

Inheritance of weighted average-convexity in communication situations

Pre-print version of the following publication: | Versione pre-print della seguente pubblicazione:

Original Citation/Citazione:

Skoda, Alexandre; Venel, Xavier Mathieu Raymond. (2025). Inheritance of weighted average-convexity in communication situations. INTERNATIONAL JOURNAL OF GAME THEORY, (ISSN: 0020-7276), 54:1, ---. Doi: 10.1007/s00182-025-00947-1.

Availability/Disponibilità:

This version is available at: [11385/254040](https://iris.luiss.it/11385/254040) since: 2025-10-17T13:33:24Z - Questa versione è disponibile alla pagina: [11385/254040](https://iris.luiss.it/11385/254040) dal: 2025-10-17T13:33:24Z

Publisher/Casa editrice:

Published version/Pubblicato:

DOI: <https://dx.doi.org/10.1007/s00182-025-00947-1>

License/Licenza:

Tutti i diritti riservati

Availability/Termini d'uso:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. Works made available under a Creative Commons license can be used according to the terms and conditions of said license. For all terms of use and more information see the publisher's website. | I termini e le condizioni relativi al riutilizzo della presente versione della pubblicazione sono disciplinati dalla politica editoriale. Le opere messe a disposizione con licenze Creative Commons possono essere utilizzate conformemente ai termini e alle condizioni previste da tali licenze. Per l'insieme delle condizioni di utilizzo e per ulteriori informazioni si rinvia al sito web dell'editore.

This item was downloaded from IRIS Luiss (<https://iris.luiss.it/>). When citing, please refer to the published version. | Questo documento è stato scaricato da IRIS Luiss (<https://iris.luiss.it/>). Per la citazione, fare riferimento alla versione pubblicata sul sito dell'editore.

(Article begins on next page | Il contributo inizia nella pagina successiva)

Inheritance of Weighted Average-convexity in Communication Situations

Alexandre Skoda*, Xavier Venel †

October 17, 2025

Abstract

We study the relation between weighted average-convexity and communication situations. We exhibit necessary conditions for a communication situation to preserve the weighted average-convexity from the initial game to the graph-restricted game and we provide a complete characterization.

AMS Classification: 91A12, 91A43, 90C27, 05C75.

Keywords: TU-games, convexity, average-convexity, weighted Shapley value, communication.

1 Introduction

We consider a situation where a set of players $N = \{1, \dots, n\}$ can cooperate to generate some profit. It is usually assumed that any coalition $S \subseteq N$ of players is feasible and that the profit generated by the players in S can be freely distributed among them. This kind of situation corresponds to a cooperative game with transferable utility. One of the key questions in this framework is how to allocate to each player his share of the profit generated by the grand coalition N . Any share of the total worth of N between the n players is called a solution of the game.

Lots of solution concepts have been introduced and they can be divided into two different families: single-valued solutions and set-valued solutions. The best-known single-valued solution in cooperative game theory is the Shapley value introduced in Shapley

*Université de Paris I, Centre d'Economie de la Sorbonne, 106-112 Bd de l'Hôpital, 75013 Paris, France, askoda@univ-paris1.fr

†Dipartimento di Economia e Finanza, LUISS, Viale Romania 32, 00197 Rome, Italy, xvenel@luiss.it, <https://orcid.org/0000-0003-1150-9139>. Xavier Venel is part of the Italian MIUR PRIN 2022 project “Learning in Markets and Society” (project 2022EKNE5K).

(1953b) and generalized to weighted Shapley values in Shapley (1953a) and Kalai and Samet (1987). A typical set-valued solution is the core (Gillies (1959)). The core is the set of allocations that are efficient and coalitionally rational, two particularly desirable properties in the context of cooperative games.

A key question is when the weighted Shapley value is in the core since it is not true in general. Shapley (1971) introduced the class of convex games. In a convex game, the marginal contribution of a player is an increasing function with respect to the coalition size. If a game is convex, then the weighted Shapley value belongs to the core (Shapley (1971); Monderer et al. (1992)). Moreover, Monderer et al. (1992) showed that for any game one can obtain all the elements of the core as weighted Shapley values by varying the weight system. A weight system is a pair composed of a weight vector and an ordered partition of the players. The partition represents the relative priorities of the players whereas the weights represent the importance of each player in each priority level.

In fact, the convexity assumption is very strong and can be relaxed to ensure that the Shapley value is a core element. Iñarra and Usategui (1993) introduced a weaker notion than convexity, called average-convexity¹ and proved that it is a sufficient condition for the Shapley value to be in the core. Skoda and Venel (2023) introduced weighted average-convexity and proved that it is a sufficient condition for the weighted Shapley value to be in the core.

In many situations, the cooperation of players may be restricted by some communication or social structures. Then, the worths of coalitions have to be modified to take these restrictions into account, leading to the introduction of restricted games. Myerson (1977) introduced the notion of communication situation defined as a game and a graph on the same set of players. Players in a coalition have to be connected either directly or by mean of other players in the coalition to be able to cooperate. Formally, he defines a graph-restricted game where each coalition is split into connected subcoalitions and its value in the graph-restricted game is the sum of the values of the subcoalitions in the initial game. Then, the classical solution concepts can be applied to this new game. In particular, the Myerson value of the communication situation is the Shapley value of the graph-restricted game. Slikker and van den Nouweland (2000) extended the Myerson value to weight systems.

A central question is the inheritance of properties from the original game (without the graph structure) to the associated graph-restricted game. For a general overview, we refer to Slikker (2000) and Slikker and van den Nouweland (2001). More recent results are given for the inheritance of some properties related to the nucleolus by Schouten et al. (2022).

Our focus will be on convexity and its variants. van den Nouweland and Borm (1991)

¹Sprumont (1990) established a result similar to the one of Iñarra and Usategui (1993) but in a different framework as he investigated population monotonic allocation schemes (PMAS).

proved that there is inheritance of convexity if and only if the graph is cycle-complete. Slikker (2000) showed that there is inheritance of average-convexity if and only if connected components of the graph are complete graphs or stars. We investigate the question of inheritance of weighted average-convexity for communication situations. We exhibit necessary and sufficient conditions on the underlying graph to preserve weighted average-convexity from any game to the associated graph-restricted game.

When there is a unique priority, we prove that the conditions given in Slikkers's characterization are also necessary using more general counter-examples taking into account the different weights of the players. In this case we get the same characterization as Slikker (2000) for inheritance of weighted average-convexity.

When there are several priorities, we provide necessary conditions giving restrictions on some specific subgraphs (cycles, paths, pans, and forks) and the players' priorities. These conditions imply that each subgraph associated with a priority level has to be composed of stars or complete graphs. Moreover, they give insights into how connected components lying in different priority levels are connected. Finally, we provide a full characterization of the class of graphs preserving weighted-average convexity.

As weighted-average convexity implies that the weighted Shapley value belongs to the core (Skoda and Venel (2023)), we obtain sufficient conditions on the game and on the graph to guarantee that the weighted Myerson value belongs to the core.

The outline of the article is the following. Section 2 provides classical definitions in cooperative game theory. We also recall the definitions of the weighted Shapley value and of weighted average-convexity. We then investigate the class of graphs ensuring inheritance of weighted average-convexity for communication situations. Section 3 provides a complete characterization (Theorem 1) of this class in the case of weight systems with a unique priority. In Section 4, we establish necessary conditions when weight systems have several priorities (Propositions 4, 5, 6, and 7). Finally, we provide a complete characterization (Theorem 2) using these previous conditions.

2 TU-games, weighted Shapley value, and weighted average-convexity

2.1 TU-Games

A *TU-game* is defined by a set of players N and a characteristic function v from 2^N to $\mathbb{R}$ that satisfies $v(\emptyset) = 0$. An *allocation* is a vector $x \in \mathbb{R}^N$ that represents the respective payoff of each player. It is said to be *efficient* if

$$\sum_{i \in N} x_i = v(N).$$

and *individually rational* if every player receives at least the value that he could guarantee alone

$$\forall i \in N, x_i \geq v(\{i\}).$$

This last property can be extended to coalitions. An allocation is *coalitionally rational* if for every coalition the sum of payoffs of its members is at least the value of the coalition

$$\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S).$$

The *core* is the set of efficient and coalitionally rational allocations. Formally,

$$\mathcal{C}(v) = \left\{ x \in \mathbb{R}^N, \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S), \forall S \subset N \right\}.$$

A player i in N is a *null player* if $v(S \cup \{i\}) = v(S)$ for every coalition $S \subseteq N \setminus \{i\}$.

A game (N, v) is *superadditive* if, for all $A, B \in 2^N$ such that $A \cap B = \emptyset$, $v(A \cup B) \geq v(A) + v(B)$.

For any given subset $\emptyset \neq S \subseteq N$, the unanimity game (N, u_S) is defined by

$$u_S(A) = \begin{cases} 1 & \text{if } A \supseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

A game (N, v) is *convex* if for all $i \in N$, and all $S \subseteq T \subseteq N$ with $i \in S$

$$v(S) - v(S \setminus \{i\}) \leq v(T) - v(T \setminus \{i\}).$$

2.2 Weighted Shapley value

We follow the definition of Kalai and Samet (1987) that contains the one of Shapley (1953a). Formally, a weight system is a pair (ω, Σ) where $\omega \in \mathbb{R}_{++}^N$ and $\Sigma = (N_1, \dots, N_m)$ is an ordered partition of N . It is called simple if Σ is equal to the singleton (N) . m will be the size of the partition. Given an element i in N , we define its priority denoted by $p(i)$ as the unique integer such that $i \in N_{p(i)}$. Given a set S , we define the priority of S , denoted by $p(S)$, as the largest $k \in \{1, \dots, m\}$ such that $N_k \cap S \neq \emptyset$. Notice that this coincides with the previous definition if S is a singleton. Moreover, priority is a nondecreasing mapping. We also define $\bar{S}$ as the elements in S with the highest priority, i.e., $\bar{S} = \{i \in S, p(i) = p(S)\}$.

Definition 1. *The weighted Shapley value with weight system (ω, Σ) is the unique linear function $\Phi^{(\omega, \Sigma)}$ from the set of TU-games to $\mathbb{R}^n$ satisfying for each unanimity game (N, u_S) and for every $i \in N$,*

$$\Phi^{(\omega, \Sigma)}(u_S)_i = \begin{cases} \frac{\omega_i}{\sum_{i \in \bar{S}} \omega_i} & \text{if } i \in \bar{S}, \\ 0 & \text{otherwise.} \end{cases}$$

It is convenient for the following results to define the relative weight of the agents in the unanimity game on S . Given a set S , one defines for all $i \in N$

$$\bar{\omega}_i^S = \begin{cases} \omega_i & \text{if } i \in \bar{S}, \\ 0 & \text{if } i \in S \setminus \bar{S}, \\ 0 & \text{if } i \notin S. \end{cases}$$

We also define $\bar{\omega}^S$ as the sum for all $i \in S$ of $\bar{\omega}_i^S$. Notice that it is equal to the sum of the weights of elements in $\bar{S}$. Moreover, the unanimity games on S and $\bar{S}$ lead to the same allocation vector.

2.3 Weighted average-convexity

Shapley (1953b) showed that the Shapley value of a convex game belongs to the core. Looking for a weaker condition insuring that the Shapley value of a game lies in the core, Iñarra and Usategui (1993) relaxed the convexity assumption by introducing the notion of average-convexity which was then extended by Skoda and Venel (2023) to the (ω, Σ) -convexity.

