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RESEARCH

On star-*k*-PCGs: exploring class boundaries for small *k* values

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Abstract

A graph $G = (V, E)$ is a star-k-pairwise compatibility graph (star-k-PCG) if there exists a weight function $w : V \to \mathbb{R}^+$ and k mutually exclusive intervals $I_1, I_2, \dots I_k$, such that there is an edge $uv \in E$ if and only if $w(u) + w(v) \in \bigcup_i I_i$. These graphs are related to two important classes of graphs: pairwise compatibility graphs (PCGs) and multithreshold graphs. It is known that for any graph G there exists a k such that G is a star-k-PCG. Thus, for a given graph G it is interesting to know which is the minimum k such that G is a star-k-PCG. We define this minimum k as the *star number* of the graph, denoted by $\gamma(G)$. Here we investigate the star number of simple graph classes, such as graphs of small size, caterpillars, cycles and grids. Specifically, we determine the exact value of $\gamma(G)$ for all the graphs with at most 7 vertices. By doing so we show that the smallest graphs with star number 2 are only 4 and have exactly 5 vertices; the smallest graphs with star number 3 are only 3 and have exactly 7 vertices. Next, we provide a construction showing that the star number of caterpillars is one. Moreover, we show that the star number of cycles and two-dimensional grid graphs is 2 and that the star number of 4-dimensional grids is at least 3. Finally, we conclude with numerous open problems.

1 Introduction

The categorization of graphs into different classes is fundamental in graph theory and its applications, as it allows for a more structured and focused study of their properties and behaviors. Indeed, each class of graphs, possess unique characteristics that make them suitable for specific problems and applications. In essence, the diversity of graph classes reflects the diversity of real-world problems they are used to model and solve. In this paper, we concentrate on a specific class of graphs, referred to as *star-k-pairwise compatibility graphs*. A graph $G = (V, E)$ is a star-k-PCG if there exists a weight function $w : V \to \mathbb{R}^+$ and

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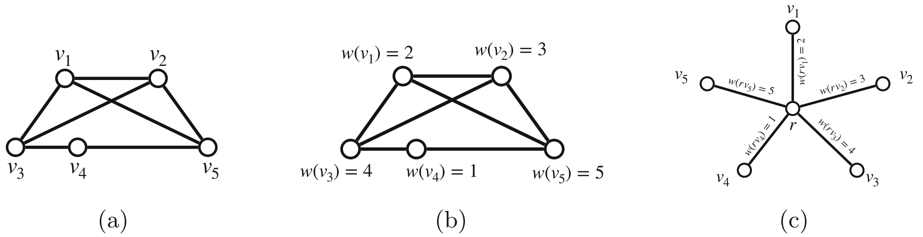
k mutually exclusive intervals $I_1, I_2, \ldots I_k$, such that there is an edge $uv \in E$ if and only if $w(u) + w(v) \in \bigcup_i I_i$. This class serves as a bridge between two well-established graph classes: pairwise compatibility graphs and multithreshold graphs.

Connection with pairwise compatibility graphs (PCGs) A graph G is a k-PCG (known also as multi-interval PCG) if and only if there exists a non-negative edge weighted tree T and k mutually exclusive intervals $I_1, I_2, \ldots, I_k$ of non-negative reals such that each vertex of G corresponds to a leaf of T and there is an edge between two vertices in G if and only if the distance between their corresponding leaves in T lies in $\bigcup_i I_i$ (see e.g. [2]). Such tree T is called the *k-witness* tree of G. The concept of 1-PCGs, also known as PCGs, originated from the problem of reconstructing phylogenetic trees [3]. Moreover, PCGs are a generalization of the well-known *k-leaf power* graphs [4] and have proven valuable in describing and analyzing evolutionary processes [5].

