

# Effects of graph operations on star pairwise compatibility graphs

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## Abstract

A graph  $G = (V, E)$  is defined as a star- $k$ -pairwise compatibility graph (PCG) when it is possible to assign a positive real number weight  $w$  to each vertex  $V$ , and define  $k$  distinct intervals  $I_1, I_2, \dots, I_k$ , in such a way that there is an edge  $uv$  in  $E$  if and only if the sum of the weights of vertices  $u$  and  $v$  falls within the union of these intervals. The star- $k$ -PCG class is connected to two significant graph categories: PCGs and multithreshold graphs. The star number of a graph  $G$ , is the smallest  $k$  for which  $G$  is a star- $k$ -PCG. In this paper, we study the effects of various graph operations, such as the addition of twins, pendant vertices, universal vertices, or isolated vertices, on the star number of the graph resulting from these operations. As significant applications of our findings, we determine the star number of lobster graphs and provide an upper bound for the star number of acyclic graphs. This is particularly interesting as determining the star number is notoriously difficult and is known only for a few classes of graphs. Indeed, for acyclic graphs, the exact value of the star number is currently known only for caterpillars [1].

## 1. INTRODUCTION

A  $k$ -pairwise compatibility graph  $G$  (shortly  $k$ -PCG), is graph for which there exists a non-negative edge-weighted tree  $T$  and  $k$  distinct intervals  $I_1, I_2, \dots, I_k$  of non-negative real numbers such that each vertex of  $G$  corresponds to a leaf of  $T$ , and there is an edge between two vertices in  $G$  if and only if the distance between their corresponding leaves in  $T$  falls within  $I_1 \cup I_2 \cup \dots \cup I_k$  [2]. The tree  $T$  serving this function is known as the  $k$ -witness tree. Figure 1 illustrates an example of a graph that is 1-PCG and another graph that is 2-PCG (but not 1-PCG).

The concept of 1-PCGs, also known as PCGs, originated from the problem of reconstructing phylogenetic trees [4] and it has proven interesting in different contexts in graph theory [2, 5–9]. Despite the research efforts, a complete characterization of the  $k$ -PCG class remains an open problem even for  $k = 1$  [5, 6]. To reduce the problem's complexity and make it more approachable, one approach is to impose restrictions on the topology of the witness tree. Stars are one of the most basic types of trees, and therefore it is natural to explore the case where the witness tree in the  $k$ -PCG is restricted to be a star. This leads to the notion of a star- $k$ -PCG graph [10], that is  $k$ -PCGs for which there exists a witness tree that is a star.

Notice that for a  $k$ -PCG the witness tree is not unique. An example of a graph  $G$  together with two different witness trees is provided in Fig. 2. For a graph to be a star- $k$ -PCG it is sufficient that one of its witness trees is a star.

Although stars are a relatively simple topology, the recognition of star-PCGs is not an easy problem. It was only in [11] that Xiao and Nagamochi developed an algorithm for recognizing

star-1-PCGs with a time complexity of  $O(n^3m)$ , where  $n$  and  $m$  denote the number of vertices and edges in the graph, respectively. Subsequently, Kobayashi et al. [12] were able to improve the time complexity for recognizing star-1-PCGs by proposing a linear-time recognition algorithm. For  $k > 1$ , the problem of recognizing star- $k$ -PCGs remains open. Nonetheless, these works laid the foundation for the introduction of multithreshold graphs [1]. Notably, star-1-PCGs are also known as double-threshold graphs.

The class of star- $k$ -PCGs is strongly connected to multithreshold graph class, which has gained considerable interest within the research community since its introduction in [1], as shown by the following studies [1, 13–16]. In particular [10, 12], it is shown that the class of star- $k$ -PCGs is equivalent to the class of  $2k$ -threshold graphs. Multithreshold graphs form a class with relatively few known graph families, and thus the results of this paper also contribute to advance understanding in this area.

To study star- $k$ -PCGs, it is natural to define the *star number* of a graph  $G$ , denoted as  $\gamma(G)$ , which is the smallest positive integer  $k$  such that  $G$  is a star- $k$ -PCG [10, 17].

In this study, we determine the star number of various simple graph classes. Our approach involves analyzing how specific graph operations impact the star number. Specifically, we consider the following operations: adding universal and isolated vertices, incorporating true/false twins, appending a pendant vertex, and complementing a graph. This method is chosen because many graph classes arise from simpler ones through these operations (see, e.g. [18]). Threshold graphs, for instance, can be constructed from a single-vertex graph by iteratively adding either a universal or an isolated vertex. Distance-hereditary graphs can

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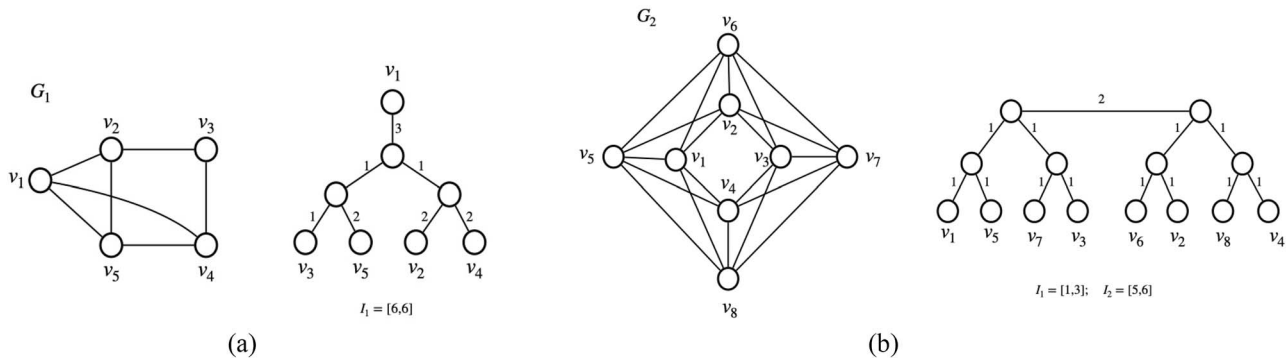


Figure 1. (a) Graph  $G_1$  that is a 1-PCG, shown with its 1-witness tree and the corresponding interval, and (b) graph  $G_2$  that is not 1-PCG (as proven in [3]) but is 2-PCG, displayed with its 2-witness tree and the corresponding intervals.

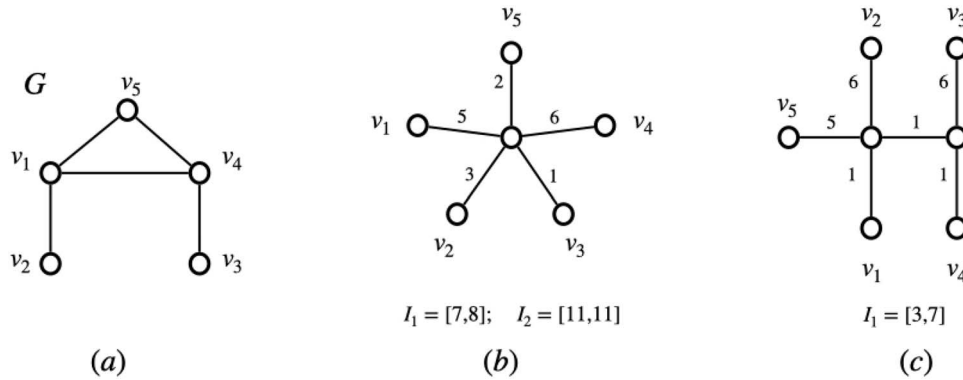


Figure 2. (a) Graph  $G$ ; (b) Star 2-witness tree for  $G$  with the corresponding intervals; (c) 1-witness tree for  $G$  with the corresponding interval.

be recursively built from the one vertex graph by adding either an isolated vertex or a new pendant vertex or a new vertex that is a false or true twin for an existing vertex  $v$ . Moreover, these operations also have practical relevance in real-world applications. Twins often arise in social and biological networks, where they represent entities with identical connections. Pendant vertices frequently occur in hierarchical structures or as terminal nodes in network topologies, such as phylogenetic trees or computer networks. Universal vertices, which act as central hubs, are critical for understanding connectivity and robustness in transportation and communication networks.