Definition 2. Let (ω, Σ) be a weight system, we say that the game is (ω, Σ) -convex if for every $S \subset T \subseteq N$,

$$\left(\sum_{i \in S} \bar{\omega}_i^T (v(T) - v(T \setminus \{i\}) - v(S) + v(S \setminus \{i\})) \right) \geq 0 \quad (1)$$

In particular, notice that if $p(S) < p(T)$, then for every $i \in S$, $\bar{\omega}_i^T = 0$ and the corresponding inequality is satisfied by any game. If $\Sigma = (N)$ and if all weights are equal, then (ω, Σ) -convexity corresponds to average-convexity. Let us also note that if a game is convex then it is (ω, Σ) -convex for any weight system (ω, Σ) .

In fact, one can exclude null players in Definition 2.

Proposition 1. Let (ω, Σ) be a weight system and let (N, v) be a game. Let N^* be the subset of N excluding null players. (N, v) is (ω, Σ) -convex if Inequality (1) is satisfied for every $S \subseteq T \subseteq N^*$.

Proof. For any $U \subseteq N$, we denote by U^* the subset of U without null players, i.e., $U^* = U \cap N^*$. Let us consider subsets $S \subseteq T \subseteq N$. Then,

$$\sum_{i \in S} \bar{\omega}_i^T (v(T) - v(T \setminus \{i\}) - v(S) + v(S \setminus \{i\})) \quad (2)$$

$$= \sum_{i \in S^*} \bar{\omega}_i^T (v(T) - v(T \setminus \{i\}) - v(S) + v(S \setminus \{i\})) \quad (3)$$

$$= \sum_{i \in S^*} \bar{\omega}_i^T (v(T^*) - v(T^* \setminus \{i\}) - v(S^*) + v(S^* \setminus \{i\})). \quad (4)$$

Indeed, if $i \in S$ is a null player then $v(T) - v(T \setminus \{i\}) - v(S) + v(S \setminus \{i\}) = 0$ and we can restrict the sum to S^* . Moreover, by definition of a null player, we have $v(U) = v(U^*)$ for any $U \subseteq N$, and $(U \setminus \{i\})^* = U^* \setminus \{i\}$ for every i in U leading to the last equation. Finally, there are two cases concerning the weights:

- if $p(T) > p(T^*)$, then $\overline{w}_i^T = 0$ for all i in S^* . Elements of S^* have a too low priority. Thus the inequality is trivially satisfied and can be forgotten.
- if $p(T) = p(T^*)$, then $\overline{w}_i^T = \overline{w}_i^{T^*}$ for all i in S^* and one can replace T by T^* . $\square$

Skoda and Venel (2023) obtained the following results.

Proposition 2. *Let (ω, Σ) be a weight system. If the game is (ω, Σ) -convex then the (ω, Σ) -weighted Shapley value is in the core.*

Proposition 3. *Let (ω, Σ) be a weight system. If (N, v) is an (ω, Σ) -convex game, then (N, v) is superadditive.*

3 Weighted-convexity and Communication situations: The singleton case

We first introduce some definitions of graph theory. Let $G = (N, E)$ be a graph. We say that an edge $e = \{v, w\}$ in E links the nodes v and w in N . v and w are the end-nodes of e .

Definition 3. *A path $P = (v_1, v_2, \dots, v_m)$ in G is a sequence of distinct nodes with $m \geq 2$ such that $\{v_k, v_{k+1}\} \in E$ for $k = 1, \dots, m-1$. When $m = 4$, we call it a 4-path.*

A subpath of P is a contiguous² subsequence of nodes of $(v_1, \dots, v_m)$.

A cycle $C = (v_1, v_2, \dots, v_m)$ in G is a sequence of nodes with $m \geq 2$ such that $\{v_k, v_{k+1}\} \in E$ for $k = 1, \dots, m$, $v_1 = v_{m+1}$, and $v_1, \dots, v_m$ are distinct nodes.

We denote by $V(P)$ (resp. $V(C)$) the set of nodes in P (resp. C).

Given a path $P = (v_1, \dots, v_m)$, we say that P connects the nodes v_1 and v_m . In order to simplify notation, we thereafter use index i to refer to a vertex v_i . Given two nodes i and j in P , we denote by P_{ij} the subpath of P connecting i and j .

We now define connected components in order to define formally the graph-restricted game. A graph $G = (N, E)$ is *connected* if for every pair $\{i, j\}$ of nodes in N there is a path connecting i and j . A *subgraph* of G is a graph $G' = (N', E')$ with $N' \subseteq N$ and $E' \subseteq E$. Let S be a subset of N . The *graph induced by S* , denoted G_S , is the subgraph of G containing only the nodes of S and the edges of G with both end-nodes in S . If G_S is connected, then S is a *connected subset*. A *connected component* of a graph G is a maximal connected subgraph of G .

²A contiguous subsequence is a subsequence of consecutive nodes with no gap.

Definition 4. A communication situation is a triple (N, v, E) such that (N, v) is a TU-game and $G = (N, E)$ is a graph. The graph-restricted game (N, v^G) associated with this communication situation is the TU-game with characteristic function

$$v^G(S) = \sum_{A \in C_G(S)} v(A), \quad \forall S \subseteq N,$$

where $C_G(S)$ is the set of all connected components of the induced graph G_S .

Definition 5. Let (ω, Σ) be a weight system. We say that a graph $G = (N, E)$ preserves the (ω, Σ) -convexity if for every game (N, v) that is (ω, Σ) -convex then the graph-restricted game (N, v^G) is also (ω, Σ) -convex.

We investigate what type of graph preserves (ω, Σ) -convexity as a function of ω and Σ . Slikker (2000) shows that a connected graph preserves average-convexity if and only if it is a complete graph or a star. If the graph is not connected, the condition becomes for each connected component to be a complete graph or a star. In our formalism, it corresponds to the case where $\Sigma = (N)$ and ω is a constant vector. In order to prove this result, he shows that a graph G preserving average-convexity has to satisfy the following conditions:

1. G has to be cycle-complete and,
2. There is no induced 4-path or induced 3-pan in G .

The formal definitions of these notions are the following. Let $G = (N, E)$ be a graph.

Definition 6. A cycle C in G is complete (resp. non-complete) if the graph $G_{V(C)}$ is complete (resp. non-complete). G is cycle-complete if any cycle C in G is complete.

Definition 7. A 3-pan $F = (i^*, j, k^*, l)$ in G is a sequence of four nodes $i, j, k, l \in N$ such that $\{i, j\}$, $\{j, k\}$, $\{k, l\}$ and $\{i, k\}$ belong to E . It is a 4-path augmented by an additional edge whose end-nodes are indicated by the stars.

We denote by $V(F)$ the set of nodes in F .

Definition 8. A path $P = (v_1, v_2, \dots, v_m)$ in G is an induced path if each two non-adjacent nodes in the sequence are not linked by any edge in G .

A 3-pan $F = (i^*, j, k^*, l)$ in G is an induced 3-pan if $\{i, l\}$ and $\{j, l\}$ do not belong to E .

An induced 4-path and an induced 3-pan are represented in Figures 1a and 1b respectively.

Let us start with the simple case where Σ is a singleton, i.e., $\Sigma = (N)$. Then all players have the same priority and we will prove that we obtain the same characterization as Slikker (2000) for inheritance of average-convexity. Let us note that, as all players have the same priority, we have $\bar{\omega}_i^T = \omega_i$ for any player $i \in T \subseteq N$ throughout this section. Hence, the difference with Slikker (2000) is the non-homogeneity of the weights.



Figure 1

Theorem 1. *Let $G = (N, E)$ be a connected graph and let (ω, Σ) be a weight system with $\Sigma = (N)$. Then the following statements are equivalent.*

1. *G preserves (ω, Σ) -convexity.*
2. *(a) G is cycle-complete.
(b) There is no induced 4-path and no induced 3-pan in G .*
3. *G is a complete graph or a star.*

The proof relies on the adaptation of previous results in Slikker (2000) and the design of specific counter-examples if the graph is not cycle-complete, or if there is an induced 3-pan or an induced 4-path. We will only present them in the more general framework when several priorities are possible.

Proof. See Appendix A. □

Remark 1. *In fact, it is enough to establish that Statement 1 implies Condition 2a (resp. Condition 2b) in Theorem 1 for a graph that is exactly a non-complete cycle (resp. an induced 4-path or a induced 3-pan). For a graph containing one of these graphs as subgraph, one can consider a game where only agents involved in the specific subgraph contribute to the utility and all remaining agents are null players. By Proposition 1, such a game is (ω, Σ) -convex if its restriction to players in the non-complete cycle (resp. induced 4-path or induced 3-pan) is (ω, Σ) -convex.*

4 Weighted convexity and Communication situations: The case with several priorities

The situation with several priorities is much more complex. We will see that the previous conditions have to be refined to take into account the priorities. In particular, whereas the examples in Slikker (2000) are only based on the graph structure, our examples require conditions simultaneously on the structure and the priorities. We start by presenting

these examples. Then, we deduce from these examples some necessary conditions on the structure of the graphs. Finally, we provide a complete characterization by necessary and sufficient conditions.

4.1 Examples with priorities

4.1.1 Cycles and Chords

The first example focuses on cycles. We show that the neighbors of a node with maximal priority in any cycle have to be linked.

Example 1 (Weighted Non-complete cycle with priorities). Let $C = \{1, 2, \dots, m, 1\}$ be a non-complete cycle in G and let l^* be a node of C satisfying

$$p(l^*) = p(V(C)).$$

Let j and k be the neighbors of l^* in C and let us assume $\{j, k\} \notin E$. We consider the convex game (N, v) defined by $v(S) = |S| - 1$ for all $S \subseteq N$, $S \neq \emptyset$. As (N, v) is convex, it is also (ω, Σ) -convex for any weight system (ω, Σ) . Let us consider $S = \{j, l^*, k\}$ and $T = V(C)$ as represented in Figure 2. We get

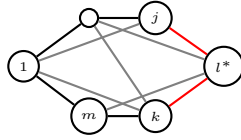


Figure 2: Non-complete cycle C , $\{j, k\} \notin E$, $p(l^*) = p(V(C))$.

$$\sum_{i \in S} \omega_i^T (v^G(S) - v^G(S \setminus \{i\})) = \omega_j^T + 2\omega_{l^*}^T + \omega_k^T > \omega_j^T + \omega_{l^*}^T + \omega_k^T = \sum_{i \in S} \omega_i^T (v^G(T) - v^G(T \setminus \{i\})).$$

This contradicts (ω, Σ) -convexity of the graph-restricted game (N, v^G) as $p(l^*) = p(T)$.

4.1.2 Induced 3-pan

The aim of this section is to exhibit a counter-example for the induced 3-pan as depicted in Figure 3 under the following condition on the priorities:

$$p(2) \geq \max(p(1), p(4)) \geq p(3). \quad (5)$$

Our example is different and much more general than the one given by Slikker (2000) for average-convexity since it needs to take into account the different weights. We present the counter-example here and highlight only that the graph-restricted game is not weighted average-convex.

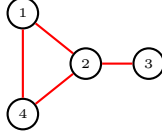


Figure 3: Induced 3-pan $(1^*, 4, 2^*, 3)$.

Example 2 (Induced 3-pan). We consider the induced 3-pan $(1^*, 4, 2^*, 3)$ as represented in Figure 3. We assume inequality (5) and present our example in the case $p(1) \geq p(4)$. A similar proof works with $p(1) < p(4)$ by symmetry.