One of the most important open problems in the field is, whether given an integer k, the k-PCG can be recognized in polynomial time, and it is unknown whether this problem can be solved in polynomial time, even for the case of $k = 1$. To make progress towards the solution of this problem, restrictions have been made on the topology of the k-witness tree into two main directions: a star and a caterpillar (see e.g. [6–8]). The class of star-k-PCGs is exactly the class of k-PCGs for which the witness tree is a star [1, 9]. Figure 1 depicts an example of a graph that is a star-1-PCG. This topology constraint on the witness tree, has been proven valuable as for star witness trees, the decision problem becomes simpler compared to the general case. Indeed, Xiao and Nagamochi [10] introduced the first polynomial-time algorithm for identifying graphs that are star-1-PCGs. Next, Kobayashi et al. in [11] improved upon this result by introducing a new characterization of star-1-PCGs that led to a linear time recognition algorithm.

Connection with multithreshold graphs Multithreshold graphs were introduced by Jamison and Sprague [12] in 2020 as a generalization of the class of threshold graphs introduced by Chvátal and Hammer [13] in 1977 and has since become one of the most prominent and well-studied graph classes (see [14]). In a similar way, multithreshold graphs have gained considerable interest within the research community since their introduction, as evidenced by the following studies [12, 15, 16]. Given real numbers $\theta_1, \theta_2, \ldots, \theta_k$, with $\theta_1 < \theta_2 < \ldots < \theta_k$ we say that a graph $G = (V, E)$ is a k-threshold graph with thresholds $\theta_1, \theta_2, \ldots, \theta_k$ if there exist an assignment $r : V \to \mathbb{R}$ of real ranks to the vertices such that for every pair of distinct vertices $u, v \in V$ we have $uv \in E$ if and only if the inequality $\theta_i \leq r(u) + r(v)$ holds for an odd number of indices i. It was shown in [11] that 2-threshold graphs are exactly



(a) (b) (c)

Fig. 1 **a** An example of a graph G that is a star-1-PCG graph. In **b** and **c** the 1-witness graph and the 1-witness star for which G is a star-1-PCG with $I_1 = [5, 8]$

star-1-PCGs. Here we directly extend this claim by showing in Observation 1 that the class of star-k-PCGs is equivalent to the class of $2k$-threshold graphs.

Results It is already established that every graph G is a star-k-PCG for some positive integer k [2]. Hence, we introduce the following notation.

Definition 1 Given a graph G, we define *the star number*, $\gamma(G)$, to be the smallest positive integer k, such that G is a star-k-PCG.

The star number is not known even for graphs belonging to simple classes. In this paper we firstly focus on n-vertex graphs for small values of n. Identifying the smallest graphs that are excluded from a class not only helps to define the boundaries of that class but could also help towards a characterization of the graph class through forbidden subgraphs. In this framework we consider the following question: *what is the smallest value of n for which there exists an n-vertex graph that is not a star-k-PCG*? This question has been already investigated for related graphs classes. Indeed, it is known that the smallest graphs that are not 1-PCGs have 8 vertices [17, 18] while the smallest graphs that are not 2-PCGs must have at least 9 vertices [19]. Here we determine the exact value of $\gamma(G)$ for all the graphs G with at most 7 vertices. By doing so we show that the smallest graphs with star number 2 are only 4 and have exactly 5 vertices; the smallest graphs with star number 3 are only 3 and have exactly 7 vertices.

Next we study some simple graph classes: caterpillars, cycle graphs, grid graphs. Caterpillars have already been shown to be 2-multithreshold [20] and thus they are also star-1-PCGs. Here we provide a different construction in the context of star-k-PCGs. While path graphs are known to be star-1-PCGs [10], cycle graphs are only shown to be 1-PCGs of caterpillar witness trees [21]. However, when considering a star tree structure, we prove that the star number of cycle graphs is 2. In [22] it was proved that 2-dimensional grid graphs are 1-PCGs of a caterpillar. Here we prove that the star number of two dimensional grid graphs G_{n_1,n_2}, is 1 if $\min\{n_1, n_2\} \leq 2$ and 2 otherwise. From the results in [20] it can be easily shown that the star number d-dimensional grids is at least $d - 3$. Here we improve this result for $d = 4$, by showing that the star number of 4-dimensional grids is at least 3. All our constructions can be obtained in linear time.