The effect of graph operations on graph parameters is well studied, such as for bandwidth in [19], for the number of spanning trees [20], for graph labeling [21, 22]. Graph operations have been also considered in the context of PCGs. For instance, Xiao *et al.* [23] have explored the impact of different graph operations on PCGs. Building on this research, we examine the effects of certain graph operations on the star number in star-PCGs.

As an application of our results, we determine the star number of lobsters and provide an upper bound for the star number of acyclic graphs. More specifically, we show that the star number of any lobster of radius at least three, is 2, while the star number of any acyclic graph is at most its radius number. This represents an advancement beyond the well-known general result stating that for any graph  $G = (V, E)$ ,  $\gamma(G) \leq |E|$  [2]. A direct corollary of our findings is that the multithreshold number [1] of lobsters ranges between 3 and 4, and for acyclic graphs, it is at most twice their radius number. This is particularly interesting because obtaining good bounds on the multithreshold number is notoriously difficult. Indeed, for acyclic graphs, the exact answer is known only for caterpillars [1].

## 2. PRELIMINARIES

In this paper, we only consider simple graphs, that is graphs that contain no loops or multiple edges. We focus on undirected graphs and for simplicity, we use a notational shorthand, writing  $uv$  to represent the unordered pair  $\{u, v\}$ . For a graph  $G = (V, E)$  and a vertex  $u \in V$ , the set  $N(u) = \{v : uv \in E\}$  is called the *neighborhood* of  $u$ .

Two distinct vertices  $u$  and  $v$  in  $G$  are called *true twins* if  $N(u) \cup \{u\} = N(v) \cup \{v\}$ , and *false twins* if  $N(u) = N(v)$ .

A vertex  $u$  of a graph  $G = (V, E)$  is said *isolated* if it has degree 0, i.e.  $|N(u)| = 0$ ; is said *universal* if it has degree  $|V| - 1$ , i.e.  $|N(u)| = |V| - 1$ ; and is said *pendant* if it has degree 1, i.e.  $|N(u)| = 1$ . For a pendant vertex  $v$  we will denote by  $p(v)$  the only vertex adjacent to  $v$ . A pendant edge is an edge that is incident to a pendant vertex.

For any integer  $n \geq 1$ , we denote by  $P_n$  the path on  $n$  vertices.

A *caterpillar* is a tree in which the removal of all pendant vertices results in a simple path.

A *lobster* is a graph such that when we delete its pendant vertices, we obtain a caterpillar. In unweighted graphs the *length* of a path is the number of edges in it. If the edges of the graph are weighted, the length of the path is the sum of the weights of the edges in it. The *distance* between two vertices  $u$  and  $v$  is the minimum length a path between  $u$  and  $v$ . If no path exists between  $u$  and  $v$  the distance is set to be infinite.

For a connected graph the *eccentricity* of a vertex  $v$ , denoted as  $\text{ecc}(v)$ , is defined as the maximum distance from  $v$  to any other vertex in the graph. The *radius* of a graph  $G$ , denoted as  $\text{rad}(G)$ , is the minimum eccentricity among all the vertices in  $G$ . A vertex of minimum eccentricity is called *center* of the graph. For a disconnected graph  $G$ , we use the convention that the eccentricity

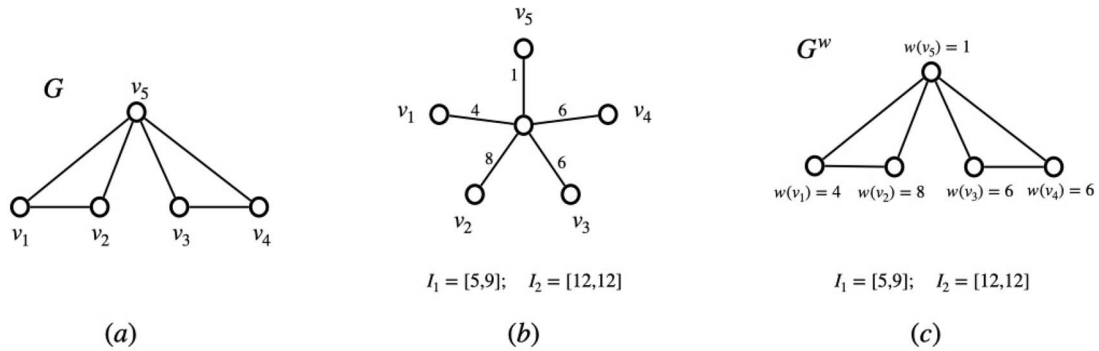


Figure 3. (a) Graph  $G$ ; (b) Star 2-witness tree for  $G$  with the corresponding intervals; (c) 2-witness graph  $G^w$  and the corresponding intervals.

(radius) of  $G$  is the maximum eccentricity (radius) of its connected components.

For star- $k$ -PCGs, the distance between any two leaves in the witness star tree is the sum of the weights of the two edges incident to these vertices. Therefore, it is possible to eliminate the star tree and use a definition that employs weights on the vertices of the graph (see Fig. 3). Thus, throughout the paper, we will use the following equivalent definition of star- $k$ -PCGs.

**Definition 1.1.** A graph  $G = (V, E)$  is a star- $k$ -PCG if there exists a weight function  $w : V \rightarrow \mathcal{R}^+$  and  $k$  mutually exclusive intervals  $I_1, I_2, \dots, I_k$ , such that there is an edge  $uv \in E$  if and only if  $w(u) + w(v) \in \bigcup_i I_i$  [10]. Such a vertex weighted graph  $G^w$  is called the  $k$ -witness of  $G$ .

For a  $k$ -witness graph  $G^w$  and for each vertex  $v$ ,  $w(v)$  denotes the weight of the vertex  $v$  and for each pair of vertices  $u$  and  $v$  of  $G$  we denote by  $w(uv)$  the sum  $w(u) + w(v)$ .

Given a  $k$ -witness graph  $G^w$ , for the sake of simplicity we will always assume that the  $k$  disjoint intervals  $I_1 = [a_1, b_1], \dots, I_k = [a_k, b_k]$  are ordered in increasing order, that is for all  $1 \leq i \leq k - 1$ ,  $b_i < a_{i+1}$ .

**Claim 1.** Let  $G$  be a graph with star number  $k > 1$  and let  $G^w$  be a  $k$ -witness graph and  $I_i = [a_i, b_i]$  for  $1 \leq i \leq k$  the corresponding intervals. Then for any  $1 \leq i \leq k - 1$  there exist a non-edge  $e \notin E$  such that  $b_i < w(e) < a_{i+1}$ .