Let us prove that there is no conservation of (ω, Σ) -convexity. Let us define

$$\begin{aligned} X &= \max \left(1 + \frac{\omega_2}{\omega_4}, 1 + \frac{\omega_3}{\omega_4} \right), \\ Y &= 1 + \frac{\omega_1}{\omega_4}, \\ Z &= X + 2Y + (1 - \alpha_p) \left[\frac{\omega_1}{\omega_2 + \omega_3 + \omega_4} X - \frac{\omega_1}{\omega_4} \right], \\ \Theta &= Z + X - 1, \end{aligned}$$

where

$$\alpha_p = \begin{cases} 1 & \text{if } p(2) = p(4) > p(3), \\ 0 & \text{otherwise.} \end{cases}$$

We consider the following game (N, v) and the graph-restricted game (N, v^G) induced by (N, v) and the communication structure given by G .

$$v(S) = \begin{cases} \alpha_p(X - 1) & \text{if } S = \{1, 2, 3\}, \\ X & \text{if } S = \{1, 4\}, \\ \alpha_p Y & \text{if } S = \{2, 4\}, \\ Y & \text{if } S = \{3, 4\}, \\ X + \alpha_p(Y - 1) & \text{if } S = \{1, 2, 4\}, \\ X + Y - 1 & \text{if } S = \{1, 3, 4\}, \\ Z & \text{if } S = \{2, 3, 4\}, \\ \Theta & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases} \quad v^G(S) = \begin{cases} \alpha_p(X - 1) & \text{if } S = \{1, 2, 3\}, \\ X & \text{if } S = \{1, 4\}, \\ \alpha_p Y & \text{if } S = \{2, 4\}, \\ 0 & \text{if } S = \{3, 4\}, \\ X + \alpha_p(Y - 1) & \text{if } S = \{1, 2, 4\}, \\ X & \text{if } S = \{1, 3, 4\}, \\ Z & \text{if } S = \{2, 3, 4\}, \\ \Theta & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

We represent in Figure 4 the lattices of coalitions of $\{1, 2, 3, 4\}$ ordered by inclusion for (N, v) and (N, v^G) with the corresponding values. Let us establish a few facts.

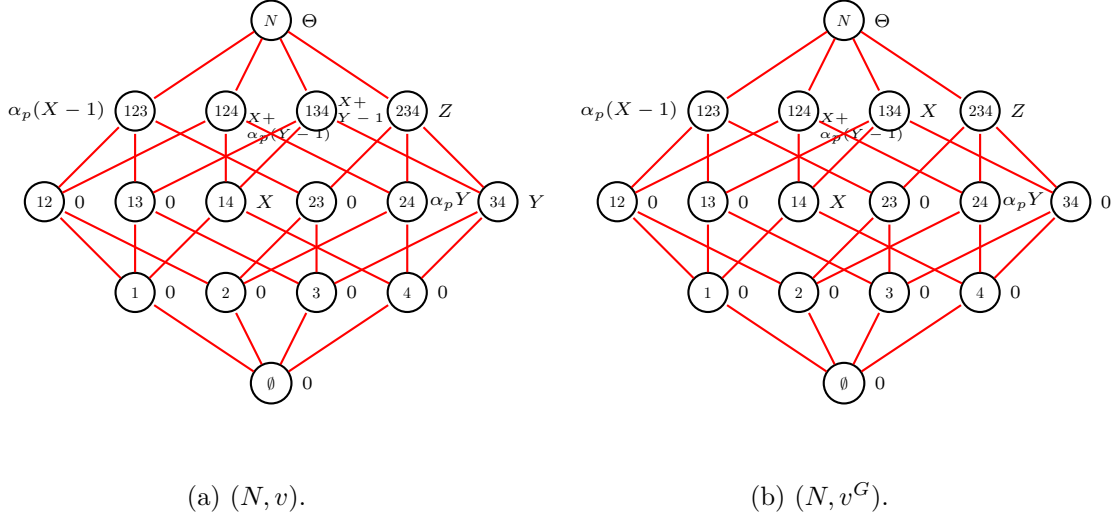


Figure 4

The game (N, v) is not convex but is (ω, Σ) -convex

A detailed proof is provided in Appendix C.

The graph-restricted game (N, v^G) on the induced 3-pan is not (ω, Σ) -convex:

Let us insist on the coalitions whose value changed compared to the original game (N, v) :

- the value of $\{3, 4\}$ changes from Y to 0 hence loosing Y ,
- the value of $\{1, 3, 4\}$ changes from $X + Y - 1$ to X hence loosing $Y - 1$.

The key element is to notice that the value of $\{1, 3, 4\}$ loses less than the value of $\{3, 4\}$ allowing for a contradiction to (ω, Σ) -convexity. Let us consider $S = \{2, 3, 4\}$ and $T = N$:

$$\sum_{i \in S} \omega_i^T(v^G(S) - v^G(S \setminus i)) = \omega_2^T(Z - 0) + \omega_3^T(Z - \alpha_p Y) + \omega_4^T(Z - 0), \quad (6)$$

whereas

$$\begin{aligned} \sum_{i \in S} \omega_i^T(v^G(T) - v^G(T \setminus i)) &= \omega_2^T(\Theta - X) + \omega_3^T(\Theta - X - \alpha_p(Y - 1)) + \omega_4^T(\Theta - \alpha_p(X - 1)) \\ &= \omega_2^T(Z - 1) + \omega_3^T(Z - \alpha_p Y - (1 - \alpha_p)) \\ &\quad + \omega_4^T(Z + (1 - \alpha_p)(X - 1)). \end{aligned} \quad (7)$$

If $\alpha_p = 1$, we get a contradiction as 2 has maximal priority. If $\alpha_p = 0$, then (6) and (7) are equivalent to

$$\sum_{i \in S} \omega_i^T(v^G(S) - v^G(S \setminus i)) = (\omega_2^T + \omega_3^T + \omega_4^T)Z, \quad (8)$$

and

$$\sum_{i \in S} \omega_i^T (v^G(T) - v^G(T \setminus i)) = (\omega_2^T + \omega_3^T + \omega_4^T)Z - \omega_2^T - \omega_3^T + \omega_4^T(X - 1). \quad (9)$$

Let us first assume $p(4) < p(N)$. Then we have $\omega_4^T = 0$ and get a contradiction as $p(2) = p(N)$. Let us now assume $p(4) = p(N)$. By assumption and as $\alpha_p = 0$, we also have $p(3) = p(N)$. Then, by definition of X , we get either $-\omega_2^T + \omega_4^T(X - 1) = -\omega_2 + \omega_4(X - 1) = 0$ or $-\omega_3^T + \omega_4^T(X - 1) = -\omega_3 + \omega_4(X - 1) = 0$. In both cases we still obtain a contradiction.

4.1.3 Induced 4-path

The aim of this section is to exhibit a counter-example for the induced 4-path as represented in Figure 5 under the following condition on the priorities:

$$\min(p(2), p(4)) \geq \max(p(1), p(3)). \quad (10)$$

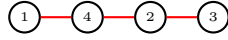


Figure 5: Induced 4-path $\{1, 4, 2, 3\}$.

Depending on the precise order of priorities, several counter-examples are necessary.

1. Assume that

$$p(2) > p(4) \geq \max(p(1), p(3)).$$

Example 2 with $\alpha_p = 0$ provides a counter-example (See Remark 2 for further details and explanations).

2. Assume that

$$p(4) > p(2) \geq \max(p(1), p(3)).$$

One can relabel the nodes as follows, 1 becomes $3'$, 2 becomes $4'$, 3 becomes $1'$, and 4 becomes $2'$. Then Case 1 applied to the induced 4-path $\{1', 4', 2', 3'\}$ provides a counter-example.

3. Assume that

$$p(2) = p(4) = \max(p(1), p(3)).$$

We have two cases. If $p(3) \leq p(1)$, then Example 2 with $\alpha_p = 0$ provides a counter-example (cf. Remark 2). If $p(1) < p(3)$, Example 2 with $\alpha_p = 0$ still provides a counter-example after relabeling the nodes as in Case 2.

4. Finally, assume that

$$p(2) = p(4) > \max(p(1), p(3)). \quad (11)$$

One needs a new counter-example. Example 3 will be our counter-example.

Remark 2. Example 2 can be extended to the induced 4-path if $\alpha_p = 0$ but not if $\alpha_p = 1$. Let us consider the induced 4-path corresponding to $\{1, 4, 2, 3\}$ as represented in Figure 5. Starting from a game (N, v) , the only differences between the graph-restricted games associated with the specified induced 3-path and induced 4-path are the values of $\{1, 2, 3\}$ and $\{1, 2\}$. In the induced 4-path, the value of coalition $\{1, 2, 3\}$ (resp. $\{1, 2\}$) is equal to $v(\{1\}) + v(\{2, 3\}) = 0$ (resp. $v(\{1\}) + v(\{2\}) = 0$). (6) is not modified and (7) becomes

$$\begin{aligned} \sum_{i \in S} \omega_i^T (v^G(T) - v^G(T \setminus i)) &= \omega_2^T (\Theta - X) + \omega_3^T (\Theta - X - \alpha_p(Y - 1)) + \omega_4^T (\Theta - 0) \\ &= \omega_2^T (Z - 1) + \omega_3^T (Z - \alpha_p Y - (1 - \alpha_p)) \\ &\quad + \omega_4^T (Z + X - 1). \end{aligned} \quad (12)$$

If $\alpha_p = 0$, then (6) and (12) are equivalent to (8) and (9) and we get the same contradiction as before. If $\alpha_p = 1$, then we would need $\omega_2 > \omega_4(X - 1)$ to get a contradiction. This is not possible as $X \geq 1 + \frac{\omega_2}{\omega_4}$ by definition.

Example 3. We consider the induced 4-path $\{1, 4, 2, 3\}$ as represented in Figure 5.

We assume inequality (11). We consider the following game (N, v) and the graph-restricted game (N, v^G) induced by (N, v) and the communication structure given by G .

$$v(S) = \begin{cases} 1 & \text{if } S \in \{\{1, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$v^G(S) = \begin{cases} 1 & \text{if } S \in \{\{1, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\}, \\ 0 & \text{otherwise.} \end{cases}$$

We represent in Figure 6 the lattices of coalitions of $\{1, 2, 3, 4\}$ ordered by inclusion for (N, v) and (N, v^G) with the corresponding values.

The game (N, v) is not convex but (ω, Σ) -convex: See the proof in Appendix D.

The graph-restricted game (N, v^G) is not (ω, Σ) -convex:

Let us insist on the coalitions whose value changed compared to the original game (N, v) :

- the value of $\{3, 4\}$ changes from 1 to 0,
- the value of $\{1, 2, 3\}$ (resp. $\{1, 3, 4\}$) becomes equal to the value of $\{2, 3\}$ (resp. $\{1, 4\}$) and therefore stays equal to 0 (resp. 1).

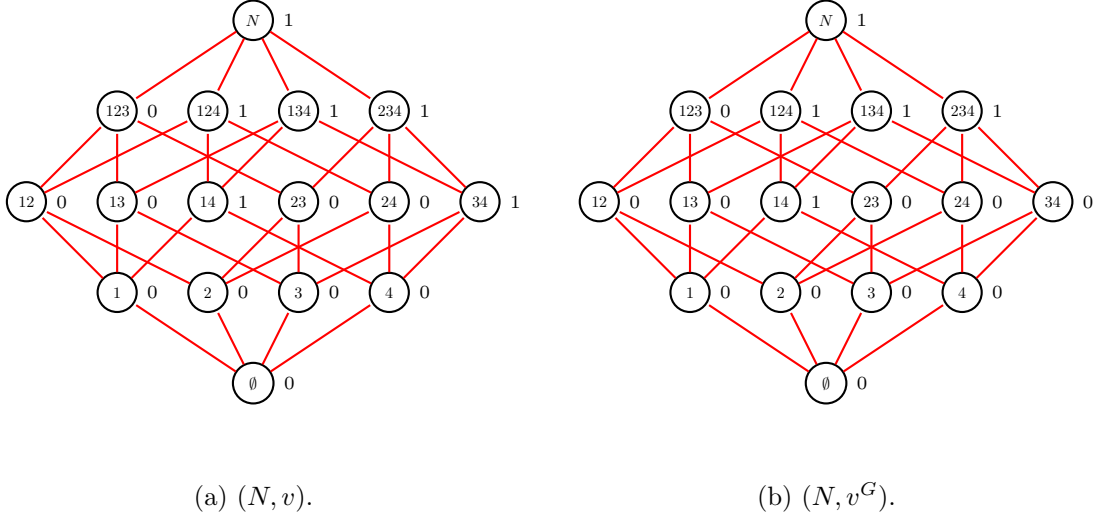


Figure 6

We consider $S = \{2, 3, 4\}$ and $T = N$:

$$\sum_{i \in S} \omega_i^T (v^G(S) - v^G(S \setminus i)) = \omega_2^T + \omega_3^T + \omega_4^T,$$

whereas

$$\sum_{i \in S} \omega_i^T (v^G(T) - v^G(T \setminus i)) = \omega_2^T(1 - 1) + \omega_3^T(1 - 1) + \omega_4^T(1 - 0) = \omega_4^T.$$

As $p(2) = p(N) > p(3)$, we have $\omega_3^T = 0$ but $\omega_2^T > 0$ implying a contradiction to (ω, Σ) -convexity of (N, v^G) .