2 Preliminaries

In this paper we only consider simple graphs, that is graphs that contain no loops or multiple edges. Additionally, we focus only undirected graphs and thus for simplicity, we use a notational shorthand, writing uv to represent the unordered pair $\{u, v\}$. For a graph $G = (V, E)$ and a vertex $u \in V$, the set $N(u) = \{v : uv \in E\}$ is called the *neighborhood* of u. A *cycle* graph, denoted as C_n, $n \geq 3$, is a graph that consists of a single cycle of n vertices. A *caterpillar* is a tree in which all the vertices are within distance 1 of a central path. The central path contains only vertices of degree at least 2 (i.e. vertices that are not leaves).

For any integer $n \geq 1$ we denote by $[n]$ the set $\{0, 1, 2, \ldots, n-1\}$. A *d-dimensional grid* graph $G_{n_1,\ldots,n_d}$, is a graph such that the vertex set is given by $[n_1] \times [n_2] \times \ldots \times [n_d]$ and there is an edge between two vertices if and only if they differ in exactly one coordinate and the difference is 1. More formally, a vertex u is described by its coordinates $(i_1, \ldots, i_d)$. For any dimension j we denote by u_j the coordinate of u in the j-th dimension. Two vertices u and u' are adjacent if there is a dimension i such that $|u_i - u'_i| = 1$ and for all $l \neq i$, $u_l = u'_l$ (see Fig. 8 for an example).

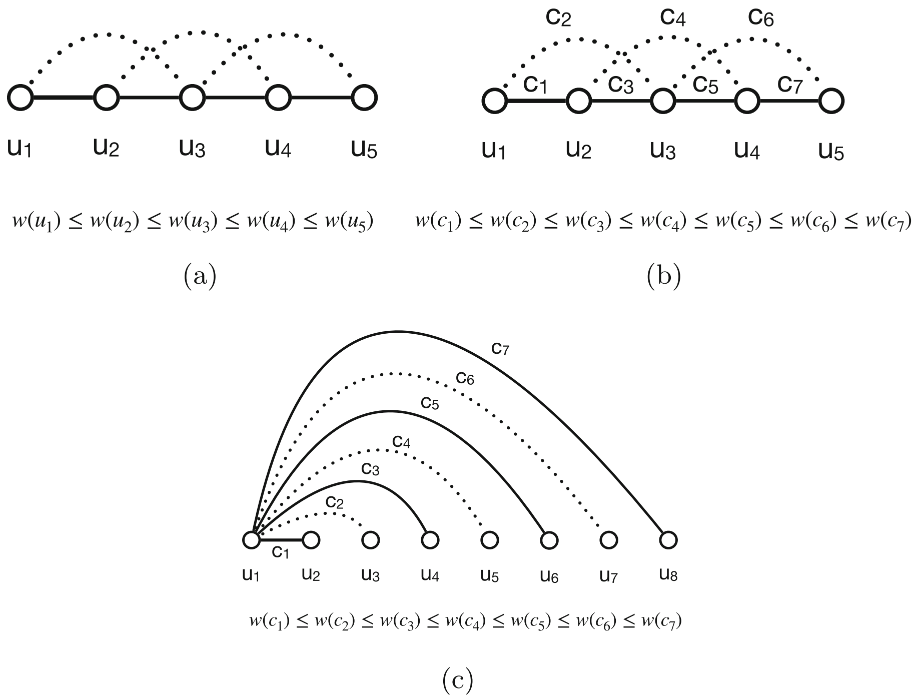


Fig. 2 **a** An example of a 3-FSeq and **b** the corresponding 3-FP sequence. **c** Another example of a 3-FP sequence which does not lead to the identification of a 3-FSeq. The dashed lines depict edges that are not in G, whereas the solid lines illustrate edges that are included in G

Given a graph $G = (V, E)$ and a function $w : V \to \mathbb{R}^+$, we denote by G^w its weighted version, that is, G supplemented with the weight function w (see Fig. 1a, b). Moreover for each vertex $v \in V$, $w(v)$ denotes the weight of the vertex v and for each pair of vertices u and v of G we denote by $w(uv)$ the sum $w(u) + w(v)$.

