mu}(t)) \cdot (\widetilde{\mathsf{SC}}\circ\boldsymbol{\mu})'(t)}{(\widetilde{\mathsf{SC}}(\boldsymbol{\mu}(t)))^2}. \tag{5.12}$$

Using the continuity of SC^* at $\boldsymbol{\mu}(\bar{t})$ and the fact that $\widetilde{\mathsf{SC}}$ is C^1 at $\boldsymbol{\mu}(\bar{t})$ (Lemmas 3.3 and 3.4), we see that the statements regarding the price of anarchy in Conjecture 4.2 also hold with weak inequalities. □

In order to use Lemma 5.1 one needs to ensure the existence of smooth equilibrium flows to the left and to the right of a breakpoint. Having a smooth equilibrium load profile (which can be derived from the results in Section 3) allows us to choose a smooth equilibrium flow, as shown in the following lemma. Note that even when the equilibrium loads are unique, there may exist multiple path flows that induce the same loads. In order to single out a smooth selection of path flows we use the Moore–Penrose pseudoinverse. We recall that the pseudoinverse generalizes the classical inverse of nonsingular square matrices, and provide a tool for solving an ill-posed linear system $A\,\boldsymbol{x} = \boldsymbol{b}$, where A is an $m \times n$ rectangular real matrix and the right-hand side $\boldsymbol{b} \in \mathbb{R}^m$ may not even belong to the range of A so that the system could have no solution. The pseudo-inverse is a linear map $\boldsymbol{b} \mapsto A^\dagger \boldsymbol{b}$ that associates to each $\boldsymbol{b} \in \mathbb{R}^m$ the least norm solution of the equation $A\,\boldsymbol{x} = \bar{\boldsymbol{b}}$ with $\bar{\boldsymbol{b}}$ the projection of b onto $\mathrm{Range}(A)$. Thus, if $\boldsymbol{b} \in \mathrm{Range}(A)$ it gives the least norm solution of $A\,\boldsymbol{x} = \boldsymbol{b}$ and when A is an invertible square matrix it reduces to the classical inverse $A^\dagger \boldsymbol{b} = A^{-1}\boldsymbol{b}$. In general, $A^\dagger \boldsymbol{b}$ is defined as the limit when $\varepsilon \to 0^+$ of the unique minimizer of the strongly convex function $\boldsymbol{x} \mapsto \|A\boldsymbol{x} - \mathbf{b}\|^2 + \varepsilon\|\boldsymbol{x}\|^2$, that is, $A^\dagger \boldsymbol{b} = \lim_{\varepsilon\to 0^+}(A^\top A + \varepsilon I)^{-1} A^\top \boldsymbol{b}$. This defines a linear map whose representative matrix is denoted by $A^\dagger$. In our setting we use the pseudoinverse to associate with each vector of loads $\boldsymbol{x}$ the vector of path flows $\boldsymbol{f}$ that induce the given $\boldsymbol{x}$ and has minimum norm $\|\boldsymbol{f}\|$.

Lemma 5.2. *Let $(G, c, \mathcal{H}, \boldsymbol{\mu}(\cdot))$ be a nonatomic routing game with a C^1 demand map $\boldsymbol{\mu}(\cdot)$. Let $t \mapsto \boldsymbol{x}(\boldsymbol{\mu}(t))$ be a curve of equilibrium loads which is defined and C^1 on some interval I. Then, $\boldsymbol{x}(\boldsymbol{\mu}(t))$ can be decomposed into a path flow $\boldsymbol{f}(\boldsymbol{\mu}(t))$ that is also C^1 for $t \in I$.*

Proof. Let $\boldsymbol{f}^0 \in \mathbb{R}^P$ be an arbitrary path flow decomposition of $x(\boldsymbol{\mu}(t^0))$ with $t^0 \in I$ fixed, and set

$$\boldsymbol{f}(\boldsymbol{\mu}(t)) := \boldsymbol{f}^0 + \begin{bmatrix} \Delta \\ S \end{bmatrix}^\dagger \begin{pmatrix} \boldsymbol{x}(\boldsymbol{\mu}(t)) - \boldsymbol{x}(\boldsymbol{\mu}(t^0)) \\ \boldsymbol{\mu}(t) - \boldsymbol{\mu}(t^0) \end{pmatrix}, \tag{5.13}$$

where Δ and S are as in (2.4), and the symbol $L^\dagger$ denotes the Moore–Penrose pseudoinverse of the matrix L. Since the pseudoinverse is also a matrix, we get the conclusion. □

With these preliminaries, we may now proceed with the proof of Theorem 4.4.