4.2 Necessary conditions

We can establish several necessary conditions based on Examples 1, 2, and 3 of Section 4.1. These examples directly imply three necessary conditions stated in the next proposition.

Proposition 4. *Let (ω, Σ) be a weight system and $G = (N, E)$ be a graph preserving the (ω, Σ) -convexity. Then G satisfies the following conditions.*

Cycle: *For any cycle C , and for any j, l^*, k contiguous nodes in C , either the edge $\{j, k\}$ belongs to E or $p(l^*) < p(V(C))$.*

3-Pan: *For any set of four nodes $S = \{i, j, k, l\}$ in N satisfying*

$$p(k) \geq \max(p(i), p(j)) \geq p(l),$$

the subgraph induced by S is not the induced 3-pan (i^, j, k^*, l) .*

4-Path: For any set of four nodes $S = \{i, j, k, l\}$ in N satisfying

$$\min(p(k), p(j)) \geq \max(p(i), p(l)),$$

the subgraph induced by S is not the induced 4-path (i, j, k, l) .

Remark 3. If all nodes in the graph G have the same priority then Proposition 4 implies the absence of any induced 4-path, any induced 3-pan, and the cycle-completeness of any cycle. Indeed, if the graph is not cycle-complete, one can consider the smallest cycle that is not complete. By minimality, there exists three nodes l^* , j and k such that j and k are neighbors of l^* in C and $\{j, k\} \notin E$.

We establish two supplementary conditions extending the 4-Path and 3-Pan conditions to situations where there may be some intermediaries. The first, called Expanded-4-Path, is a direct generalization of the 4-Path adding possible nodes between the four main nodes. An extension of the 3-Pan is a little less easy. Observe that the key elements of the 3-Pan are that i and j can collaborate without k , but k is necessary for collaboration between i and l (resp. j and l). A subgraph called Fork linking these four nodes and satisfying these properties also leads to a counter-example as with the induced 3-pan. It provides the Fork Condition as defined below.

Definition 9. An (induced) expanded-4-path is an (induced) path containing at least 4 nodes. In practice, given $i, j, k, l \in N$, we denote by $(i, \dots, j, \dots, k, \dots, l)$ any (induced) expanded-4-path starting from i , arriving at l and passing through j and k in this order.

Definition 10. A fork F is the union of an expanded-4-path $P = (i, \dots, j, \dots, k, \dots, l)$ and a path $P' = (m, \dots, j)$ such that $V(P) \cap V(P') = \{j\}$. We will denote this fork by $F = [(i, \dots, j, \dots, k, \dots, l) | (m, \dots, j)]$.

We denote by $V(F)$ the set of nodes in F .

Proposition 5. Let (ω, Σ) be a weight system and let $G = (N, E)$ be a graph preserving (ω, Σ) -convexity. Then G satisfies the following conditions

a) **Expanded-4-path:** For any set of four nodes $S = \{i, j, k, l\}$ in N satisfying

$$\min(p(j), p(k)) \geq \max(p(i), p(l)),$$

there is no induced expanded 4-path of type $(i, \dots, j, \dots, k, \dots, l)$.

b) **Fork:** For any set of five nodes $S = \{i, j, k, l, m\}$ of nodes in N satisfying

$$p(k) \geq \max(p(i), p(m)) \geq p(l),$$

there is no fork of type $F = [(i, \dots, j, \dots, k, \dots, l) | (m, \dots, j)]$ where k is an articulation point³ of $G_{V(F)}$ and $p(k) = p(V(F))$.

³An articulation point in a graph is a vertex whose deletion increases the number of connected components.

Proof. **a)** By contradiction, let $P = (1, \dots, 4, \dots, 2, \dots, 3)$ be a path in G with $p(2) > p(4) \geq \max(p(1), p(3))$ (as represented in Figure 7b). The proofs are similar for the other cases described in Section 4.1.3.

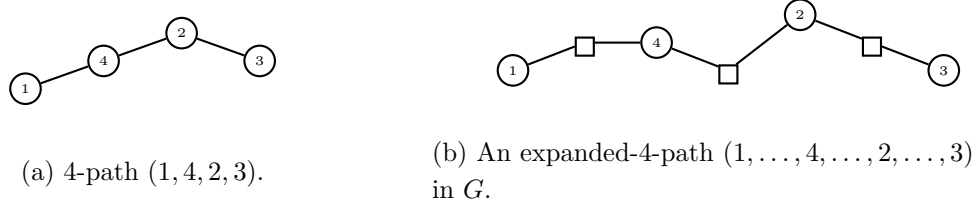


Figure 7: $p(2) > p(4) \geq \max(p(1), p(3))$.

We select 2 and 4 with highest possible priorities in P . We consider a game (N, v) with $N = \{1, 2, 3, 4\}$ and we denote by P_4 the induced 4-path $(1, 4, 2, 3)$ as represented in Figure 7a. We extend the game v to $N' = V(P)$ taking all nodes in $V(P) \setminus \{1, 2, 3, 4\}$ as null players. The restricted games will be denoted by v^{P_4} and v^P respectively.

We set $S = \{2, 3, 4\}$, $T = \{1, 2, 3, 4\}$, $S' = V(P_{3,4})$, and $T' = V(P)$. By the choice of 2 and 4, we have $p(i) < \max(p(2), p(4)) = p(T) = p(T')$ for any $i \in V(P) \setminus \{1, 2, 3, 4\}$. We get

$$\sum_{i \in S'} \omega_i^{T'} (v^P(S') - v^P(S' \setminus i)) = \sum_{i \in S} \omega_i^T (v^{P_4}(S) - v^{P_4}(S \setminus i)), \quad (13)$$

and

$$\sum_{i \in S'} \omega_i^{T'} (v^P(T') - v^P(T' \setminus i)) = \sum_{i \in S} \omega_i^T (v^{P_4}(T) - v^{P_4}(T \setminus i)). \quad (14)$$

We take the game (N, v) with $N = \{1, 2, 3, 4\}$ defined in Example 2 with $\alpha_p = 0$. (N, v) is (ω, Σ) -convex and by Proposition 1, (N', v) is also (ω, Σ) -convex. Therefore we get the same contradiction as in Example 2.

b) By contradiction, let $F = [(1, \dots, 5, \dots, 2, \dots, 3)|(4, \dots, 5)]$ be a fork in G satisfying

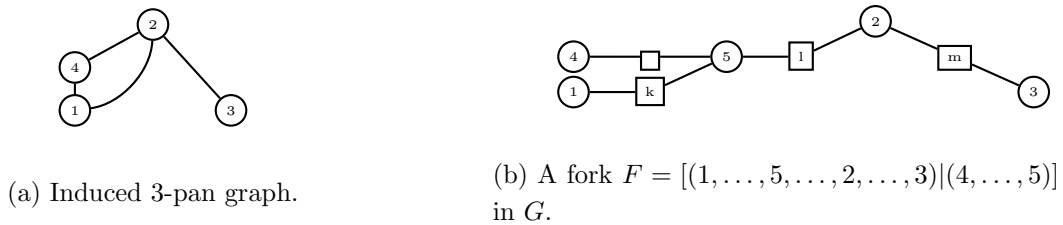


Figure 8: $p(2) \geq \max(p(1), p(4)) \geq p(3)$.

$p(2) = p(V(F)) \geq \max(p(1), p(4)) \geq p(3)$, as represented in Figure 8b and such that 2 is an articulation point of $G_{V(F)}$. We set $P = (1, \dots, 5, \dots, 2, \dots, 3)$ and $P' = (4, \dots, 5)$.

We first show that we can replace F by a fork satisfying the following statements.

- $P_{3,5}$ is an induced path. We select $P_{2,3}$ as short as possible. We select P and P' such that the node 5 is as close as possible to node 2 and such that $P_{2,5}$ is as short as possible. As 2 is an articulation point and by the choice of P , $P_{5,3}$ is a shortest path and therefore an induced path.
- 1 and 4 are such that $p(i) < p(1)$ and $p(i) < p(4)$ for any node $i \in V(P_{1,5}) \setminus \{1, 5\}$ and $i \in V(P') \setminus \{4, 5\}$ respectively (up to shortening the paths).

We now prove that the selected fork satisfies

$$p(i) < p(2), \forall i \in V(F) \setminus \{1, 2, 3, 4\} \quad (15)$$

- By the preceding selection of 1 and 4 and as $\max(p(1), p(4)) \leq p(2)$ the inequality is true for any node $i \in V(P_{1,5}) \setminus \{1, 5\}$ and $i \in V(P') \setminus \{4, 5\}$.
- Now for the nodes in $V(P_{5,3}) \setminus \{5, 2, 3\}$ we use Claim a). $P_{5,3}$ is an induced path. If there exists $i \in V(P_{5,3}) \setminus \{5, 2, 3\}$ with $p(i) \geq p(2)$, then as $p(2) = p(V(F))$ we have $\min(p(i), p(2)) \geq \max(p(5), p(3))$, contradicting Claim a).
- It remains to show that $p(5) < p(2)$.

By contradiction, we assume $p(5) = p(2)$. Let k (resp. l) be the neighbor of 5 in $P_{1,5}$ (resp. $P_{5,2}$). Let m be the neighbor of 2 in $P_{2,3}$. If there is no edge linking k to a node in $V(P_{5,2}) \setminus \{5\}$, then we get a contradiction by Claim a) as $p(2) = p(5) \geq \min(p(k), p(3))$. If there is an edge linking k to a node in $V(P_{5,2}) \setminus \{5, 2\}$, it contradicts the choice of 5. If $\{k, 2\}$ exists then there is a cycle C passing through 5 with $p(5) = p(V(C))$ and we have $\{k, l\} \in E$ by Proposition 4. By the choice of 5, we have $l = 2$. Then $(k^*, 5, 2^*, m)$ is an induced 3-pan with $p(2) = \max(p(5), p(k)) \geq p(m)$, contradicting Proposition 4.

We consider the game (N, v) as in Case a) and extend it to $N' = V(F)$ taking all nodes in $V(F) \setminus \{1, 2, 3, 4\}$ as null players. We also consider the same subsets S, T , but take $S' = V(P') \cup V(P_{5,3})$, and $T' = V(F)$. Then (13) and (14) are satisfied with the restricted game associated with F instead of P and the restricted game associated with the induced 3-pan $(1^*, 4, 2^*, 3)$ (as represented in Figure 8a) instead of P_4 . We get the same contradictions as with the induced 3-pan using Example 2. $\square$

4.3 Conditions on graph layers

Here we present conditions implied by the necessary conditions introduced in Section 4.2. These supplementary conditions give insight into the particular situations of players at a given level of priority or linked between two different levels.