**Proof.** If there exists an  $1 \leq i \leq k - 1$  for which the condition of the claim does not hold, then the two intervals  $I_i, I_{i+1}$  could be merged in a single interval  $[a_i, b_{i+1}]$  and  $G^w$  would be a  $(k - 1)$ -witness, contradicting the hypothesis that the star number of  $G$  is  $k$ .

Consider a graph  $G$  with star number  $k$  and let  $G^w$  be a  $k$ -witness of  $G$ , with intervals  $I_i = [a_i, b_i]$  for  $1 \leq i \leq k$ . For any non-edge  $xy \notin E$  exactly one of the followings must hold: (i)  $w(xy) < a_1$ , (ii) there exists an  $1 \leq i \leq k - 1$  and  $b_i < w(xy) < a_{i+1}$ , (iii)  $w(xy) > b_k$ . Thus, the set of non-edges can be partitioned into three subsets based on which of the conditions (i)–(iii) is satisfied. From the Claim 1 the subset of non-edges that satisfies (ii) is always non empty for any  $k > 1$ . Based on all of this, we define three types of  $k$ -witness graphs. ■

**Definition 1.2.** Consider a graph  $G$  with star number  $k$  and let  $G^w$  be a  $k$ -witness for  $G$ . We say that:

- $G^w$  is *left-free* if for every non-edge  $e$  we have  $w(e) > b_1$ .
- $G^w$  is *right-free* if for every non-edge  $e$  we have  $w(e) < a_k$ .

- $G^w$  is *double-free* if it is both *right- and left-free*, that is, if for every non-edge  $e$  we have  $b_1 < w(e) < a_k$ .

In Fig. 4(a)–(c), we present an example of each of the three types and in Fig. 4(d), we present a witness graph that is neither left-free nor right-free.

**Lemma 1.1.** Let  $G$  be a graph with  $\gamma(G) = k$ , then there exists a left-free  $k$ -witness  $G^w$  if and only if there exists a right-free  $k$ -witness  $G^{w'}$ .

**Proof.** Let  $G = (V, E)$  be a graph with star number  $k$  and let  $G^w$  be a  $k$ -witness that is right-free with  $I_i = [a_i, b_i]$  for  $1 \leq i \leq k$  the corresponding intervals. If  $G^w$  is also left-free, the proof follows. Otherwise notice that for all pairs of vertices  $u, v$  (that not necessarily correspond to edges) we have  $w(uv) \leq b_k$  and thus for any  $u \in V$ ,  $w(u) \leq b_k$ . We define now a left-free  $G^{w'}$  as follows. For all  $u \in V$  we set  $w'(u) = b_k - w(u)$  and for any  $1 \leq i \leq k$ , we define  $I'_i = [2b_k - b_i, 2b_k - a_i]$ . Notice that all the new weights and intervals are positive. Moreover, the new intervals  $I'_i$  are ordered in increasing order. Now observe that for any edge  $uv \in E$  there exists an  $i$  such that  $a_i \leq w(uv) \leq b_i$ . Then  $2b_k - b_i \leq w'(uv) = 2b_k - w(uv) \leq 2b_k - a_i$  and thus  $w'(uv) \in I'_i$ . Next, for any  $uv \notin E$ , as  $G^w$  is right-free we have that either  $w(uv) < a_1$  or there exists and  $1 \leq i \leq k - 1$  for which  $b_i < w(uv) < a_{i+1}$ . Then we have  $w'(uv) = 2b_k - w(uv)$  and thus either  $w'(uv) > 2b_k - a_1$  or  $2b_k - a_i < w'(uv) < 2b_k - b_{i+1}$ . Hence, we have  $w'(uv) \notin \bigcup_i I'_i$ . Thus,  $G^{w'}$  is demonstrated to be a left-free  $k$ -witness.

Similarly, we can show that if  $G^w$  is left-free, then it is possible to construct a  $G^{w'}$  that is right-free. Fig. 5(a)–(c) illustrates this result. ■

Based on Lemma 1.1 from now on we will simply use *free witness* without specifically mentioning “left” or “right.” The next result shows that there exist graphs  $G$  that do not have a free  $\gamma(G)$ -witness.

**Theorem 1.1.** Let  $G$  be a graph with  $\gamma(G) = 1$  that contains  $P_4$  as an induced subgraph. Then every 1-witness of  $G$  is not free.

**Proof.** To prove the claim, it is enough to prove that  $P_4$  has no free 1-witness. Indeed, if we assume on the contrary that there exists a free 1-witness  $P_n^w$ , then as  $\gamma(G) = \gamma(P_4) = 1$  it is not difficult to see that a subgraph of  $G^w$  would serve as a free 1-witness for  $P_4$ . Therefore, we proceed to prove that  $P_4$  has no free 1-witness. Let  $P_4 = v_1, v_2, v_3, v_4$  where  $v_i, v_{i+1} \in E(P_4)$  with  $1 \leq i \leq 3$ . Consider any 1-witness  $P_4^w$  with corresponding interval

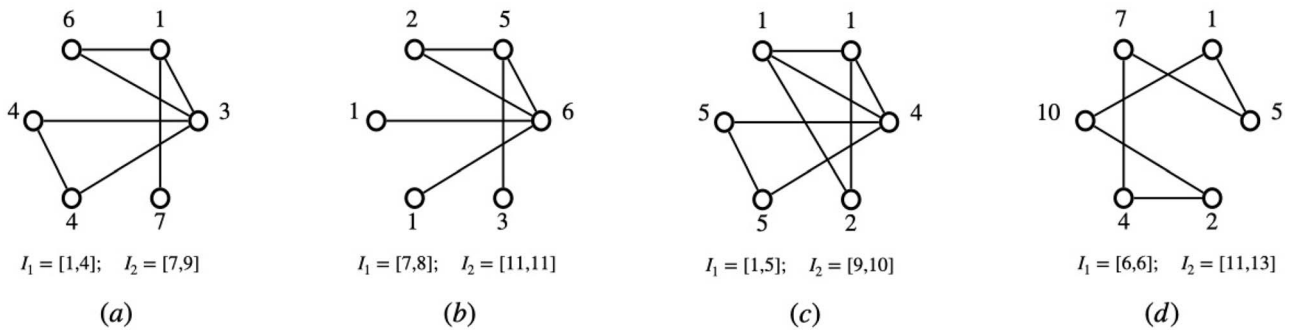


Figure 4. (a) 2-witness graph that is left-free but not right-free; (b) 2-witness graph that is right-free but not left-free; (c) 2-witness graph that is double-free; (d) 2-witness graph that is neither left-free nor right-free.

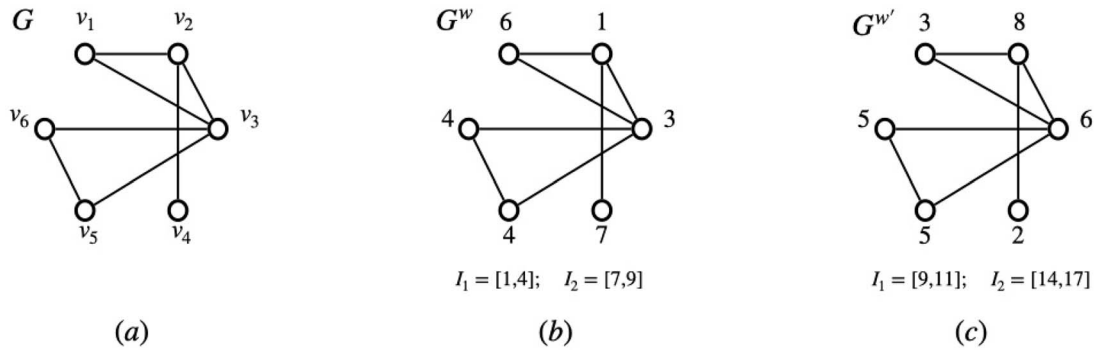


Figure 5. (a) Graph  $G$ ; (b) 2-witness graph  $G^w$  that is left-free but not right-free; (c) 2-witness graph  $G^{w'}$  constructed based on Lemma 1.1 that is right-free but not left-free.