Proof of Theorem 4.4(i). Using Lemma A.1 we have that for every regime $\mathcal{R}$ one can find a C^1 solution $t \mapsto \boldsymbol{x}_{\mathcal{R}}(\boldsymbol{\mu}(t))$ to the problem $(\mathsf{P}_{\boldsymbol{\mu}}^{\mathcal{R}})$, defined in an open neighborhood of $\bar{t}$, and such that $\boldsymbol{x}_{\mathcal{R}}(\boldsymbol{\mu}(t))$ coincides with the equilibrium $\boldsymbol{x}(\boldsymbol{\mu}(t))$ whenever $\widehat{\mathcal{P}}(\boldsymbol{\mu}(t))$ is equal to $\mathcal{R}$. Furthermore, by applying Lemma 5.2 we can find C^1 flows associated to such load profiles. If we do that in the two cases of $\mathcal{R} = \widehat{\mathcal{P}}(\bar{t}^-)$ and $\mathcal{R} = \widehat{\mathcal{P}}(\bar{t}^+)$, we obtain differentiable flows in a neighborhood of $\bar{t}$ that are equilibria wherever the active regime coincides with $\mathcal{R}$. Now, because the edge costs are strictly increasing, the equilibrium load vector $\bar{\boldsymbol{x}}$ is unique, and then it follows by continuity that the flows satisfy $\boldsymbol{f}^-(\boldsymbol{\mu}(\bar{t})) = \boldsymbol{f}^+(\boldsymbol{\mu}(\bar{t}))$, so that we can apply Lemma 5.1 to conclude the proof. □

Proof of Theorem 4.4(ii). When the cost functions are affine, to the left and to the right of $\bar{t}$ we can choose equilibrium flows in affine form

$$\boldsymbol{f}^{-}(\boldsymbol{\mu}(t)) = w^{-} \cdot t + z^{-}, \quad \boldsymbol{f}^{+}(\boldsymbol{\mu}(t)) = w^{+} \cdot t + z^{+}, \tag{5.14}$$

with $\boldsymbol{f}^{-}(\boldsymbol{\mu}(\bar{t})) = \boldsymbol{f}^{+}(\boldsymbol{\mu}(\bar{t}))$. A proof of this in the case of a single commodity can be found in Cominetti et al. (2024b, proposition 4.1). The reasoning for the multi-commodity case is completely analogous.

These functions are differentiable as functions of t so that Lemma 5.1 implies that (a) and (b) in Conjecture 4.2 hold with weak inequalities. It remains to show that the inequalities are strict. By arguing as in the proof of Lemma 5.1, we need to show that the inequality between $\theta(\bar{t}^{-})$ and $\theta(\bar{t}^{+})$ is strict, which requires to compare the values of the problems (5.8) and (5.11). As one can deduce looking at (5.7), the two problems are exactly the problems $\left(\mathsf{P}^{\mathcal{R}^{+}}_{\boldsymbol{\mu}(\bar{t})}\right)$ and $\left(\mathsf{P}^{\mathcal{R}^{-}}_{\boldsymbol{\mu}(\bar{t})}\right)$ as in Lemma A.1 for a game on the graph G, where the cost functions are the linear functions $\bar{c}_e(x) = c_e'(\bar{x}_e) \cdot x$, with $\bar{x}_e = \sum_{p \ni e} f_p^{-}(\mu(\bar{t})) = \sum_{p \ni e} f_p^{+}(\mu(\bar{t}))$.

In these two problems, the objective function is the same, but there are more constraints in the case associated to the smaller regime between $\mathcal{R}^{-}$ and $\mathcal{R}^{+}$. Suppose for instance that $\mathcal{R}^{-}$ is strictly contained in $\mathcal{R}^{+}$. Then the extra constraints in (5.11) with respect to (5.8) are $y_p = 0$ for $p \in \mathcal{R}^{+} \setminus \mathcal{R}^{-}$, with associated Lagrange multipliers ν_p given as in the proof of Lemma 5.1, that is, denoting $h(p)$ the commodity associated with the route p,

$$\nu_p = (\boldsymbol{c}_p \circ \boldsymbol{x}^{+})'(\bar{t}) - (\lambda^{(h(p))})'(\bar{t}^{+}). \tag{5.15}$$

The path p is active when $t > \bar{t}$, but not when $t < \bar{t}$. Hence, there exists $\varepsilon > 0$ such that

$$(c_p \circ \boldsymbol{x}^{+})(t) > \lambda^{(h(p))}(t), \quad \text{for } t \in (\bar{t} - \varepsilon, \bar{t}) \tag{5.16}$$

because p is not active, and

$$(c_p \circ \boldsymbol{x}^{+})(\bar{t}) = \lambda^{(h(p))}(\bar{t}) \tag{5.17}$$

by continuity of path costs.

By assumption, the cost functions are affine; hence the path costs and equilibrium costs are also affine in the variable $t \in (\bar{t}-\varepsilon, \bar{t})$. This implies that the difference $(c_p \circ \boldsymbol{x}^{+})'(t) - (\lambda^{(h(p))})'(t)$ is a strictly negative constant on $(\bar{t} - \varepsilon, \bar{t})$. Hence, the Lagrange multiplier ν_p assumes a strictly negative value at the optimum solution. Since the multipliers are unique for the problem $\left(\mathsf{P}^{\mathcal{R}^{-}}_{\boldsymbol{\mu}(\bar{t})}\right)$ by Lemma A.1, relaxing the constraint $y_p = 0$ in (5.11) will produce a strict decrease in the optimum value of the objective function, that is, $\theta(\bar{t}^{-}) > \theta(\bar{t}^{+})$, which gives in turn the strict inequalities in the statement (a) of Conjecture 4.2.