A first necessary condition is for each level of priority to be composed of disjoint complete subgraphs or stars. For any graph G and given a priority k , we denote by G_k the subgraph induced by the set of players with priority k .

Proposition 6. *Let (ω, Σ) be a weight system and G be a graph satisfying the **Cycle**, **3-Pan**, and **4-Path** conditions. Then given a priority k*

1. G_k is cycle-complete and there is no induced 3-pan or induced 4-path in G_k .

Or equivalently

2. G_k corresponds to a collection of disconnected star/complete subgraphs.

Proof. The Cycle, 3-Pan, and 4-Path conditions immediately imply Claim 1. The equivalence between Claims 1 and 2 was already established in Slikker (2000). $\square$

Let us insist that when restricting to the set of players in layer k , one can loose the connectivity. Hence *a priori* G_k may be composed of several stars and/or complete subgraphs.

Moreover, we obtain that the connection between two components in different layers depends on the type of the component of larger priority.

Proposition 7. *Let (ω, Σ) be a weight system and G be a graph satisfying the **Cycle**, **3-Pan**, and **4-Path** conditions. If a player s of priority k is linked to a player a in a layer with higher priority then one of the following statements is satisfied*

- a is part of a complete component K in the graph restricted to its priority and s is then linked to each node of K ,
- a is the center of a star S in the graph restricted to its priority and s cannot be linked to any leaf of S if $|V(S)| \geq 3$.

Remark 4. *Connected components with one or two nodes play special roles. Indeed a component of size one is at the same time a star with no leaf and a complete component. Similarly, a connected component of size two is at the same time a star centered on one of the two nodes, a star centered on the second one and a complete component.*

Proof. Let s be a player of priority k connected to a player a of priority $k' > k$. Let C_a be the connected component of $G_{k'}$ containing a . By Proposition 6, it is necessarily a star or a complete component. We can assume that C_a contains at least three nodes otherwise Proposition 7 is trivially satisfied.

By contradiction, let us assume that a is not the center of a star and not part of a complete component. Then C_a is a star and a is a leaf of this star. Let c be the center of C_a and let $b \neq a$ be another leaf. Then $\{s, a, c, b\}$ corresponds to a path in G (but not necessarily an induced 4-path as some more edges between nodes s, a, c, b could exist). As C_a is a star, there is no edge linking a and b .

- Let us assume s non-linked to b . If s is not linked (resp. linked) to c then (s, a, c, b) (resp. (s^*, a, c^*, b)) is an induced 4-path (resp. an induced 3-path) as represented in Figure 9a (resp. Figure 9b) contradicting the 4-Path (resp. 3-Pan) Condition.

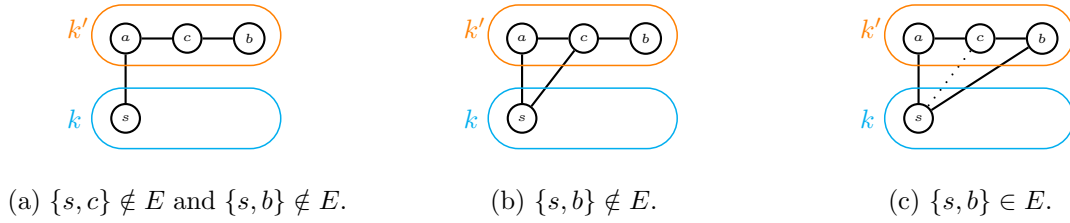


Figure 9: $k < k'$.

- Let us assume s linked to b . Then we have a cycle with $\{a, b\} \notin E$ as represented in Figure 9c, contradicting the Cycle Condition.

Let us now turn to the case where C_a is a complete component.

- Assume that there is exactly one node in C_a that is not linked to s denoted by t . Let b be a node of C_a different from a and t as represented in Figure 10a. Then (s, a, t, b, s) corresponds to a cycle with $\{s, t\} \notin E$, contradicting the Cycle Condition.

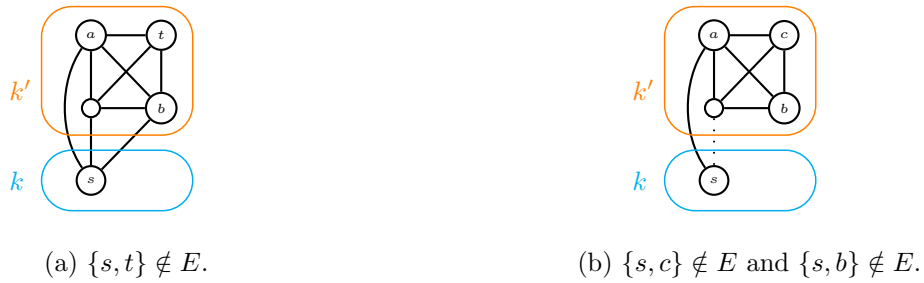


Figure 10: $k < k'$.

- Assume that there exist at least two nodes b and c in C_a that are not linked to s as represented in Figure 10b. Then (c^*, b, a^*, s) corresponds to an induced 3-pan where s has degree 1, contradicting the 3-Pan Condition. $\square$

Finally, we establish the following lemma implying that some subgraphs are necessarily paths with increasing priorities.

Proposition 8. *Let $G = (N, E)$ be a graph satisfying the **Cycle**, **3-Pan**, **Expanded-4-Path**, and **Fork** conditions. Let s_1, i, s_2 be three nodes such that $\{s_1, i\} \in E$, $\{s_2, i\} \in E$, and*

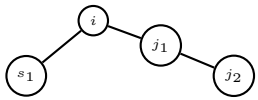
$$p(s_1) \leq p(s_2) \leq p(i).$$

Let S_2 be the set of nodes of the connected component of $G_{N \setminus \{i\}}$ containing s_2 . Assume $s_1 \notin S_2$ and $p(S_2) \leq p(i)$. If $|S_2| > 1$ then G_{S_2} is an induced path $(j_1, \dots, j_q)$ with $q \geq 2$, $s_2 = j_1$, and

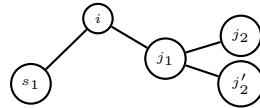
$$p(s_2) = p(j_1) < \dots < p(j_q) \leq p(i). \quad (16)$$

Proof. Set $j_1 = s_2$. Let j_2 be a neighbor of j_1 in S_2 . There are several cases but the following are excluded by the conditions.

- $\{i, j_2\} \in E$ contradicts the 3-Pan as $p(i) \geq \max(p(j_1), p(j_2)) \geq p(s_1)$.
- $\{i, j_2\} \notin E$ and $p(j_2) \leq p(j_1)$ (Figure 11a) contradicts the Expanded-4-Path as



(a) $\{i, j_2\} \notin E$ and $p(j_2) \leq p(j_1)$.



(b) $p(j_2) > p(j_1)$.

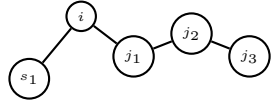
Figure 11: $j_1 = s_2$ and $p(s_1) \leq p(j_1)$.

$$\min(p(i), p(j_1)) \geq \max(p(s_1), p(j_2)).$$

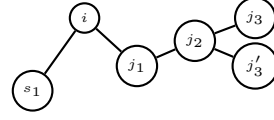
- $\{i, j_2\} \notin E$, $p(j_2) > p(j_1)$, and there exists $j'_2 \neq j_2$ neighbor of j_1 in S_2 (Figure 11b), contradicts the Fork as $p(i) \geq \max(p(j_2), p(j'_2)) \geq p(s_1)$.

Thus we assume now $p(j_2) > p(j_1)$, $\{i, j_2\} \notin E$, and j_2 is the unique neighbor of j_1 in S_2 . Let j_3 be a neighbor of j_2 in $S_2 \setminus \{j_1\}$.

- $\{i, j_3\} \in E$ contradicts the Cycle Condition applied with (i, j_1, j_2, j_3, i) as $\{j_1, j_3\} \notin E$.



(a) $\{j_1, j_3\} \notin E$, $\{i, j_3\} \notin E$, and $p(j_3) \leq p(j_2)$.



(b) $p(j_3) > p(j_2)$.

Figure 12: $j_1 = s_2$, $\{i, j_2\} \notin E$, and $p(s_1) \leq p(j_1) < p(j_2)$.

- $\{i, j_3\} \notin E$, and $p(j_3) \leq p(j_2)$ (Figure 12a), contradicts the Expanded-4-Path as $\min(p(i), p(j_2)) \geq \max(p(s_1), p(j_3))$.
- $\{i, j_3\} \notin E$, $p(j_3) > p(j_2)$, and there exists $j'_3 \neq j_3$ neighbor of j_2 in $S_2 \setminus \{j_1\}$ (Figure 12b) contradicts the Fork as $p(i) \geq \max(p(j_3), p(j'_3)) \geq p(s_1)$.

Thus $\{j_1, j_3\} \notin E$, $\{i, j_3\} \notin E$, $p(j_3) > p(j_2)$ and j_3 is the unique neighbor of j_2 in $S_2 \setminus \{j_1\}$.

We can repeat this reasoning with the neighbors of j_3 in $S_2 \setminus \{j_1, j_2\}$, and so on. $\square$

4.4 Necessary and sufficient conditions

We will now provide a characterization of the class of graphs preserving (ω, Σ) -convexity. We need a technical lemma and a proposition.

Lemma 1. *Let (N, v) be an (ω, Σ) -convex game. Let us consider $S, T, U \subseteq N$ such that $S \cap T = \emptyset$, $U \subseteq T$ and for all $s \in S$, $p(s) > p(U)$ then*

$$v(S \cup T) - v(T) \geq v(S \cup U) - v(U). \quad (17)$$

Proof. See the proof in Appendix B. $\square$

Proposition 9. *Let (N, v) be an (ω, Σ) -convex game and $G = (N, E)$ a graph satisfying the **Cycle**, **3-Pan**, **Expanded-4-Path**, and **Fork** conditions. Then, for any connected subsets $S \subseteq T \subseteq N$ with $p(S) = p(T)$ and any $i \in \bar{S}$,*

$$v(S \setminus \{i\}) - \sum_{j=1}^p v(S_j) \leq v(T \setminus \{i\}) - \sum_{j=1}^q v(T_j), \quad (18)$$

where $\{S_1, \dots, S_p\}$ (resp. $\{T_1, \dots, T_q\}$) with $p \geq 1$ (resp. $q \geq 1$) is the partition of $S \setminus \{i\}$ (resp. $T \setminus \{i\}$) into connected subsets.

Proof. Let S and T be connected subsets with $S \subseteq T \subseteq N$ and $p(S) = p(T)$. Let us consider $i \in \bar{S}$ and let $\{S_1, \dots, S_p\}$ (resp. $\{T_1, \dots, T_q\}$) with $p \geq 1$ (resp. $q \geq 1$) be the partition of $S \setminus \{i\}$ (resp. $T \setminus \{i\}$) into connected subsets.

We first establish that

$$S_j \subseteq T_j \text{ for all } j \text{ with } 1 \leq j \leq p \leq q, \quad (19)$$

after renumbering if necessary. By contradiction, let T_j be a subset containing two subsets S_1 and S_2 of the partition of S . Let j (resp. k) be a node in S_1 (resp. S_2) such that $\{i, j\}$ (resp. $\{i, k\}$) belongs to E . As T_j is connected, there exists a path P in T_j connecting j and k . Then the concatenation of (i, j) , P and (k, i) forms a cycle C in G with $p(i) = p(V(C))$ and $\{j, k\} \notin E$, a contradiction. Therefore (19) is satisfied.