$I_1 = [a_1, b_1]$ . W.l.o.g. assume  $w(v_1) < w(v_4)$ . Then we have  $a_1 \leq w(v_1v_2) < w(v_1v_4)$  and thus necessarily it must hold  $w(v_1v_4) > b_1$ . Moreover  $b_1 \geq w(v_3v_4) > w(v_3v_1)$  and we must have  $w(v_3v_1) < a_1$ . Thus,  $P_4^w$  cannot be free. ■

In [10, 11] it is shown that  $\gamma(P_n) = 1$  and thus we have the following

**Corollary 1.1.** Every 1-witness of  $P_n$ , with  $n \geq 4$ , is not free.

We define a normal form for the  $k$ -witness graphs.

**Definition 1.3.** Let  $G = (V, E)$  be a graph with  $\gamma(G) = k$ . A  $k$ -witness  $G^w$  is said to be in *normal form* if for any vertex  $x \in V$ , both of the followings are true

- (i) for any vertex  $u \neq x$  in  $V$ , it holds  $w(x) \neq w(u)$ .
- (ii) for any two distinct vertices  $u_1$  and  $u_2$  in  $V$  it holds  $2w(x) \neq w(u_1) + w(u_2)$

The next lemma shows that every graph  $G$  has a  $\gamma(G)$ -witness in normal form.

**Lemma 1.2.** Let  $G = (V, E)$  be a graph with  $\gamma(G) = k$ , then there exists a  $k$ -witness  $G^w$  in normal form.

**Proof.** Notice that in the extreme case where the graph  $G$  is complete (or empty), the claim trivially holds as it is always possible to assign different weights to the vertices and set the single interval  $I$  appropriately in such a way both (i) and (ii) are satisfied. So we consider now graphs that have at least one edge

and one non-edge. Let  $G = (V, E)$  be a graph with star number  $k$  and let  $G^w$  a  $k$ -witness graph and  $I_i = [a_i, b_i]$  for  $1 \leq i \leq k$  the corresponding intervals. Let  $x$  be a vertex in  $V$  for which at least one among (i) and (ii) does not hold. If no such vertex  $x$  exists, we are done. However, if  $x$  does exist, we adjust the weight of  $x$  and the intervals  $I_i$ , as we describe next, to ensure that, within this updated  $k$ -witness configuration,  $x$  now meets both conditions (i) and (ii). Furthermore, vertices which already met these conditions continue to do so. By repeating this process, we consistently reduce the number of vertices that violate the conditions in (i) and (ii), and thus the proof will follow. We define the following three quantities:

$$\delta_1 = \min_{\substack{xy \in E \\ 1 \leq i \leq k}} \{ |w(xy) - b_i|, |w(xy) - a_i| \} > 0$$

Notice that for every non-edge  $xy$ , its weight is at least  $\delta_1$  apart from the endpoints of each of the intervals  $I_i$ . Moreover,  $\delta_1 > 0$  as there is at least one non-edge in  $G$ .

$$\delta_2 = \min_{\substack{y \in V \\ w(y) \neq w(x)}} \{ |w(x) - w(y)| \} > 0$$

Notice that all weights different from  $w(x)$  differ from it by at least  $\delta_2$ . This will be useful in proving that condition (i) is met for  $x$ . Moreover,  $\delta_2 > 0$  as the only case where all the vertices have equal weight is where the graph is complete or empty.

$$\delta_3 = \min_{\substack{u_1, u_2, u_3 \in V \\ x \in \{u_1, u_2, u_3\}; 2w(u_1) \neq w(u_2u_3)}} \{ |2w(u_1) - w(u_2u_3)| \}$$

The way we defined  $\delta_3$  will help proving that condition (ii) is met for  $x$ . Finally, we set  $\epsilon = \frac{1}{4} \min\{\delta_1, \delta_2, \delta_3\}$  if  $\delta_3 > 0$ , otherwise  $\epsilon = \frac{1}{4} \min\{\delta_1, \delta_2\}$ .

Then define  $w' : V \rightarrow \mathcal{R}^+$  as follows

$$w'(v) = \begin{cases} w(v) + \epsilon & \text{if } v = x, \\ w(v) & \text{otherwise} \end{cases}$$

Finally, the new intervals  $I'_i$  are defined by slightly enlarging the original intervals to accommodate the updated weights. Intuitively, each interval is extended by  $\epsilon$  on the right. Since  $\epsilon$  is carefully chosen as a fraction of the smallest separation between the interval boundaries and the weights of non-edges, this adjustment ensures that the intervals can be tuned without problems to consistently satisfy all the required conditions.

Formally, for  $1 \leq i \leq k$  we set  $I'_i = [a_i, b_i + \epsilon]$ . Notice that as  $\epsilon < \delta_1$ , from Claim 1, we have that the intervals  $I'_i$  do not overlap. Moreover, by construction it is not difficult to see that  $x$  satisfies both condition (i) and (ii). In addition, as we changed only the weight of  $x$  then all the vertices  $y$  that satisfied both conditions (1) and (2) still continue to satisfy them.

It remains to prove that  $G^{w'}$  is a  $k$ -witness with respect to the intervals  $I'_1, \dots, I'_k$ . Given two vertices  $u$  and  $v$  there are two cases to consider:

- $u \neq x$  and  $v \neq x$ . Then as  $\epsilon < \delta_1$  it is easy to see that  $w(uv) \in \cup_i I_i$  if and only if  $w'(uv) \in \cup_i I'_i$ .
- $u = x$ . Then we have  $w'(xu) = w(xv) + \epsilon$ . Clearly, if  $w(xv) \in I_i$  for some  $1 \leq i \leq k$ , then  $a_i \leq w(xv) \leq b_i$ . Thus we have  $a_i \leq w'(xu) = w(xv) + \epsilon \leq b_i + \epsilon$ . Thus  $w'(xu) \in I'_i$ . Otherwise if  $w(xv) \notin \cup_i I_i$  since  $\epsilon < \delta_1$  we have that  $w'(xu) = w(xv) + \epsilon \notin \cup_i I_i$ .

This concludes the proof.  $\blacksquare$

It is not difficult to see that the proof of Lemma 1.2 preserves the free property of a witness, that is, if the graph  $G^w$  is a left-free (or right-free)  $k$ -witness then  $G^{w'}$  is a left-free (or right-free)  $k$ -witness in normal form. Thus we have the following.

**Corollary 1.2.** For any graph  $G$  with  $\gamma(G) = k$ , if there exists a left-free (or right-free)  $k$ -witness, then there also exists a left-free (or right-free)  $k$ -witness in normal form.