A similar argument holds for case (b) in Conjecture 4.2 when $\mathcal{R}^{-}$ strictly contains $\mathcal{R}^{+}$. $\square$

Remark 5.3. To have statements (a) and (b) in Conjecture 4.2, linearity of the demand function is necessary. Indeed, as shown by Example 4.3, Lemma 5.1 can fail even if the demand function is affine (and even when restricting to affine cost functions). Hence Theorem 4.4 cannot be applied.

CRediT authorship contribution statement

Roberto Cominetti: Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Valerio Dose:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Marco Scarsini:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors did not use any AI.

Acknowledgments

We thank two reviewers for their careful reading of the paper and their useful remarks and suggestions. Valerio Dose and Marco Scarsini are members of GNAMPA-INdAM. Their work was partially supported by the GNAMPA project CUP_E53C22001930001 *"Limiting behavior of stochastic dynamics in the Schelling segregation model"*. Marco Scarsini's work is partially supported by the MIUR PRIN 2022EKNE5K grant *"Learning in markets and society"*. Roberto Cominetti's research was supported by Proyecto Anillo ANID/PIA/ACT192094.

Appendix A. Missing proofs

Proof of Lemma 3.4. The assumptions on the cost functions imply that the optimum flows are the Wardrop equilibria of the game $(G, \widetilde{c}, \mathcal{H})$, where

$$\widetilde{c}_e(x_e) := c_e(x_e) + x_e c_e'(x_e). \tag{A.1}$$

For every $e \in \mathcal{E}$, the function $\widetilde{c}_e$ is continuous and nondecreasing. For this modified game, the minimal value of the Beckmann potential is

$$\widetilde{V}(\boldsymbol{\mu}) = \min_{\boldsymbol{x}\in\mathcal{X}_\mu} \sum_{e\in\mathcal{E}} x_e c_e(x_e), \tag{A.2}$$

which is the optimum social cost $\widetilde{\mathsf{SC}}(\boldsymbol{\mu})$. The result then follows by noting that the minimal value of the Beckmann potential is continuously differentiable with its gradient equal to the vector equilibrium costs, which are continuous (see Lemma 3.3). □

The following technical lemma is used in the proof of Theorem 3.2. In addition to the optimization problem $(\mathsf{P}_{\mu}^{\mathcal{R}})$ we consider the optimal value function for the perturbed minimization problem

$$\begin{aligned} V_{\mathcal{R}}(\boldsymbol{\mu}, \boldsymbol{\xi}, \boldsymbol{\omega}) = \min_{(\boldsymbol{x},\boldsymbol{f})} & \ \Phi(\boldsymbol{x}) \\ \text{s.t. } & S\boldsymbol{f} = \boldsymbol{\mu}, \ \boldsymbol{x} = \Delta\boldsymbol{f} + \boldsymbol{\xi}, \ \text{and } f_p = \omega_p \text{ for all } p \notin \mathcal{R}, \end{aligned} \tag{$\mathsf{P}_{\mu,\xi,\omega}^{\mathcal{R}}$}$$

where $\boldsymbol{\xi}$ is a perturbation of the load vector and $\boldsymbol{\omega}$ is the flow perturbation vector.

Lemma A.1. *Let $(G, \boldsymbol{c}, \mathcal{H})$ be a routing game structure and $\mathcal{R}$ a given fixed regime. Then,*

(a) *For each $\boldsymbol{\mu} \in \mathbb{R}_+^{\mathcal{H}}$ the minimum in $(\mathsf{P}_{\mu}^{\mathcal{R}})$ is attained at some $\boldsymbol{x} \in \mathcal{X}_{\mu}^{\mathcal{R}}$. The edge costs $\eta_e := c_e(x_e)$ are the same for every optimum $\boldsymbol{x}$, and for each $h \in \mathcal{H}$ there exists a path cost $m^{(h)}$ such that $\sum_{e\in p} \eta_e = m^{(h)}$ for all $p \in \mathcal{P}^{(h)} \cap \mathcal{R}$. Moreover, the optimal value function $V_{\mathcal{R}}(\cdot)$ is everywhere finite, convex, and differentiable at $(\boldsymbol{\mu}, \mathbf{0}, \mathbf{0})$, with $\nabla V_{\mathcal{R}}(\boldsymbol{\mu}, \mathbf{0}, \mathbf{0}) = (\boldsymbol{m}, \boldsymbol{\eta}, \boldsymbol{v})$ where*

$$v_p = \sum_{e\in p} \eta_e - m^{(h)} \quad \forall p \in \mathcal{P}^{(h)} \setminus \mathcal{R}. \tag{A.3}$$