As S is connected, there exist $t_1^1, \dots, t_1^p$ elements in $S_1, \dots, S_p$ respectively such that $\{i, t_1^k\} \in E$ for every $k \in \{1, \dots, p\}$. As $i \in \bar{S}$, we have

$$p(t_1^1) \leq p(t_1^2) \leq \dots \leq p(t_1^p) \leq p(i), \quad (20)$$

after renumbering if necessary.

We now prove that all nodes in S_k with $k \in \{2, \dots, p\}$ have a priority strictly smaller than the nodes in $T_k \setminus S_k$. We have $t_1^k \in T_k$, $t_1^1 \notin T_k$, and $p(T_k) \leq p(i)$. As inequality (20) is satisfied, Proposition 8 gives that G_{T_k} is an induced path $(t_1^k, \dots, t_{m_k}^k)$ with $m_k \geq 1$ satisfying

$$p(t_1^k) < \dots < p(t_{m_k}^k) \leq p(i). \quad (21)$$

As $t_1^k \in S_k$ and S_k is connected, G_{S_k} is a subpath of G_{T_k} starting at t_1^k . Thus (21) implies

$$p(r) \geq p(t_1^k) \text{ for any node } r \in S_k, \quad (22)$$

and

$$p(r) > p(S_k) \text{ for any node } r \in T_k \setminus S_k. \quad (23)$$

As (23) is satisfied, Lemma 1 applied with $T_k \setminus S_k$ and $S_k \subseteq \bigcup_{j=1}^{k-1} T_j \cup \bigcup_{j=k}^p S_j$ gives

$$v(T_k) - v(S_k) \leq v\left(\bigcup_{j=1}^k T_j \cup \bigcup_{j=k+1}^p S_j\right) - v\left(\bigcup_{j=1}^{k-1} T_j \cup \bigcup_{j=k}^p S_j\right). \quad (24)$$

Adding (24) for every $k \in \{2, \dots, p\}$, we get

$$\sum_{k=2}^p (v(T_k) - v(S_k)) \leq v\left(\bigcup_{j=1}^p T_j\right) - v\left(T_1 \cup \bigcup_{j=2}^p S_j\right). \quad (25)$$

Moreover, by Proposition 3, v is superadditive and therefore

$$v\left(\bigcup_{j=1}^p T_j\right) \leq v(T \setminus \{i\}) - \sum_{j=p+1}^q v(T_j). \quad (26)$$

To conclude it will suffice to show the following inequality when $k = 1$.

$$v(S \setminus \{i\}) - v(S_1) \leq v\left(T_1 \cup \bigcup_{j=2}^p S_j\right) - v(T_1). \quad (27)$$

Indeed summing (25), (26), and (27) imply (18).

To prove (27) we consider two cases.

Case 1 $p(S_1) < p(t_1^2)$ By (20) and (22), $p(r) > p(S_1)$ for any $r \in \bigcup_{j=2}^p S_j$. Then Lemma 1 applied with $\bigcup_{j=2}^p S_j$ and $S_1 \subseteq T_1$ gives (27) since $\bigcup_{j=1}^p S_j = S \setminus \{i\}$.

Case 2 $p(S_1) \geq p(t_1^2)$ We first prove

$$p(t) > p(S_1), \forall t \in T_1 \setminus S_1. \quad (28)$$

By contradiction, let t be a node in $T_1 \setminus S_1$ with $p(t) \leq p(S_1)$. Let s be a node in $\overline{S_1}$. By connectivity of S_1 , there exists a shortest path P linking s to t_1^1 (P may be reduced to $s = t_1^1$). By connectivity of T_1 , there exists a shortest path P' linking t to a node t' in P . If there are several possibilities, we select P' such that t' is as close as possible to t_1^1 (t' may be equal to t_1^1).

If $t' \neq s$, the fork $[(s, \dots, t', \dots, i, t_1^2)|(t, \dots, t')]$ is generated by concatenating P and (t_1^1, i, t_1^2) and then combining it with P' as represented in Figure 13a where i is an articulation point with maximum priority and with $p(i) \geq \max(p(s), p(t)) = p(s) \geq p(t_1^2)$, contradicting the Fork condition. Therefore $t' = s$ as represented in Figure 13b.

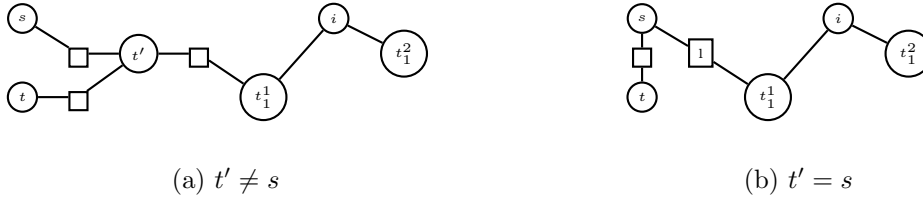


Figure 13: $p(i) \geq p(s) = p(S_1) \geq p(t_1^2)$ and $p(t) \leq p(S_1)$.

We consider the following subcases.

- If the concatenation of P' , P and (t_1^1, i, t_1^2) is an induced path, then it contradicts the Expanded-4-path condition as $\min(p(s), p(i)) \geq \max(p(t), p(t_1^2))$.
- If the concatenation of P' , P and (t_1^1, i, t_1^2) is not an induced path, then there exists an edge linking i to a node j in $V(P') \cup V(P) \setminus \{t_1^1\}$. Note that there is no edge between $V(P') \setminus \{s\}$ and $V(P) \setminus \{s\}$, as it would contradict the minimality of P' or the position of t' . We select j as close as possible to s (j may be equal to s). We consider different cases depending on the position of j .

If $j \in V(P') \setminus \{s\}$, then there is a cycle passing through i as represented in Figure 14a and the Cycle Condition implies $\{j, t_1^1\} \in E$. If $s \neq t_1^1$, it contradicts the choice of t' . If $s = t_1^1$, (s^*, j, i^*, t_1^2) is an induced 3-pan with $p(i) \geq \max(p(j), p(s)) \geq p(t_1^2)$, contradicting the 3-pan condition.

If $j \in V(P) \setminus \{t_1^1\}$, then the concatenation of P' , $P_{s,j}$ and (j, i, t_1^2) is an induced path as represented in Figure 14b, contradicting the Expanded-4-path condition as $\min(p(s), p(i)) \geq \max(p(t), p(t_1^2))$.



Figure 14: $p(i) \geq p(s) = p(S_1) \geq p(t_1^2)$ and $p(t) \leq p(S_1)$.

Thus (28) is satisfied and Lemma 1 applied with $T_1 \setminus S_1$ and $S_1 \subseteq S \setminus \{i\}$ gives (27). $\square$

We can now establish our characterization of graphs preserving (ω, Σ) -convexity.

Theorem 2. *Let (ω, Σ) be a weight system and $G = (N, E)$ be a graph. The following statements are equivalent:*

1. G preserves (ω, Σ) -convexity.
2. G satisfies the **Cycle**, **3-Pan**, **Expanded-4-Path**, and **Fork** conditions.

Proof. For any subset $S \subseteq N$ and any $i \in S$, we set

$$\Delta_i v(S) = v(S) - v(S \setminus \{i\}).$$

By Propositions 4 and 5, **1** $\Rightarrow$ **2**. We now prove **2** $\Rightarrow$ **1**. Let (N, v) be an (ω, Σ) -convex game. We have to prove that Equation (1) is satisfied by the graph-restricted game (N, v^G) . We assume $p(S) = p(T) = k$ otherwise (1) is trivially satisfied. S and T are composed of connected components. We only have to consider components of highest priority. Let $S' = \{S_1, \dots, S_p\}$ (resp. $T' = \{T_1, \dots, T_q\}$) be the set of components of S (resp. T) satisfying $p(S_j) = k$ (resp. $p(T_j) = k$) for all j , $1 \leq j \leq p$ (resp. $1 \leq j \leq q$). We set $L = \{1, \dots, p\}$ and $M = \{1, \dots, q\}$. By Proposition 6, G_k is a collection of disconnected star/complete subgraphs. Each component in S' and T' intersects at least one of the subgraphs in G_k .

Let us focus on a component S_j . We show that deleting a node i of priority k in S_j can disconnect S_j only if i is the center of a star in G_k .

- If S_j belongs to G_k , then S_j is part of a star or a complete subgraph of G_k and the result is obvious. Therefore we can assume the existence of at least one node in S_j with priority strictly smaller than k .
- Let i be a leaf of a star in G_k . By Proposition 7, a node with priority strictly smaller than k can be linked to a star of G_k only at its center. Hence, deleting i cannot disconnect S_j .
- Let i belong to a complete subgraph Q in G_k containing at least two nodes. By Proposition 7, any node linked to Q and with priority strictly smaller than k is linked to all nodes of Q . Let a be a node in S_j linked to Q with $p(a) < k$. By contradiction, assume that deleting i disconnects S_j . Then there exists a node $b \neq a$ in $S_j \setminus Q$ linked to i (and all nodes of Q) with $p(b) < k$ and $\{a, b\} \notin E$, as represented in Figure 15. As Q contains at least two nodes, there exists a node $j \in Q$ with $j \neq i$.

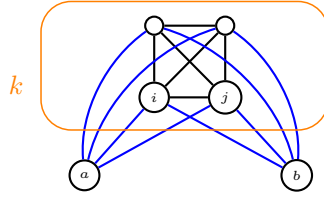


Figure 15: a and b linked to complete subgraph Q of G_k .

Then (i, a, j, b, i) is a cycle where i has highest priority and the Cycle Condition implies $\{a, b\} \in E$, a contradiction.

For any j in L (resp. M), we denote by S_j^* (resp. T_j^*) the set of star centers of G_k contained in S_j (resp. T_j). S_j^* and T_j^* may be empty. We get

$$\sum_{i \in S} \bar{\omega}_i^T \Delta_i v^G(S) = \sum_{j \in L} \left(\sum_{i \in S_j^*} \omega_i (v(S_j) - v^G(S_j \setminus \{i\})) + \sum_{i \in S_j \setminus S_j^*} \omega_i \Delta_i v(S_j) \right). \quad (29)$$

For any i in S_j^* , we denote by $S_{j,1}^{-i}, \dots, S_{j,\alpha_j^{-i}}^{-i}$ with $\alpha_j^{-i} \geq 1$ the components of $S_j \setminus \{i\}$. We get

$$\sum_{i \in S_j^*} \omega_i (v(S_j) - v^G(S_j \setminus \{i\})) = \sum_{i \in S_j^*} \omega_i \left(v(S_j) - \sum_{a=1}^{\alpha_j^{-i}} v(S_{j,a}^{-i}) \right). \quad (30)$$

(29) and (30) give

$$\sum_{i \in \bar{S}} \bar{\omega}_i^T \Delta_i v^G(S) = \sum_{j \in L} \left(\sum_{i \in \bar{S}_j} \omega_i \Delta_i v(S_j) + \sum_{i \in S_j^*} \omega_i \left(v(S_j \setminus \{i\}) - \sum_{a=1}^{\alpha_j^{-i}} v(S_{j,a}^{-i}) \right) \right). \quad (31)$$

As each S_j is contained in one T_r , (31) is equivalent to

$$\sum_{i \in \bar{S}} \bar{\omega}_i^T \Delta_i v^G(S) = \sum_{r \in M} \sum_{S_j \subseteq T_r} \left(\sum_{i \in \bar{S}_j} \omega_i \Delta_i v(S_j) + \sum_{i \in S_j^*} \omega_i \left(v(S_j \setminus \{i\}) - \sum_{a=1}^{\alpha_j^{-i}} v(S_{j,a}^{-i}) \right) \right). \quad (32)$$

Denoting for any i in T_r^* the components of $T_r \setminus \{i\}$ by $T_{r,1}^{-i}, \dots, T_{r,\beta_r^{-i}}^{-i}$ with $\beta_r^{-i} \geq 1$, one obtains similarly

$$\sum_{i \in \bar{S}} \bar{\omega}_i^T \Delta_i v^G(T) = \sum_{r \in M} \sum_{S_j \subseteq T_r} \left(\sum_{i \in \bar{S}_j} \omega_i \Delta_i v(T_r) + \sum_{i \in S_j^*} \omega_i \left(v(T_r \setminus \{i\}) - \sum_{b=1}^{\beta_r^{-i}} v(T_{r,b}^{-i}) \right) \right). \quad (33)$$

As $p(S_j) = p(T_r) = k$, (ω, Σ) -convexity of (N, v) gives for any $S_j \subseteq T_r$

$$\sum_{i \in \bar{S}_j} \omega_i \Delta_i v(S_j) \leq \sum_{i \in \bar{S}_j} \omega_i \Delta_i v(T_r). \quad (34)$$

By Proposition 9 and as $p(S_j) = p(T_r)$, the following inequality is also satisfied:

$$v(S_j \setminus \{i\}) - \sum_{a=1}^{\alpha_j^{-i}} v(S_{j,a}^{-i}) \leq v(T_r \setminus \{i\}) - \sum_{b=1}^{\beta_r^{-i}} v(T_{r,b}^{-i}), \quad (35)$$

for any $i \in S_j^*$. Then, (32), (33), (34), and (35) give (1) for (N, v^G) . $\square$

A Proof of Theorem 1

Proof of Theorem 1. **3 $\Rightarrow$ 1** Let $G = (N, E)$ be a star or a complete graph and (N, v) be an (ω, Σ) -convex game. If G is complete then $v^G = v$ and (N, v^G) is (ω, Σ) -convex. If G is a star, then (N, v^G) is also (ω, Σ) -convex by Slikker (2000)⁴.