### 3. GRAPH OPERATIONS ON STAR- $k$ -PCGS

In this section, we consider how the star number of a graph changes after we perform one of the following operations:

- (a) Adding an isolated vertex;
- (b) Adding a universal vertex;
- (c) Adding a pendant vertex;
- (d) Adding a vertex that is a false twin for the old vertex  $v$ ;
- (e) Adding a vertex that is a true twin for the old vertex  $v$ ;
- (f) Graph complement;

Figure 6 illustrates an example for each one of these graph operations.

**Theorem 2.1.** Given a graph  $G$  with  $\gamma(G) = k$ , let  $G'$  be the graph obtained from  $G$  by adding an isolated vertex. It holds that  $\gamma(G') = k$ .

**Proof.** Let  $G = (V, E)$  be a graph with star number  $k$  and let  $G^w$  be a  $k$ -witness graph with  $I_i = [a_i, b_i]$  for  $1 \leq i \leq k$ . Now consider

the graph  $G'$  obtained from  $G$  by adding an isolated vertex  $x$ . We construct the  $k$ -witness  $G^{w'}$  from  $G^w$  by assigning the weight  $b_k + 1$  to the new vertex  $x$  and keeping the intervals unchanged, that is  $I'_i = I_i$  for  $1 \leq i \leq k$ . Notice that as for any vertex  $v \in V$ ,  $w(vx) = w(v) + w(x) \geq b_k + 1 \notin I'_i$  for any  $1 \leq i \leq k$ ,  $G^{w'}$  is a  $k$ -witness for  $G'$ .  $\blacksquare$

**Theorem 2.2.** Given a graph  $G$  with  $\gamma(G) = k$ , let  $G'$  be the graph obtained from  $G$  by adding a universal vertex. If there exists a  $G^w$  that is free then  $\gamma(G') = k$ , otherwise  $\gamma(G') \leq k + 1$ .

**Proof.** Let  $G = (V, E)$  be a graph with star number  $k$  and let  $G^w$  be a  $k$ -witness graph with  $I_i = [a_i, b_i]$  for  $1 \leq i \leq k$  and let  $G' = (V \cup \{x\}, E')$ , where  $E' = E \cup \{xv : v \in V\}$ . First assume there exists  $G^w$  that is free. By Lemma 1.1, we can assume w.l.o.g. that  $G^w$  is right-free and thus for any non-edge  $uv$  it holds  $w(uv) < a_k \leq b_k$ . This clearly means that for all  $u \in V$ ,  $w(u) < b_k$ . Then we define the graph  $G^{w'}$  from  $G^w$  by extending the weight  $w'$  from  $w$  as follows

$$w'(u) = \begin{cases} b_k + 1 & \text{if } u = x, \\ w(u) & \text{otherwise} \end{cases}$$

For  $1 \leq i \leq k - 1$  we set  $I'_i = I_i$  and  $I'_k = [a_k, 2b_k + 1]$ .

For any  $uv \in E$  we have that there exists an  $i$  such that  $w(uv) \in I_i$  and we have  $w'(uv) = w(uv) \in I_i \subseteq I'_i$ . Next, for any edge  $xu$  where  $u \in V$ , we have  $a_k < b_k + 1 \leq w'(u) + w'(x) \leq b_k + b_k + 1 = 2b_k + 1$  and thus  $w'(xu) \in I'_k$ . Finally, for any non-edge  $uv \notin E$ , we have that  $w'(uv) = w(uv) \notin \cup_{1 \leq i \leq k-1} I_i = \cup_{1 \leq i \leq k-1} I'_i$ . Moreover as  $G^w$  is right-free we have  $w'(uv) = w(uv) < a_k \notin I'_k$ . Thus,  $G^{w'}$  is a  $k$ -witness for  $G'$ .

Now, assume all the  $k$ -witnesses of  $G$  are not free. In that case, let  $M = \max_{e \notin E} w(e)$  and we define the weight  $w'$  from  $w$  as follows

$$w'(u) = \begin{cases} M + 1 & \text{if } u = x, \\ w(u) & \text{otherwise} \end{cases}$$

For all  $1 \leq i \leq k$ , we set  $I'_i = I_i$  and  $I'_{k+1} = [M, 2M]$ . Clearly for any  $uv \in E$  as the weights of the vertices and the  $k$  intervals  $I_1, \dots, I_k$  remain the same, it continues to hold that there exists an  $1 \leq i \leq k$  such that  $w(uv) \in I_i$ .

Next, for any edge  $xu$  where  $u \in V$ , we have  $M \leq w'(xu) = w(u) + M + 1 \leq 2M + 1$  where the last inequality follows by the fact that  $w(u) \leq M$ . Hence,  $w'(xu) \in I'_{k+1}$ . Finally, for any non-edge  $uv \notin E$ , we have that first  $w'(uv) = w(uv) \notin \cup_{i=1}^k I_i$  and  $w(uv) < M + 1$  and thus  $w'(uv) \notin \cup_{1 \leq i \leq k+1} I'_i$ . This concludes the proof.  $\blacksquare$

In Fig. 7a, we show a path  $P_4$ , which has no free 1-witnesses (see Theorem 1.1), and where the addition of a universal vertex increases the number of intervals. However, in Fig. 7(b) we have another star-1-PCG which has no free 1-witnesses (the proof is almost identical to the one in Theorem 1.1) where the addition of a universal vertex does not increase the number of intervals.

**Theorem 2.3.** Given a graph  $G = (V, E)$  with  $\gamma(G) = k$ , let  $X$  be a set of vertices for which  $V \cap X = \emptyset$  and let  $G'$  be the graph obtained from  $G$  by adding the vertices of  $X$  as pendant. If there exists a  $G^w$  that is free then  $\gamma(G') = k$ , otherwise  $\gamma(G') \leq k + 1$ .

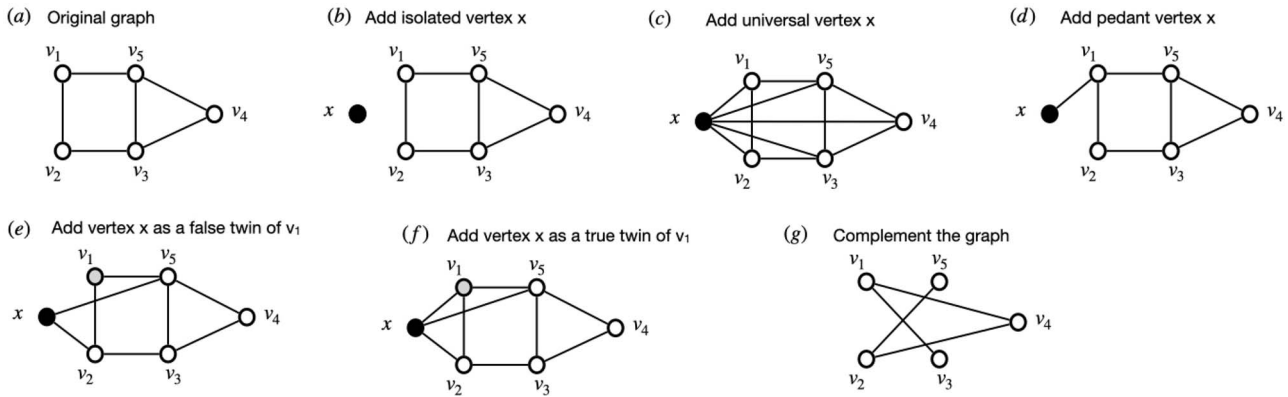


Figure 6. The original graph (a) and the application of graph operations considered in this paper.