(b) *If the cost functions are strictly increasing, then $(\mathsf{P}_{\mu}^{\mathcal{R}})$ has a unique optimal solution $\boldsymbol{x}_{\mathcal{R}}(\boldsymbol{\mu})$. Moreover, if the costs c_e are C^1 with strictly positive derivative, then $\boldsymbol{\mu} \to \boldsymbol{x}_{\mathcal{R}}(\boldsymbol{\mu})$ is also C^1.*

Proof. (a) In view of our extension of the costs $c_e(\cdot)$ to $\mathbb{R}_-$, we have $\lim_{x\to-\infty} c_e(x) < 0$ and also $\lim_{x\to\infty} c_e(x) > 0$. Hence, using recession analysis, for each nonzero direction $\boldsymbol{d} \in \mathbb{R}^{\mathcal{E}} \setminus \{0\}$ we have

$$\Phi^{\infty}(\boldsymbol{d}) = \lim_{t\to\infty} \Phi(t\boldsymbol{d})/t = \sum_{e\in\mathcal{E}} \lim_{t\to\infty} c_e(td_e)d_e > 0, \tag{A.4}$$

so that Φ is inf-compact, and therefore the set of minima of $(\mathsf{P}_{\mu,\xi,\omega}^{\mathcal{R}})$ is nonempty (see for example Rockafellar (1997, Theorem 9.2)). This shows in particular that the value function $V_{\mathcal{R}}(\cdot)$ is finite everywhere. Hence, by convex duality, this function is convex and the subdifferential $\partial V_{\mathcal{R}}(\boldsymbol{\mu}, \mathbf{0}, \mathbf{0})$ coincides with the optimum solution set of the dual problem. We next characterize this dual and prove that it has a unique solution, from where we deduce that it is differentiable with $\nabla V_{\mathcal{R}}(\boldsymbol{\mu}, \mathbf{0}, \mathbf{0})$ given as in the statement of part (a).

To write the dual problem we write explicitly the problem $(\mathsf{P}_{\mu}^{\mathcal{R}})$ as

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{f}} \sum_{e\in\mathcal{E}} C_e(x_e) \quad \text{s.t. } & \sum_{p\in\mathcal{P}^{(h)}} f_p = \mu^{(h)} \text{ for all } h \in \mathcal{H}, \text{ and } f_p = 0 \text{ for } p \notin \mathcal{R}, \\ & x_e = \sum_{p\in\mathcal{P}} \delta_{ep} f_p \text{ for all } e \in \mathcal{E}. \end{aligned} \tag{A.5}$$

Introducing multipliers $\boldsymbol{m} = (m^{(h)})_{h\in\mathcal{H}}$, $\boldsymbol{v} = (v_p)_{p\notin\mathcal{R}}$, and $\boldsymbol{\eta} = (\eta_e)_{e\in\mathcal{E}}$, and the Lagrangian

$$\begin{aligned} \mathcal{L}(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{m}, \boldsymbol{v}, \boldsymbol{\eta}) &= \sum_{e\in\mathcal{E}} C_e(x_e) + \sum_{h\in\mathcal{H}} m^{(h)} \left(\mu^{(h)} - \sum_{p\in\mathcal{P}^{(h)}} f_p \right) - \sum_{p\notin\mathcal{R}} v_p f_p \\ &\quad + \sum_{e\in\mathcal{E}} \eta_e \left(\sum_{p\in\mathcal{P}} \delta_{ep} f_p - x_e \right) \\ &= \sum_{e\in\mathcal{E}} C_e(x_e) - \eta_e x_e + \sum_{h\in\mathcal{H}} m^{(h)}\mu^{(h)} + \sum_{p\in\mathcal{R}} f_p \left(\sum_{e\in p} \eta_e - m^{(h(p))} \right) \\ &\quad + \sum_{p\notin\mathcal{R}} f_p \left(\sum_{e\in p} \eta_e - m^{(h(p))} - v_p \right), \end{aligned} \tag{A.6}$$

where $h(p)$ is the commodity of path p, the dual problem becomes

$$\sup_{\boldsymbol{m},\boldsymbol{\nu},\boldsymbol{\eta}} \min_{\boldsymbol{x},\boldsymbol{f}} \mathcal{L}(\boldsymbol{x},\boldsymbol{f},\boldsymbol{m},\boldsymbol{\nu},\boldsymbol{\eta}). \tag{A.7}$$

The inner minimum over $(\boldsymbol{x},\boldsymbol{f})$ can be solved explicitly to obtain the dual in final form as

$$\begin{aligned} \sup_{\boldsymbol{m},\boldsymbol{\nu},\boldsymbol{\eta}} -\sum_{e\in\mathcal{E}} C_e^*(\eta_e) + \sum_{h\in\mathcal{H}} m^{(h)}\mu^{(h)} \quad \text{s.t.} \quad & m^{(h(p))} = \sum_{e\in p} \eta_e \quad \text{for every } p \in \mathcal{R}, \\ & m^{(h(p))} = \sum_{e\in p} \eta_e - \nu_p \quad \text{for every } p \notin \mathcal{R}, \end{aligned} \tag{A.8}$$

where C_e^* is the Fenchel conjugate of C_e.