1 $\Rightarrow$ 2 By Proposition 4 and Remark 3, a graph preserving (ω, Σ) -convexity has to be cycle-complete and there is no induced 4-path or 3-pan.

2 $\Rightarrow$ 3 Following Slikker (2000)⁵, we know that the only connected cycle-complete graphs without any induced 4-path or 3-pan are the star and the complete graphs. $\square$

⁴The result can be straightforwardly obtained from the proof given in Slikker (2000) (Theorem 4.3) replacing average-convexity by (ω, Σ) -convexity.

⁵Lemmas 4.2 and 4.3 in Slikker (2000).

B Proof of Lemma 1

We first recall the following result by Skoda and Venel (2023).

Proposition 10. *Let (N, v) be an (ω, Σ) -convex game. Let us consider $S, T, U \subseteq N$ such that $S \cap T = \emptyset$, $U \subseteq T$ and for all $s \in S$, $p(s) > p(U)$ and $p(s) \geq p(T)$ then*

$$v(S \cup T) - v(T) \geq v(S \cup U) - v(U). \quad (36)$$

The next lemma gives a more general result than Proposition 10.

Lemma 1. *Let (N, v) be an (ω, Σ) -convex game. Let us consider $S, T, U \subseteq N$ such that $S \cap T = \emptyset$, $U \subseteq T$ and for all $s \in S$, $p(s) > p(U)$ then*

$$v(S \cup T) - v(T) \geq v(S \cup U) - v(U). \quad (17)$$

Proof. We consider the partition of S into its priority subsets $\{S_1, \dots, S_k\}$ with $k \geq 1$ and $p(i) = p(S_l)$ for any $i \in S_l$, $1 \leq l \leq k$. If $k \geq 2$, we assume $p(S_1) < \dots < p(S_k)$ after renumbering if necessary. We consider the partition of T into the following subsets : $T_0 = \{j \in T, p(j) \leq p(S_1)\}$, $T_l = \{j \in T, p(S_l) < p(j) \leq p(S_{l+1})\}$ for any l with $1 \leq l \leq k-1$, and $T_k = \{j \in T, p(S_k) < p(j)\}$. Let us note that T_0 contains U , but any subset T_l with $l \geq 1$ may be empty. Applying Proposition 10 with S_1 and $U \subseteq T_0$, we get

$$v(S_1 \cup U) - v(U) \leq v(S_1 \cup T_0) - v(T_0). \quad (37)$$

Applying Proposition 10 with T_1 and $T_0 \subseteq S_1 \cup T_0$, we get

$$v(T_0 \cup T_1) - v(T_0) \leq v(S_1 \cup T_0 \cup T_1) - v(S_1 \cup T_0). \quad (38)$$

(37) and (38) imply

$$v(S_1 \cup U) - v(U) \leq v(S_1 \cup T_0 \cup T_1) - v(T_0 \cup T_1). \quad (39)$$

If $k = 1$, (39) corresponds to (17). Otherwise, Proposition 10 applied with S_2 and $S_1 \cup U \subseteq S_1 \cup T_0 \cup T_1$ gives

$$v(S_1 \cup S_2 \cup U) - v(S_1 \cup U) \leq v(S_1 \cup S_2 \cup T_0 \cup T_1) - v(S_1 \cup T_0 \cup T_1). \quad (40)$$

(39) and (40) imply

$$v(S_1 \cup S_2 \cup U) - v(U) \leq v(S_1 \cup S_2 \cup T_0 \cup T_1) - v(T_0 \cup T_1). \quad (41)$$

Then, applying successively:

- Proposition 10 with T_h and $\bigcup_{d=0}^{h-1} T_d \subseteq \bigcup_{d=1}^h S_d \cup \bigcup_{d=0}^{h-1} T_d$,
- Proposition 10 with S_{h+1} and $\bigcup_{d=1}^h S_d \cup U \subseteq \bigcup_{d=1}^h S_d \cup \bigcup_{d=0}^h T_d$,

for $h \in \{2, \dots, k-1\}$, we get

$$v(S \cup U) - v(U) \leq v\left(S \cup \bigcup_{d=0}^{k-1} T_d\right) - v\left(\bigcup_{d=0}^{k-1} T_d\right). \quad (42)$$

Finally using Proposition 10 with T_k and $\bigcup_{d=0}^{k-1} T_d \subseteq S \cup \bigcup_{d=0}^{k-1} T_d$, we get (17). $\square$

C Proof of Counter-example induced 3-pan (Example 2)

We show that Example 2 described page 10 provides a counter-example if

$$\max(p(1), p(4)) = p(1), p(2) \geq p(4), \text{ and } p(4) \geq p(3).$$

The game (N, v) is not convex:

The marginal contribution of player 1 is strictly smaller to $\{3, 4\}$ than to $\{4\}$.

$$v(\{1, 4\}) - v(\{4\}) = X - 0 > (X + Y - 1) - Y = v(\{1, 3, 4\}) - v(\{3, 4\}).$$

The game (N, v) is (ω, Σ) -convex:

We need to check the inequalities for any pair of non-empty sets S and T such that $S \subset T$.

- Let us first notice that the function v is monotonic since $(Y - 1) \geq 0$, $(X - 1) \geq 0$ and $\Theta \geq Z \geq Y$. Hence if S has value 0, we have

$$\sum_{i \in S} \omega_i^T (v(S) - v(S \setminus i)) = 0 - 0 = 0. \quad (43)$$

Since the function v is monotonic, each marginal contribution has to be positive, hence the (ω, Σ) -convexity inequality is satisfied for any T containing S . (43) is satisfied if S is a singleton, and if S is equal to $\{1, 2\}$, $\{1, 3\}$, or $\{2, 3\}$.

In the following cases, we give details only when the inequality is not trivially satisfied.

- If S is equal to $\{1, 4\}$, then

$$\sum_{i \in S} \omega_i^T (v(S) - v(S \setminus i)) = \omega_1^T X + \omega_4^T X = (\omega_1^T + \omega_4^T) X.$$

There are 3 possible cases for T : $\{1, 2, 4\}$, $\{1, 3, 4\}$ and N .

- Assume $T = \{1, 2, 4\}$,

$$\begin{aligned} \sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)) &= \omega_1^T (X + \alpha_p(Y - 1) - \alpha_p Y) + \omega_4^T (X + \alpha_p(Y - 1) - 0), \\ &= (\omega_1^T + \omega_4^T) X + \alpha_p \omega_4^T Y - \alpha_p (\omega_1^T + \omega_4^T). \end{aligned}$$

If $\alpha_p = 1$, then by definition of Y and as $p(4) \geq p(1)$, we have $\omega_4^T Y - (\omega_1^T + \omega_4^T) \geq 0$ and the (ω, Σ) -convexity inequality is satisfied.

- Assume $T = \{1, 3, 4\}$,

$$\begin{aligned} \sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)) &= \omega_1^T (X + Y - 1 - Y) + \omega_4^T (X + Y - 1 - 0), \\ &= (\omega_1^T + \omega_4^T) X + \omega_4^T Y - (\omega_1^T + \omega_4^T). \end{aligned}$$

Then we can conclude as in the previous case with $\alpha_p = 1$.

– Let us consider $T = N$, then

$$\begin{aligned}\sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)) &= \omega_1^T (\Theta - Z) + \omega_4^T (\Theta - \alpha_p (X - 1)), \\ &= \omega_1^T (X - 1) + \omega_4^T (Z + (1 - \alpha_p)(X - 1)).\end{aligned}$$

If $\alpha_p = 1$, we get

$$\begin{aligned}\sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)) &= \omega_1^T (X - 1) + \omega_4^T (X + 2Y), \\ &= (\omega_1^T + \omega_4^T)X + \omega_4^T Y + (\omega_4^T Y - \omega_1^T).\end{aligned}$$

As $p(4) = p(N)$, we have $\omega_4^T Y - \omega_1^T = \omega_4 + \omega_1 - \omega_1^T \geq 0$ and the inequality is satisfied. If $\alpha_p = 0$, we have

$$\begin{aligned}\sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)) &\geq \omega_1^T (X - 1) + \omega_4^T Z, \\ &= (\omega_1^T + \omega_4^T)X + \omega_4^T + (\omega_4^T Y - \omega_1^T) + \frac{\omega_1 \omega_4^T}{\omega_2 + \omega_3 + \omega_4} X.\end{aligned}$$

If $p(4) < p(N)$ then by assumption we also have $p(1) < p(N)$ and the (ω, Σ) -convexity inequality is trivially satisfied. Otherwise, we have $\omega_4^T Y - \omega_1^T = \omega_4 + \omega_1 - \omega_1^T \geq 0$ and the inequality is also satisfied.

• If S is equal to $\{2, 4\}$, then

$$\sum_{i \in S} \omega_i^T (v(S) - v(S \setminus i)) = \alpha_p (\omega_2^T + \omega_4^T) Y.$$

There are 3 possible cases for T : $\{1, 2, 4\}$, $\{2, 3, 4\}$ and N .

– Let us assume that $T = \{1, 2, 4\}$,

$$\begin{aligned}\sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)) &= \omega_2^T (X + \alpha_p (Y - 1) - X) + \omega_4^T (X + \alpha_p (Y - 1) - 0), \\ &= \alpha_p (\omega_2^T + \omega_4^T) Y + \omega_4^T X - \alpha_p (\omega_2^T + \omega_4^T).\end{aligned}$$

If $\alpha_p = 1$, then $p(2) = p(4)$ and we have $\omega_4 X - \omega_2 - \omega_4 \geq 0$ by definition of X .

– Let us assume $T = \{2, 3, 4\}$,

$$\sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)) = \omega_2^T (Z - Y) + \omega_4^T (Z - 0).$$

If $\alpha_p = 1$, then $Z = X + 2Y$ and the inequality is satisfied.

– Finally, let $T = N$, then

$$\begin{aligned}\sum_{i \in S} \omega_i^T(v(T) - v(T \setminus i)) &= \omega_2^T(\Theta - X - Y + 1) + \omega_4^T(\Theta - \alpha_p(X - 1)), \\ &= \omega_2^T(Z - Y) + \omega_4^T(Z + (1 - \alpha_p)(X - 1)).\end{aligned}$$

If $\alpha_p = 1$, then $Z = X + 2Y$ and the inequality is also satisfied.