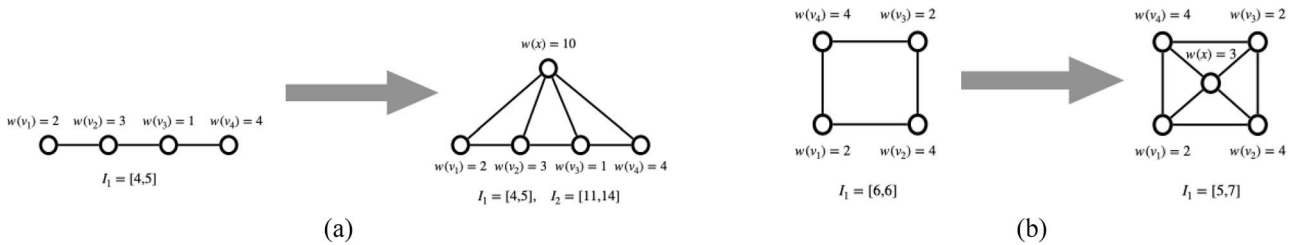


Figure 7. (a) Star-1-PCG for which no 1-witness is free and where the addition of a universal vertex necessarily increases the number of intervals (see the graph  $G_{27}$  in [10]) (b) Star-1-PCG for which no 1-witness is free and where the addition of a universal vertex does not increase the number of intervals.

**Proof.** Let  $G = (V, E)$  be a graph with star number  $k$  and let  $G^w$  be a  $k$ -witness graph in normal form with  $I_i = [a_i, b_i]$  for  $1 \leq i \leq k$ . Assume first  $G^w$  is not free. From Lemma 1.2 we can assume all the vertices have distinct weights. Denote by  $W = \max_{v \in V} w(v)$  and set  $M = \max\{W, b_k\} + 1$ . Now, let  $G' = (V \cup X, E')$  obtained from  $G$  by adding the pedant vertices in  $X$ . Then we define the graph  $G^{w'}$  from  $G^w$  as follows:

$$w'(u) = \begin{cases} w(u) & \text{if } u \in V, \\ 2M - w(v) & \text{if } u \in X \text{ and } p(u) = v \end{cases}$$

Notice that the weights of the edges in  $E$  do not change and moreover each pedant edge has weight exactly  $2M$ . For  $1 \leq i \leq k$  we set  $I'_i = I_i$  and  $I'_{k+1} = [2M, 2M]$ . We show now that  $G^{w'}$  is a  $(k+1)$ -witness for  $G'$ .

For any edge  $uv$  with  $u \in V, v \in V$ , we have that there exists an  $1 \leq i \leq k$  such that  $w(uv) \in I_i$  and clearly by definition of  $w'$  we have  $w'(uv) = w(uv) \in I_i = I'_i$ . Next, for any pedant edge  $xv$  in  $G'$  with  $x \in X$  and  $p(x) = v \in V$  we have  $w'(xv) = 2M - w(v) + w(v) = 2M \in I'_{k+1}$ .

For any non-edge  $uv$ , with  $u \in V, v \in V$  we have  $w'(uv) = w(uv) \notin \cup_{1 \leq i \leq k} I'_i$ . Moreover, as  $w(u) + w(v) < M + M$  (as the weights are all distinct in  $G^w$ ) we have that  $w'(uv) \notin I'_{k+1}$ .

Next, for any non-edge  $xu$  with  $x \in X$  and  $u \in V$ , we have  $w'(xu) = 2M - w(p(x)) + w(u)$ . As  $p(x) \neq u$  and all the vertices in  $G^w$  have distinct weights we have that  $w'(xu) \neq 2M$  and thus it is not in  $I'_{k+1}$ . Moreover,  $w'(xu) = 2M - w(p(x)) + w(u) > M + w(u) > b_k$ , where in the first and last inequality we use  $M > w(p(x))$  and  $M > b_k$ , respectively. Hence,  $w'(xu) \notin \cup_{1 \leq i \leq k} I'_i$ .

Finally, for any non-edge  $x_1x_2$  with  $x_1 \in X$  and  $x_2 \in X$ , we have  $w'(x_1x_2) = 4M - w(p(x_1)) - w(p(x_2)) > 4M - M - M = 2M$ , where the inequality holds as  $M$  is strictly greater than any weight of the vertices in  $V$ . Hence,  $w'(x_1x_2) \notin \cup_{1 \leq i \leq k+1} I'_i$ .

Thus  $G^{w'}$  is a  $(k+1)$ -witness for  $G'$ .

Finally, notice that if there exists a  $G^w$  which is free, thanks to Lemma 1.1 and Corollary 1.2 we can always assume  $G^w$  is normal and right-free. The proof follows identically as in the previous case noticing that  $I'_k$  and  $I'_{k+1}$  can be merged together meaning that  $G^{w'}$  is a  $k$ -witness for  $G'$ . ■

In Fig. 8a, we show a path  $P_5$ , which has no free 1-witnesses (see Corollary 1), and for which the addition of a pendant vertex does not increase the number of intervals (see Fig. 8b). Next, from Theorem 1 the caterpillar in Fig. 8b has no free 1-witness. By adding another pendant vertex as in Fig. 8c, we obtain an example of an *asteroidal triple*  $AT$  [24], for which in [14] it was shown that  $\gamma(AT) > 1$  and in [10] a construction was provided concluding that  $\gamma(AT) = 2$ . Thus, in this case, the addition of a pendant vertex necessarily increases the number of intervals.

**Theorem 2.4.** Given a graph  $G = (V, E)$  with  $\gamma(G) = k$ , let  $v \in V$  and let  $X$  be a set of vertices for which  $V \cap X = \emptyset$ . For any graph  $G'$  obtained from  $G$  by adding the vertices of  $X$  in such a way that  $\{v\} \cup X$  is a set of pairwise false (true) twins in  $G'$ , it holds  $\gamma(G') \leq k + 1$ .

**Proof.** Let  $G = (V, E)$  be a graph with star number  $k$  and let  $G^w$  be a  $k$ -witness graph in normal form with  $I_i = [a_i, b_i]$  for  $1 \leq i \leq k$ . Fix any  $v \in V$  and let  $X$  be such  $V \cap X = \emptyset$ . We consider first the case of false twins. Hence, let  $G' = (V', E')$  such that  $V' = V \cup X$  and  $E' = E \cup \{xu : x \in X \wedge vu \in E\}$ . Then we define  $w'$  as follows:

$$w'(u) = \begin{cases} w(u) & \text{if } u \in V, \\ w(v) & \text{if } u \in X \end{cases}$$

Notice that if  $2w(v) \notin \cup_i I_i$  then we could set for all  $1 \leq i \leq k$ ,  $I'_i = I_i$  and  $G^{w'}$  is clearly a witness for  $G'$ . Suppose now that there exists  $i$  such that  $2w(v) \in I_i$ . Notice that as  $G^w$  is normal form there is