This amounts to solving

$$\min_{\boldsymbol{m},\boldsymbol{\eta}} \sum_{e\in\mathcal{E}} C_e^*(\eta_e) - \sum_{h} m^{(h)}\mu^{(h)} \quad \text{s.t.} \quad \sum_{e\in p} \eta_e = m^{(h(p))} \quad \text{for every } p \in \mathcal{R}, \tag{A.9}$$

and then defining

$$\nu_p = \sum_{e\in p} \eta_e - m^{(h(p))} \quad \text{for every } p \notin \mathcal{R}. \tag{A.10}$$

As mentioned above, from general convex duality, the optimum solution set of this dual coincides with the sub-differential $\partial V_{\mathcal{R}}(\boldsymbol{\mu},\mathbf{0},\mathbf{0})$ of the primal optimum value function. Since $V_{\mathcal{R}}(\cdot)$ is finite everywhere, this sub-differential is nonempty and the dual has optimum solutions. On the other hand, since C_e^* is strictly convex, the dual optimum solution $(\boldsymbol{m},\boldsymbol{\nu},\boldsymbol{\eta})$ is unique and therefore $V_{\mathcal{R}}(\cdot)$ is differentiable at $(\boldsymbol{\mu},\mathbf{0},\mathbf{0})$ with $\nabla V_{\mathcal{R}}(\boldsymbol{\mu},\mathbf{0},\mathbf{0}) = (\boldsymbol{m},\boldsymbol{\nu},\boldsymbol{\eta})$. Writing the optimality conditions for (A.5) we obtain $c_e(x_e) = \eta_e$, and the characterization of the gradient follows from (A.9) and (A.10). This establishes part (a) of the lemma.

(b) Note that the relaxed constraints in ($\mathsf{P}_{\boldsymbol{\mu}}^{\mathcal{R}}$) are just linear equality constraints. Projecting these equations onto the space of pairs $(\boldsymbol{x},\boldsymbol{\mu})$, one can remove the flow variables $\boldsymbol{f}$ and rewrite the constraints as a linear system of equations in the load variables $\boldsymbol{x}$ only, that is,

$$A\boldsymbol{x} + B\boldsymbol{\mu} = \mathbf{0}, \tag{A.11}$$

where the matrices A and B depend on the regime $\mathcal{R}$, and the matrix $\begin{bmatrix} A & B \end{bmatrix}$ has linearly independent rows. Hence, the problem can be restated in the equivalent form

$$\min_{\boldsymbol{x}\in\mathbb{R}^{\mathcal{E}}} \{\Phi(\boldsymbol{x}) : A\boldsymbol{x} + B\boldsymbol{\mu} = \mathbf{0}\}. \tag{$\mathsf{P}_{\boldsymbol{\mu}}^{\mathcal{R}}$}$$

Introducing a Lagrange multiplier vector $\boldsymbol{\alpha}$, the optimality conditions are

$$\begin{aligned} \nabla\Phi(\boldsymbol{x}) + A^\top\boldsymbol{\alpha} &= \mathbf{0}, \\ A\boldsymbol{x} + B\boldsymbol{\mu} &= \mathbf{0}. \end{aligned} \tag{A.12}$$

We note that for each $\boldsymbol{\mu} \in \mathbb{R}_+^{\mathcal{H}}$ the (unique) optimum solution $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\mu})$ also has a unique corresponding multiplier $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\mu})$. This follows from $\operatorname{Ker} A^\top = \{\mathbf{0}\}$. To prove the latter, suppose that $A^\top\boldsymbol{\beta} = \mathbf{0}$ for some vector $\boldsymbol{\beta}$. For each $\boldsymbol{\mu} \in \mathbb{R}_+^{\mathcal{H}}$ we can find a feasible $\boldsymbol{x} \in \mathcal{X}_{\boldsymbol{\mu}}$ such that $A\boldsymbol{x} + B\boldsymbol{\mu} = \mathbf{0}$, by just routing any flow of demand $\boldsymbol{\mu}$ that uses the regime $\mathcal{R}$. We have then

$$0 = \langle \boldsymbol{\beta}, A\boldsymbol{x} + B\boldsymbol{\mu}\rangle = \langle B^\top\boldsymbol{\beta}, \boldsymbol{\mu}\rangle. \tag{A.13}$$

for all $\boldsymbol{\mu} \in \mathbb{R}_+^{\mathcal{H}}$, from which we deduce $B^\top\boldsymbol{\beta} = \mathbf{0}$. Hence $\begin{bmatrix} A & B \end{bmatrix}^\top \boldsymbol{\beta} = \mathbf{0}$ and therefore $\boldsymbol{\beta} = \mathbf{0}$, because the rows of $\begin{bmatrix} A & B \end{bmatrix}$ are linearly independent.