Given G^w we define $\sigma(G^w) = v_1, v_2 \dots, v_{|V|}$ as the sequence of vertices in G^w, arranged in non-descending order based on their weights. Given a family $\mathcal{S}$, of r pairwise disjoint subsets of V we will use the notation $S_1, S_2, \dots, S_r$ where $S_i \in \mathcal{S}$, to denote the partial order where for all $1 \leq i \leq r$ the weights of the vertices in S_i are all less or equal to the weights of all vertices in S_{i+1}.

We say that the neighborhood $N(u)$ is *consecutive* with respect to $\sigma(G^w)$ if for $v_i, v_j \in N(u)$ with $i < j$ implies $v_t \in N(u)$ for any $i < t < j$.

Definition 2 A graph $G = (V, E)$ is a star-k-PCG if there exists a weight function $w : V \to \mathbb{R}^+$ and k mutually exclusive intervals $I_1, I_2, \dots I_k$, such that there is an edge $uv \in E$ if and only if $w(u) + w(v) \in \bigcup_i I_i$. Such weighted graph G^w is called the *k-witness* of G.

The following result is a straightforward generalization of Observation 2.2 in [11].

Observation 1 *Given an integer k, a graph is a $2k$-threshold graph if and only if it is a star-k-PCG.*

Proof ($\leftarrow$) Let $G = (V, E)$ be a $2k$-threshold graph defined by $r : V \to \mathbb{R}$ and $\theta_1 < \theta_2 < \ldots < \theta_k$ such that for every pair of distinct vertices $u, v \in V$ we have $uv \in E$ if and only if $\theta_i \leq r(u) + r(v)$ for an odd number of indices i. Notice that *w.l.o.g.* we can assume that $\theta_i \geq 0$ for all $1 \leq i \leq 2k$ and that $r(v) > 0$ for all $v \in V$. To show that G is a k-star-PCG we construct G^w as follows. For every $v \in V$ we define $w(v) = r(v)$ and $I_i = [\theta_{2i-1}, \theta_{2i})$ with $1 \leq i \leq k$. It is easy to see that if we want to have closed intervals as in the definition of star-k-PCGs, it is sufficient to set $I'_i = [\theta_{2i-1}, \theta'_{2i}]$ where $\theta'_{2i} = \max\{w(e) : e \in E \text{ and } w(e) \in I_i\}$. Finally, by construction for all $e = uv$ it holds that $w(e) \in I_i$ if and only if $\theta_{2i-1} \leq r(u) + r(v) < \theta_{2i}$, meaning that the inequality is valid for an odd number of indices.

($\rightarrow$) Now, let G be a star-k-PCG and let G^w be its k-witness for $I_i = [a_i, b_i]$ with $1 \leq i \leq k$. Notice that we can substitute $I_i = [a_i, b_i]$ with $I_i = [a_i, b'_i)$ as follows. For $1 \leq i < k$ we set $b'_i = \frac{a_{i+1}+b_i}{2}$ if there is no edge e for which $b_i < w(e) < a_{i+1}$, otherwise we set $b'_i = \min\{w(e) : e \notin E \text{ and } b_i < w(e) < a_{i+1}\}$. If $i = k$ we set $b'_k = b_k + 1$ if there is no edge e for which $b_k < w(e)$, otherwise we set $b'_k = \min\{w(e) : e \notin E \text{ and } b_k < w(e)\}$. It is easy to see that G^w still remains a k-witness. Now *w.l.o.g.* we can assume $a_1 < b_1 < \ldots < a_k < b_k$. Then G is clearly a $2k$-threshold with $r = w$ and $1 \leq j \leq 2k$ $\theta_j = a_{(j+1)/2}$ if j is odd or $\theta_j = b_{j/2}$ otherwise. The proof follows straightforwardly by the construction. □

Given a graph $G = (V, E)$, we say that $G' = (V', E')$ is an *induced subgraph* of G if $V' \subset V$ and E' consists of all the edges in E whose endpoints are both in V'. The following lemma can be directly deduced from Definition 1.

Lemma 1 *Given a graph G it holds that $\gamma(G') \leq \gamma(G)$ for any induced subgraph G' of G.*

The next lemma follows trivially by the definition of star-k-PCG.