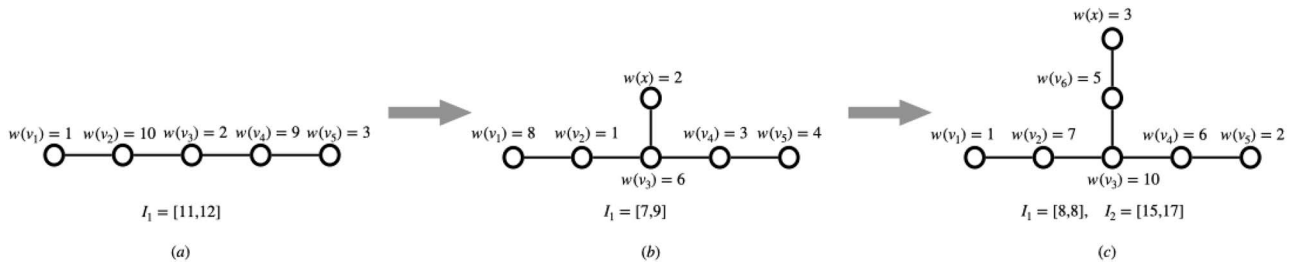


Figure 8. (a), (b) Star-1-PCG for which no 1-witness is free and where the addition of a pedant vertex does not increase the number of intervals (b), (c) Star-1-PCG for which no 1-witness is free and where the addition of a pedant vertex increases the number of intervals (see the graph  $G_{70}$  in [10]).

no edge  $e \in E$  such that  $w(e) = 2w(v)$ . We split the interval  $I_i$  into at most two smaller intervals, ensuring that the value  $2w(v)$  falls outside these new intervals. To this purpose, we define

$$\epsilon = \min_{u_1, u_2 \in V: w(u_1 u_2) \in I_i} \left\{ |2w(v) - w(u_1 u_2)| \right\}$$

Note that  $\epsilon$  represents the distance of  $2w(v)$  from the closest edge of  $G$  whose weight falls within  $I_i$ . Moreover,  $\epsilon$  is well-defined as there is at least one edge with weight in  $I_i$  (otherwise  $\gamma(G) < k$ ) and  $\epsilon > 0$  as  $G^w$  is in normal form. We consider three cases:

- $2w(v) - \epsilon/2 \leq a_i$ . In this case we set the interval  $I'_i = [2w(v) + \epsilon/2, b_i]$ .
- $2w(v) + \epsilon/2 \geq b_i$ . In this case we set the interval  $I'_i = [a_i, 2w(v) - \epsilon/2]$ .
- otherwise, we define two new intervals  $I'_i = [a_i, 2w(v) - \epsilon/2]$  and  $I''_i = [2w(v) + \epsilon/2, b_i]$ .

In the first two cases, we set the new intervals as  $I_1, \dots, I_{i-1}, I'_i, \dots, I_k$  and in the third case as  $I_1, \dots, I_{i-1}, I'_i, I''_i, \dots, I_k$ . In both cases, we have at most  $k + 1$  intervals and it is not difficult to check that  $G^{w'}$  is a  $(k + 1)$ -witness for  $G'$ .

In the case of true twins, we define the  $w'$  in the same way as in the case of false twins. Notice that if  $2w(v) \in \cup_i I_i$  then we could set for all  $1 \leq i \leq k$ ,  $I'_i = I_i$  and  $G^{w'}$  is clearly a  $k$ -witness for  $G'$ . Otherwise, we add the new interval  $I = [2w(v), 2w(v)]$ . Notice that as  $G^w$  is normal form there is no pair of vertices  $u_1, u_2$  such that  $w(u_1 u_2) = 2w(v)$ . It is easy to see that we obtain a  $(k + 1)$ -witness for  $G'$ . ■

**Lemma 2.1.** Given a graph  $G = (V, E)$  with  $\gamma(G) = k$ , let  $\bar{G}$  be its complement. It holds  $|\gamma(\bar{G}) - \gamma(G)| \leq 1$ .

**Proof.** Assume on the contrary that there exists  $G$  and  $\bar{G}$  such that  $|\gamma(\bar{G}) - \gamma(G)| \geq 2$ . Let  $\gamma(G) = k$ , and assume w.l.o.g. that  $\gamma(\bar{G}) \geq \gamma(G) + 2$ .  $G^w$  be a  $k$ -witness graph with  $I_i = [a_i, b_i]$  for  $1 \leq i \leq k$ . We construct  $\bar{G}^{w'}$  as follows:  $w' = w$  and  $I'_1, \dots, I'_k$  are the intervals defined by the set  $R^+ \setminus \cup_i I_i$ . By construction  $k' = k + 1$  and it is not difficult to see that  $\bar{G}^{w'}$  is a  $k'$ -witness for  $\bar{G}$ . Thus, we have that  $\gamma(\bar{G}) \leq k' = k + 1 = \gamma(G) + 1$  contradicting our initial hypothesis. ■

**Theorem 2.5.** Given a graph  $G = (V, E)$  with  $\gamma(G) = k$ , let  $\bar{G}$  be its complement. It holds

- if there exists a  $k$ -witness  $G^w$  that is both left- and right-free then  $\gamma(\bar{G}) = k - 1$ .
- otherwise if there exists a  $k$ -witness  $G^w$  that is free then  $\gamma(\bar{G}) \leq k$ .
- otherwise if no  $k$ -witness  $G^w$  is free then  $\gamma(\bar{G}) \leq k + 1$ .

**Proof.** First, notice that the three items cover all the possible types that a witness graph can be. Next, the three items can be proved using the same construction described in Lemma 2.1 and observing that: (i) if  $G^w$  is both left and right free, then the intervals  $I'_1$  or  $I'_k$  are not used, that is, there is no edge  $e$  of  $\bar{G}$  whose weight falls in  $I'_1$  or  $I'_k$  and thus,  $\gamma(\bar{G}) \leq k - 1$  and by the result of Lemma 2.1 we have that the equality must hold; (ii) if  $G^w$  is free then exactly one among  $I'_1$  or  $I'_k$  is not used and hence  $\gamma(\bar{G}) \leq k$ ; (iii) if  $G^w$  is not free, then the construction we provide uses all the intervals and hence we can only claim that  $\gamma(\bar{G}) \leq k + 1$ . ■

In Fig. 9, we depict a graph that provides an example for the case (i) of the previous theorem.

Notice that there are examples of graphs showing that the inequalities in (ii) and (iii) cannot be tight. Every graph that is self-complementary (i.e. a graph which is isomorphic to its complement) satisfies  $\gamma(G) = \gamma(\bar{G})$ . Thus, self-complementary graphs that have a free witness are examples for which the inequality in (ii) is tight. An example is shown in Fig. 10. Next, the complement of the cycle on 7 vertices,  $\bar{C}_7$ , has star number 3 (see [10]) and a right-free 3-witness is depicted in Fig. 11. As the star number of cycles is 2 [9] this graph provides an example for which the inequality in (ii) is strict.

Consider now the inequality in (iii). Self-complementary graphs for which no free witness exists (like  $P_4$  and  $C_5$ ) are examples for which the inequality in (iii) is strict. Moreover, by Theorem 1.1,  $C_7$  has no free 1-witnesses and thus provides an example for which the inequality in (iii) is tight.

#### 4. THE STAR NUMBER OF ACYCLIC GRAPHS

In this section, we show how the results from Section 2 can be applied to determine the star number of lobster graphs and more generally, improve the existing bounds on the star number of acyclic graphs. The following result holds.

**Theorem 3.1.** [1, 10] For any caterpillar  $T$ , it holds  $\gamma(T) = 1$ .

Since  $AT$  (see the graph in Fig. 8c) has star number 2 and is a subgraph of any binary tree of radius at least two, we have the following corollary.

**Corollary 3.1.** For any binary tree  $T$  of radius at least 2 it holds  $\gamma(T) \geq 2$ .

A lobster can be obtained from a caterpillar by adding pendant vertices. Thus, we have the following result.