The conclusion of the lemma follows from the implicit function theorem if we show that the left hand side of (A.12) has a nonsingular Jacobian at any point $(\boldsymbol{x},\boldsymbol{\alpha})$.

This amounts to proving that the following homogeneous linear system

$$\begin{aligned} \nabla^2\Phi(\boldsymbol{x})\boldsymbol{d} + A^\top\boldsymbol{\beta} &= \mathbf{0} \\ A\boldsymbol{d} &= \mathbf{0} \end{aligned} \tag{A.14}$$

admits only the trivial solution. Multiplying the first equation by $\boldsymbol{d}$ and using the second equation it follows that

$$0 = \langle \boldsymbol{d}, \nabla^2\Phi(\boldsymbol{x})\boldsymbol{d}\rangle + \langle \boldsymbol{d}, A^\top\boldsymbol{\beta}\rangle = \langle \boldsymbol{d}, \nabla^2\Phi(\boldsymbol{x})\boldsymbol{d}\rangle = \sum_{e\in\mathcal{E}} c_e'(x_e)d_e^2, \tag{A.15}$$

which implies $\boldsymbol{d} = \mathbf{0}$ because $c_e'(x_e) > 0$. Hence the first equation reduces to $A^\top\boldsymbol{\beta} = \mathbf{0}$, which implies $\boldsymbol{\beta} = \mathbf{0}$, because $\operatorname{Ker}(A^\top) = \{\mathbf{0}\}$, as shown above. This completes the proof. □

Remark A.2. Lemma A.1 and Theorem 3.2 can be generalized to games with cost functions that are just strictly increasing, assuming that the set of edges e where $c'(x_e(\boldsymbol{\mu})) = 0$ form a graph without undirected cycles. Indeed, (A.14) implies that for every active edge e, either $d_e = 0$ or $c_e'(x_e) = 0$, whereas the condition $A\boldsymbol{d} = \mathbf{0}$ tells us that the vector $\boldsymbol{d}$ is a (possibly negative) flow of zero demand on our network. Since $d_e = 0$ for all edges such that $c_e'(x_e) > 0$, the vector $\boldsymbol{d}$ induces a flow of zero demand on the subnetwork formed by edges e such that $c_e'(x_e) = 0$. If the latter subnetwork has no undirected cycles, then we can conclude that $\boldsymbol{d} = \mathbf{0}$ and complete the proof as above.

Appendix B. List of symbols

c_e	Cost of edge e
$\boldsymbol{c}$	Edge Cost function vector
c_p	Cost of path p
$C_e(x_e)$	$\int_0^{x_e} c_e(z)\,\mathrm{d}z$
$C_e^*(\cdot)$	Fenchel conjugate of $C_e(\cdot)$
$\mathsf{D}^{(h)}$	Destination of OD pair h
e	Edge
$\mathcal{E}$	Set of edges
f_p	Flow on path p
f_p^*	Equilibrium flow on path p
$\boldsymbol{f}$	Flow vector
$\boldsymbol{f}^*$	Equilibrium flow vector
$\mathcal{F}_{\boldsymbol{\mu}}$	$\{\boldsymbol{f} \in \mathbb{R}_+^{\mathcal{P}} : \sum_{p\in\mathcal{P}^{(h)}} f_p = \mu^{(h)}$ for all $h \in \mathcal{H}\}$, set of feasible flows, defined in (2.2)
G	$(\mathcal{V}, \mathcal{E})$, directed multigraph with set of vertices $\mathcal{V}$ and set of edges $\mathcal{E}$
$\mathcal{H}$	Set of OD pairs
H	Number of elements in $\mathcal{H}$
I^-	$(\bar{t} - \varepsilon, \bar{t})$, defined in Lemma 5.1
I^+	$(\bar{t}, \bar{t} + \varepsilon)$, defined in Lemma 5.1
$J_{\boldsymbol{c}}(\cdot)$	Jacobian matrix of the path costs function $\boldsymbol{c} : \mathbb{R}_+^n \to \mathbb{R}_+^n$
$m^{(h)}$	Lagrange multiplier associated to OD pair h
$\boldsymbol{m}$	OD-pairs Lagrange multiplier vector
N	Neighborhood
$\mathsf{O}^{(h)}$	Origin of OD pair h
p	Path
$\mathcal{P}^{(h)}$	Set of paths of OD pair h
$P^{(h)}$	Number of elements in $\mathcal{P}^{(h)}$
P	Number of elements in $\mathcal{P}$
$\mathcal{P}$	$\bigcup_{h\in\mathcal{H}} \mathcal{P}^{(h)}$, union of the path sets of every OD pair, defined in (2.1)
$\widehat{\mathcal{P}}(\boldsymbol{\mu})$	Active regime, defined in (2.10)
$\widehat{\mathcal{P}}(\bar{t}^-)$	$\widehat{\mathcal{P}}(\boldsymbol{\mu}(\bar{t} - \varepsilon))$, defined in (4.1)
$\widehat{\mathcal{P}}(\bar{t}^+)$	$\widehat{\mathcal{P}}(\boldsymbol{\mu}(\bar{t} + \varepsilon))$, defined in (4.1)
$\mathsf{PoA}(\boldsymbol{\mu})$	Price of anarchy with demand $\boldsymbol{\mu}$, defined in (2.13)
$r^{(h)}$	Rate of OD pair h for linearly increasing demand
$\boldsymbol{r}$	Rate vector for linearly increasing demand
$\mathcal{R}$	Regime, defined in Definition 2.2
$\mathcal{R}^-$	$\widehat{\mathcal{P}}(\bar{t}^-)$, defined in Lemma 5.1
$\mathcal{R}^+$	$\widehat{\mathcal{P}}(\bar{t}^+)$, defined in Lemma 5.1
S	Defined in (2.4)
SC	Social cost, defined in (2.6)
$\mathsf{SC}^*(\boldsymbol{\mu})$	Equilibrium social cost with demand $\boldsymbol{\mu}$, defined in (2.11)
$\widetilde{\mathsf{SC}}(\boldsymbol{\mu})$	Optimum social cost with demand $\boldsymbol{\mu}$, defined in (2.12)
$\mathcal{V}$	Set of vertices
x_e	Load on edge e
$\boldsymbol{x}$	Load vector
$\boldsymbol{x}_{\mathcal{R}}(\boldsymbol{\mu})$	Optimum solution of $(\mathsf{P}_{\boldsymbol{\mu}}^{\mathcal{R}})$
$\mathcal{X}_{\boldsymbol{\mu}}$	Set of load profiles induced by some flow of demand $\boldsymbol{\mu}$
y_p	Introduced in (5.8)
$\boldsymbol{y}$	Introduced in (5.7)
$\boldsymbol{\alpha}$	Lagrange multiplier vector
$\boldsymbol{\beta}$	Lagrange multiplier vector
$\boldsymbol{\gamma}_p$	p-th element of the canonical basis
δ_{ep}	$\begin{cases} 1 & \text{if } e \in p, \\ 0 & \text{otherwise.} \end{cases}$, defined in (2.3)
Δ	Defined in (2.4)
η_e	Lagrange multiplier associated to edge e
$\boldsymbol{\eta}$	Edge Lagrange multiplier vector