• If S is equal to $\{3, 4\}$, then

$$\sum_{i \in S} \omega_i^T(v(S) - v(S \setminus i)) = \omega_3^T Y + \omega_4^T Y = (\omega_3^T + \omega_4^T)Y.$$

There are 3 possible cases for T : $\{1, 3, 4\}$, $\{2, 3, 4\}$ and N .

– Let us assume that $T = \{1, 3, 4\}$,

$$\begin{aligned}\sum_{i \in S} \omega_i^T(v(T) - v(T \setminus i)) &= \omega_3^T(X + Y - 1 - X) + \omega_4^T(X + Y - 1 - 0), \\ &= (\omega_3^T + \omega_4^T)Y + \omega_4^T X - (\omega_3^T + \omega_4^T), \\ &\geq (\omega_3^T + \omega_4^T)Y + \omega_3 - \omega_3^T.\end{aligned}$$

The last inequality is by definition of X and as $p(4) = p(\{1, 3, 4\})$. Therefore the (ω, Σ) -convexity inequality is satisfied.

– Let us now assume that $T = \{2, 3, 4\}$,

$$\sum_{i \in S} \omega_i^T(v(T) - v(T \setminus i)) = \omega_3^T(Z - \alpha_p Y) + \omega_4^T(Z - 0).$$

If $\alpha_p = 1$, then $Z = X + 2Y$ and the inequality is also satisfied.

– Finally, let $T = N$, then

$$\begin{aligned}\sum_{i \in S} \omega_i^T(v(T) - v(T \setminus i)) &= \omega_3^T(\Theta - X - \alpha_p(Y - 1)) + \omega_4^T(\Theta - \alpha_p(X - 1)), \\ &= \omega_3^T(Z - 1 - \alpha_p(Y - 1)) + \omega_4^T(Z + (1 - \alpha_p)(X - 1)).\end{aligned}$$

If $\alpha_p = 0$, then the inequality is satisfied as $Z \geq Y + 1$. If $\alpha_p = 1$, then $Z = X + 2Y$ and the inequality is also satisfied.

• Let $S = \{1, 2, 3\}$ and $T = N$, then

$$\sum_{i \in S} \omega_i^T(v(S) - v(S \setminus i)) = \alpha_p(\omega_1^T + \omega_2^T + \omega_3^T)(X - 1),$$

and

$$\begin{aligned}\sum_{i \in S} \omega_i^T(v(T) - v(T \setminus i)) &= \omega_1^T(\Theta - Z) + \omega_2^T(\Theta - (X + Y - 1)) + \omega_3^T(\Theta - (X + \alpha_p(Y - 1))), \\ &= \omega_1^T(X - 1) + \omega_2^T(Z - Y) + \omega_3^T(Z - 1 - \alpha_p(Y - 1)).\end{aligned}$$

If $\alpha_p = 0$, then the inequality is satisfied as $Z \geq Y + 1$. If $\alpha_p = 1$, then $Z = X + 2Y$ and the inequality is also satisfied.

- Let $S = \{1, 2, 4\}$ and $T = N$, then

$$\sum_{i \in S} \omega_i^T(v(S) - v(S \setminus i)) = \omega_1^T(X - \alpha_p) + \alpha_p \omega_2^T(Y - 1) + \omega_4^T(X + \alpha_p(Y - 1)),$$

and

$$\begin{aligned} \sum_{i \in S} \omega_i^T(v(T) - v(T \setminus i)) &= \omega_1^T(\Theta - Z) + \omega_2^T(\Theta - (X + Y - 1)) + \omega_4^T(\Theta - \alpha_p(X - 1)), \\ &= \omega_1^T(X - 1) + \omega_2^T(Z - Y) + \omega_4^T(Z + (1 - \alpha_p)(X - 1)). \end{aligned}$$

If $\alpha_p = 0$, then we can conclude as in the case $S = \{1, 4\} \subseteq T = N$. If $\alpha_p = 1$, then $Z = X + 2Y$ and the inequality is also satisfied.

- Let $S = \{1, 3, 4\}$ and $T = N$

$$\begin{aligned} \sum_{i \in S} \omega_i^T(v(S) - v(S \setminus i)) &= \omega_1^T(X + Y - 1 - Y) + \omega_3^T(X + Y - 1 - X) + \omega_4^T(X + Y - 1 - 0), \\ &= \omega_1^T(X - 1) + \omega_3^T(Y - 1) + \omega_4^T(X + Y - 1). \end{aligned}$$

and

$$\begin{aligned} \sum_{i \in S} \omega_i^T(v(T) - v(T \setminus i)) &= \omega_1^T(\Theta - Z) + \omega_3^T(\Theta - X - \alpha_p(Y - 1)) + \omega_4^T(\Theta - \alpha_p(X - 1)), \\ &= \omega_1^T(X - 1) + \omega_3^T(Z - 1 - \alpha_p(Y - 1)) + \omega_4^T(Z + (1 - \alpha_p)(X - 1)). \end{aligned}$$

If $\alpha_p = 0$, then the inequality is satisfied as $Z \geq Y$. If $\alpha_p = 1$, then $Z = X + 2Y$ and the inequality is also satisfied.

- Finally, let $S = \{2, 3, 4\}$ and $T = N$,

$$\sum_{i \in S} \omega_i^T(v(S) - v(S \setminus i)) = \omega_2^T(Z - Y) + \omega_3^T(Z - \alpha_p Y) + \omega_4^T(Z - 0),$$

and

$$\begin{aligned} \sum_{i \in S} \omega_i^T(v(T) - v(T \setminus i)) &= \omega_2^T(\Theta - (X + Y - 1)) + \omega_3^T(\Theta - (X + \alpha_p(Y - 1))) \\ &\quad + \omega_4^T(\Theta - \alpha_p(X - 1)) \\ &= \omega_2^T(Z - Y) + \omega_3^T(Z - \alpha_p Y) + \omega_4^T(Z - 0) \\ &\quad - \omega_3^T(1 - \alpha_p) + \omega_4^T(1 - \alpha_p)(X - 1). \end{aligned}$$

If $\alpha_p = 1$, then the inequality is tight. Let us assume $\alpha_p = 0$. If $p(4) < p(N)$ then by assumption we also have $p(3) < p(N)$ and the inequality is trivially satisfied. If $p(4) = p(N)$, then we also have $p(3) = p(N)$ as $\alpha_p = 0$ and by definition of X we get $-\omega_3^T(1 - \alpha_p) + \omega_4^T(1 - \alpha_p)(X - 1) = -\omega_3 + \omega_4(X - 1) \geq 0$ and the inequality is satisfied.

D Proof of Counter-example induced 4-path with strict inequality

We show that Example 3 described page 13 provides a counter-example if

$$p(2) = p(4) > \max(p(1), p(3)).$$

The game (N, v) is not convex:

The marginal contribution of player 1 is strictly smaller to $\{3, 4\}$ than to $\{4\}$.

$$v(\{1, 4\}) - v(\{4\}) = 1 - 0 = 1 > 0 = 1 - 1 = v(\{1, 3, 4\}) - v(\{3, 4\}).$$

The game (N, v) is (ω, Σ) -convex:

We need to check the inequalities for any pair of non-empty sets S and T such that $S \subset T$.

- Let us first notice that the function v is monotonic. Hence if S has value 0, we have

$$\sum_{i \in S} \omega_i^T (v(S) - v(S \setminus i)) = 0 - 0 = 0. \quad (44)$$

Since the function v is monotonic, each marginal contribution is positive and the (ω, Σ) -convexity inequality is satisfied for any T containing S . (44) is satisfied if S is a singleton, and if S is equal to $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{2, 4\}$, or $\{1, 2, 3\}$.

- If S is equal to $\{1, 4\}$, then

$$\sum_{i \in S} \omega_i^T (v(S) - v(S \setminus i)) = \omega_1^T + \omega_4^T.$$

There are 3 possible cases for T : $\{1, 2, 4\}$, $\{1, 3, 4\}$ and N . As $p(4) > p(1)$, we have $\omega_1^T = 0$ and get $\sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)) = \omega_4^T$ in any case. Therefore the (ω, Σ) -convexity inequality is satisfied.

- If S is equal to $\{3, 4\}$, then

$$\sum_{i \in S} \omega_i^T (v(S) - v(S \setminus i)) = \omega_3^T + \omega_4^T.$$

There are 3 possible cases for T : $\{1, 3, 4\}$, $\{2, 3, 4\}$ and N . As $p(4) > p(3)$, we have $\omega_3^T = 0$ and get $\sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)) = \omega_4^T$ in any case.

- Let $S = \{1, 2, 4\}$ and $T = N$. As $p(2) = p(4) > p(1)$, we have

$$\sum_{i \in S} \omega_i^T (v(S) - v(S \setminus i)) = \omega_2^T (1 - 1) + \omega_4^T (1 - 0) = \sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)).$$

- Let $S = \{1, 3, 4\}$ and $T = N$. The inequality is satisfied as

$$\sum_{i \in S} \omega_i^T (v(S) - v(S \setminus i)) = \omega_1^T (1 - 1) + \omega_3^T (1 - 1) + \omega_4^T (1 - 0) = \sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)).$$

- Finally, let $S = \{2, 3, 4\}$ and $T = N$. As $p(2) = p(4) > p(3)$, we have

$$\sum_{i \in S} \omega_i^T (v(S) - v(S \setminus i)) = \omega_2^T (1 - 1) + \omega_4^T (1 - 0) = \sum_{i \in S} \omega_i^T (v(T) - v(T \setminus i)).$$

References

- Gillies, D. B. (1959). Solutions to general non-zero-sum games. *Contributions to the Theory of Games*, 4(40):47–85.
- Iñarra, E. and Usategui, J. (1993). The shapley value and average convex games. *International Journal of Game Theory*, 22:13 – 29.
- Kalai, E. and Samet, D. (1987). On weighted shapley values. *International Journal of Game Theory*, 16:205–222.
- Monderer, D., Samet, D., and Shapley, L. S. (1992). Weighted values and the core. *International Journal of Game Theory*, 21:27–39.
- Myerson, R. (1977). Graphs and cooperation in games. *Mathematics of Operations Research*, 2(3):225–229.
- Schouten, J., Dietzenbacher, B., and Borm, P. (2022). The nucleolus and inheritance of properties in communication situations. *Annals of Operations Research*, pages 1–19.
- Shapley, L. S. (1953a). *Additive and Non-Additive Set Functions*. PhD thesis, Princeton.
- Shapley, L. S. (1953b). A value for n -person games. In *Contributions to the Theory of Games (AM-28), Volume II*, pages 307–318. Princeton University Press.
- Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory*, 1(1):11–26.
- Skoda, A. and Venel, X. (2023). Weighted average-convexity and shapley values. *Games and Economic Behavior*, 140:88–98.
- Slikker, M. (2000). Inheritance of properties in communication situations. *Int. J. Game Theory*, 29(2):241–268.
- Slikker, M. and van den Nouweland, A. (2000). Communication situations with asymmetric players. *Mathematical Methods of Operations Research*, 52:39 – 56.
- Slikker, M. and van den Nouweland, A. (2001). *Social and Economic Networks in Cooperative Game Theory*, volume 27. Kluwer Academic Publishers, Norwell, MA.
- Sprumont, Y. (1990). Population monotonic allocation schemes for cooperative games with transferable utility. *Games and Economic Behavior*, 2(4):378–394.

van den Nouweland, A. and Borm, P. (1991). On the convexity of communication games.
International Journal of Game Theory, 19(4):421–30.