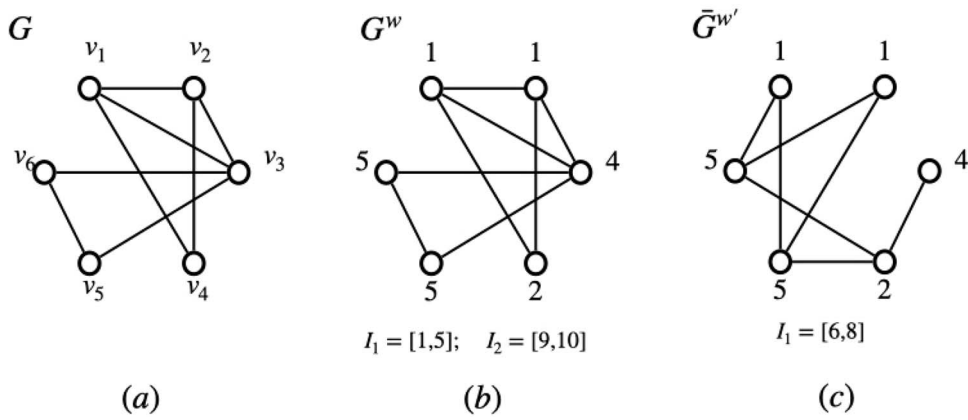


Figure 9. (a) Graph  $G$ ; (b) 2-witness graph  $G^w$  that is double-free; (c) 2-witness graph  $\bar{G}^{w'}$  constructed based on Theorem 2.5 case (i).

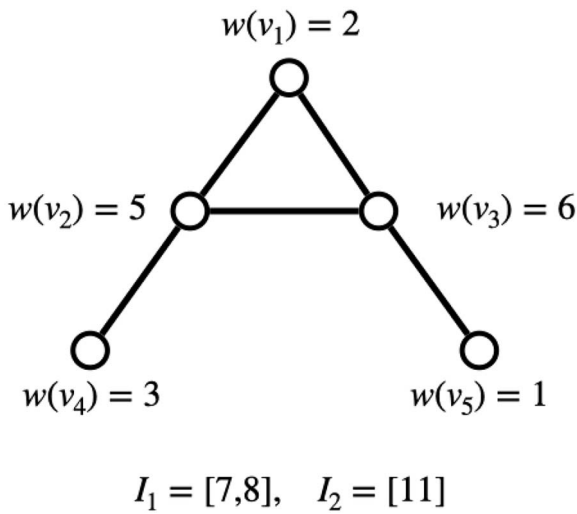


Figure 10. A right-free 2-witness graph of a self-complementary graph  $G$  with star number 2 [10].

**Theorem 3.2.** For any lobster  $L$ , with radius at least 2 it holds  $\gamma(L) = 2$ .

**Proof.** The proof follows by a direct application of Theorem 3.1, Theorem 2.3, and Corollary 3.1. ■

The following theorem shows that for any acyclic graph  $G$  it holds  $\gamma(G) \leq \text{rad}(G)$ . This represents an advancement beyond the broadly acknowledged general result stating  $\gamma(G) \leq |E|$  (see [2]).

**Theorem 3.3.** For any acyclic graph  $G$  it holds  $\gamma(G) \leq \text{rad}(G)$ .

**Proof.** Let  $G$  be an acyclic graph. To prove the upper bound, we provide a  $\text{rad}(G)$ -witness graph. Let  $\mathcal{CC}$  be the set of connected components of  $G$ . Let  $|\mathcal{CC}| = t$ , recall that  $\text{rad}(G) = \max_{CC_i \in \mathcal{CC}} \text{rad}(CC_i)$  and let  $a_i$  denote a center of graph  $CC_i$ .

We start with the graph  $G_0 = (V_0, E_0)$  where  $V_0 = \{a_1, \dots, a_t\}$  and  $E_0 = \emptyset$ . At step 1 we construct  $G_1$  by adding all the vertices adjacent to the vertices in  $V_0$  as pendant vertices. Thus, from Theorem 2.3  $G_1$  is a 1-witness. At a generic step  $i$  of this procedure, we construct the graph  $G_i$  from the graph  $G_{i-1}$  by adding all the vertices adjacent to the vertices in  $V_{i-1}$ . Clearly, these vertices are all pendant (otherwise they would have been added in a

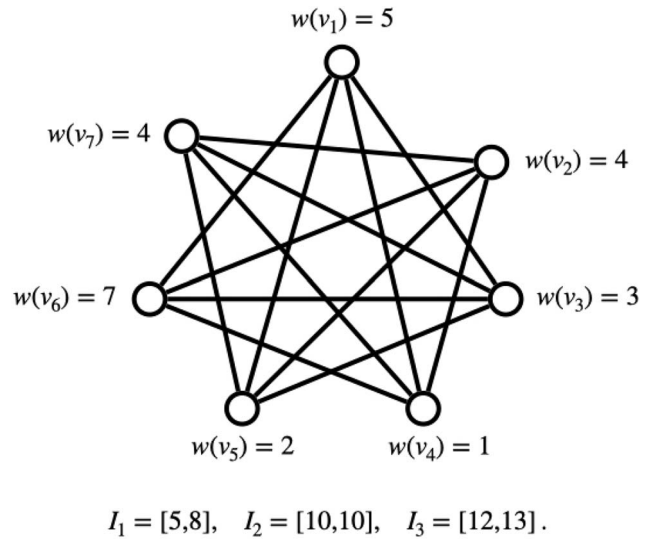


Figure 11. A right-free 3-witness of the graph  $\bar{C}_7$  with star number 3 [10].

previous step) and we could apply theorem Theorem 2.3 increasing the number of intervals by at most 1. The number of steps we have to do until we reach  $G$  is exactly  $\text{rad}(G)$ . Thus, we have  $\gamma(G) \leq \text{rad}(G)$ . ■

## 5. CONCLUSIONS AND OPEN PROBLEMS

In this work, we study how different graph operations, including the insertion of twins, pendant vertices, universal vertices, and isolated vertices, affect the graph's star number. Then we use these results to compute the star number for lobsters and establish an upper bound for the star number of acyclic graphs. Many problems remain open.

It is interesting to improve the lower bound for the star number of acyclic graphs. Indeed, the only lower bound we have is the trivial value of 2, which applies to all graphs containing an asteroidal triple.

Another interesting problem is to determine whether there exists some constant  $c$  such that all acyclic graphs are star- $c$ -PCGs.

Next, it may be worth to consider the effects of other classical graph operations such as the disjoint union of two graphs. While it is not difficult to see that  $\gamma(G_1 + G_2) \leq \gamma(G_1) + \gamma(G_2)$  improving

this result is certainly interesting. Another operation of interest is the cartesian product of two graphs  $\gamma(G_1 \times G_2)$ .

Finally, beyond the operations studied in this paper, it would be valuable to analyze how the addition or removal of edges or vertices affects the star number. This line of investigation is particularly relevant for applications involving real datasets, which often contain noise or incomplete information. Similarly, one could explore the changes in the star number when cliques, cycles, or specific motifs are added to a graph. Such investigations could provide insights into the relationship between the star number and the structural properties of the graph.

## CONFLICT OF INTEREST

None declared.

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## DATA AVAILABILITY

No new data were generated or analyzed in support of this research.

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