$\theta(t)$	$\sum_{h\in\mathcal{H}} r^{(h)} \cdot (\lambda^{(h)})'(t)$, defined in Eq. (5.2)
$\lambda^{(h)}(\boldsymbol{\mu})$	Equilibrium cost of OD pair h with demand $\boldsymbol{\mu}$
$\boldsymbol{\lambda}$	Equilibrium cost vector
$\mu^{(h)}$	Demand at OD pair h
$\boldsymbol{\mu}$	$(\mu^{(1)}, \ldots, \mu^{(H)})$, demand vector
ν_p	Lagrange multiplier associated to path p
$\boldsymbol{\nu}$	Path Lagrange multiplier vector
ξ_e	Perturbation of load x_e
$\boldsymbol{\xi}$	Perturbation of load vector $\boldsymbol{x}$
τ_e	$c_e(x_e^*)$ equilibrium cost of edge e
$\boldsymbol{\Phi}(\boldsymbol{x})$	$\sum_{e\in\mathcal{E}} C_e(x_e)$ potential
ω_p	Perturbation of flow f_p
$\boldsymbol{\omega}$	Perturbation of flow vector $\boldsymbol{f}$
$\mathbf{0}$	Zero vector
$\mathbf{1}^{(h)}$	$\sum_{p\in h} \boldsymbol{\gamma}_p$

References

Beckmann, M.J., McGuire, C., Winsten, C.B., 1956. Studies in the Economics of Transportation. Yale University Press, New Haven, CT.

Colini-Baldeschi, R., Cominetti, R., Mertikopolous, P., Scarsini, M., 2020. When is selfish routing bad? The price of anarchy in light and heavy traffic. Oper. Res. 68 (2), 411–434.

Colini-Baldeschi, R., Cominetti, R., Scarsini, M., 2019. Price of anarchy for highly congested routing games in parallel networks. Theory Comput. Syst. 63 (1), 90–113.

Cominetti, R., Dose, V., Scarsini, M., 2024a. Monotonicity of equilibria in nonatomic congestion games. European J. Oper. Res. (forthcoming).

Cominetti, R., Dose, V., Scarsini, M., 2024b. The price of anarchy in routing games as a function of the demand. Math. Program. 203 (1–2), 531–558.

Correa, J.R., Schulz, A.S., Stier-Moses, N.E., 2004. Selfish routing in capacitated networks. Math. Oper. Res. 29 (4), 961–976.

Correa, J.R., Schulz, A.S., Stier-Moses, N.E., 2007. Fast, fair, and efficient flows in networks. Oper. Res. 55 (2), 215–225.

Correa, J.R., Schulz, A.S., Stier-Moses, N.E., 2008. A geometric approach to the price of anarchy in nonatomic congestion games. Games Econom. Behav. 64 (2), 457–469.

Correa, J.R., Stier-Moses, N.E., 2011. Wardrop equilibria. In: Cochran, J.J. (Ed.), Encyclopedia of Operations Research and Management Science. Wiley.

Dafermos, S., Nagurney, A., 1984. Sensitivity analysis for the asymmetric network equilibrium problem. Math. Program. 28 (2), 174–184.

Dafermos, S.C., Sparrow, F.T., 1969. The traffic assignment problem for a general network. J. Res. Nat. Bur. Stand. Sect. B 73B, 91–118.

Dumrauf, D., Gairing, M., 2006. Price of anarchy for polynomial Wardrop games. In: WINE '06: Proceedings of the 2nd Conference on Web and Internet Economics. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 319–330.

Englert, M., Franke, T., Olbrich, L., 2010. Sensitivity of Wardrop equilibria. Theory Comput. Syst. 47 (1), 3–14.

Fisk, C., 1979. More paradoxes in the equilibrium assignment problem. Transp. Res. B 13 (4), 305–309.

Florian, M., Hearn, D., 1995. Network equilibrium models and algorithms. In: Hall, M., et al. (Eds.), Handbooks in Operation Research and Management Science. Vol. 8, North Holland, Amsterdam.

Fukushima, M., 1984. On the dual approach to the traffic assignment problem. Transp. Res. B 18 (3), 235–245.

Hall, M.A., 1978. Properties of the equilibrium state in transportation networks. Transp. Sci. 12 (3), 208–216.

Josefsson, M., Patriksson, M., 2007. Sensitivity analysis of separable traffic equilibrium equilibria with application to bilevel optimization in network design. Transp. Res. B 41 (1), 4–31.

Klimm, M., Warode, P., 2019. Computing all wardrop equilibria parametrized by the flow demand. In: Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms. SIAM, Philadelphia, PA, pp. 917–934.

Klimm, M., Warode, P., 2022. Parametric computation of minimum-cost flows with piecewise quadratic costs. Math. Oper. Res. 47 (1), 812–846.

Koutsoupias, E., Papadimitriou, C., 1999. Worst-case equilibria. In: STACS 99 (Trier). In: Lecture Notes in Comput. Sci., vol. 1563, Springer, Berlin, pp. 404–413.

Monnot, B., Benita, F., Piliouras, G., 2017. Routing games in the wild: efficiency, equilibration and regret. In: R. Devanur, N., Lu, P. (Eds.), Web and Internet Economics. Springer International Publishing, Cham, pp. 340–353.

O'Hare, S.J., Connors, R.D., Watling, D.P., 2016. Mechanisms that govern how the price of anarchy varies with travel demand. Transp. Res. B 84, 55–80.

Papadimitriou, C., 2001. Algorithms, games, and the Internet. In: Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing. ACM, New York, pp. 749–753.

Patriksson, M., 2004. Sensitivity analysis of traffic equilibria. Transp. Sci. 38 (3), 258–281.

Pigou, A.C., 1920. The Economics of Welfare, first ed. Macmillan and Co., London.

Rockafellar, R.T., 1997. Convex analysis. In: Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, p. xviii+451, Reprint of the 1970 original, Princeton Paperbacks.

Roughgarden, T., 2003. The price of anarchy is independent of the network topology. J. Comput. System Sci. 67 (2), 341–364.

Roughgarden, T., Tardos, É., 2002. How bad is selfish routing? J. ACM 49 (2), 236–259.

Roughgarden, T., Tardos, É., 2004. Bounding the inefficiency of equilibria in nonatomic congestion games. Games Econom. Behav. 47 (2), 389–403.

Takalloo, M., Kwon, C., 2020. Sensitivity of Wardrop equilibria: revisited. Optim. Lett. 14 (3), 781–796.

Tomlin, J.A., 1966. Minimum-cost multicommodity network flows. Oper. Res. 14 (1), 45–51.

Verbree, J., Cherukuri, A., 2023. Wardrop equilibrium and Braess's paradox for varying demand. Technical Report, arXiv preprint arXiv:2310.04256.

Wardrop, J.G., 1952. Some theoretical aspects of road traffic research. In: Proceedings of the Institute of Civil Engineers, Part II. Vol. 1, pp. 325–378.

Wu, Z., Möhring, R., 2023. A sensitivity analysis for the price of anarchy in non-atomic congestion games. Math. Oper. Res. 48 (3), 1364–1392.

Wu, Z., Möhring, R.H., Chen, Y., Xu, D., 2021. Selfishness need not be bad. Oper. Res. 69 (2), 410–435.

Youn, H., Gastner, M.T., Jeong, H., 2008. Price of anarchy in transportation networks: Efficiency and optimality control. Phys. Rev. Lett. 101 (12), 128701.

Youn, H., Gastner, M.T., Jeong, H., 2009. Erratum: Price of anarchy in transportation networks: Efficiency and optimality control [Phys. Rev. Lett. **101**, 128701 (2008)]. Phys. Rev. Lett. 102, 049905.