Lemma 2 *Let G be a star-k-PCG and let G^w be a k-witness for G. For any two vertices u, v in G^w for which $N(u) - \{v\} \neq N(v) - \{u\}$ it holds $w(u) \neq w(v)$.*

The following definition will be needed throughout the paper.

Definition 3 Let G^w be a vertex weighted graph and k an integer such that $k \geq 1$. Consider a sequence, $c_1, c_2, \ldots, c_{2k+1}$ of $2k + 1$ pairs of vertices in G^w. This sequence is said to be a k-FP if it satisfies the following conditions:

1. A pair c_i is an edge in G^w (i.e., $c_i \in E$) if and only if i is an odd integer.
2. The weights of the pairs follow an ascending order: $w(c_1) \leq w(c_2) \leq \ldots \leq w(c_{2k+1})$.

See Fig. 2b for an example of a 3-FP. The next lemma shows that the k-FP is a forbidden pattern (hence the name) for any graph that is a star-k-PCG.

Lemma 3 *Let G be a star-k-PCG than any k-witness G^w of G does not contain a k-FP.*

Proof Firstly, observe that an equality within the sequence of inequalities in item 2 of Definition 3 implies the existence of an edge and a non-edge with identical weight which contradicts the definition of a star-k-PCG. So we have $c_i \neq c_j$ for all $i \neq j$. The proof follows by noticing that if G contains a k-FP then no two edges c_i and c_j can belong to the same interval. Given that there are exactly $k + 1$ edges in the k-FP sequence, then at least $k + 1$ distinct intervals are required. □

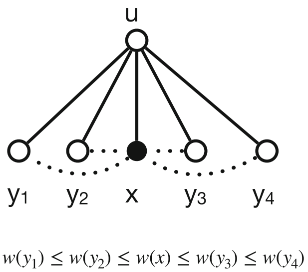


Fig. 3 An illustration of Lemma 5 for a graph G. The dashed lines depict edges that are not in G, whereas the solid lines illustrate edges that are included in G. The vertices $\{y_1, y_2, x, y_3, y_4\}$ appear consecutive in $\sigma(G^w)$

We introduce now the concept of k-FSeq.

Definition 4 Let G^w be a vertex weighted graph and k an integer such that $k \geq 1$. Consider a sequence of $k+2$ vertices in G^w, denoted as $u_1, u_2, \ldots u_{k+2}$. This sequence is said to be a *k-FSeq* if it satisfies the following conditions:

1. $u_i u_{i+1} \in E$ for $1 \leq i \leq k+1$
2. $u_i u_{i+2} \notin E$ for $1 \leq i \leq k$
3. $w(u_1) < w(u_2) < \ldots < w(u_{k+2})$

See Fig. 2a for an example of a 3-FSeq. The following lemma holds.

Lemma 4 *Let G be a star-k-PCG than any k-witness G^w of G does not contain a k-FSeq.*

Proof Notice that if G^w contains a k-FSeq then the edges

$$u_1u_2, u_1u_3, u_2u_3, u_2u_4, \ldots, u_ku_{k+1}, u_ku_{k+2}, u_{k+1}u_{k+2}$$

form a k-FP (see Fig. 2a and Fig. 2b for an example). The proof follows trivially by Lemma 3. □

Note that from the proof of Lemma 4, we can infer that a k-FSeq implies the existence of a k-FP. However, the converse is not always true. This is shown in Fig. 2c, where we present a 3-FP edge sequence that does not lead to the direct identification of a 3-FSeq.

3 The star number of *n*-vertex graphs with *n* ≤ 7

3.1 *n*-vertex graphs with *n* ≤ 5

There are 34 non isomorphic graphs with 5 vertices [23]. Let $\mathcal{G}_5 = \{G_1, G_2, \ldots, G_{34}\}$ be the set of these graphs. These graphs are depicted in Fig. 4 based on increasing number of edges. In this section, we prove that all graphs with at most 5 vertices have a star number equal to 1, with the exception of G_{15}, G_{20}, G_{25}, and G_{27}, which have a star number equal to 2. We start by proving the following lemmas.

Lemma 5 *Let $G = (V, E)$ be a star-1-PCG and let $u, x \in V$ such that $x \in N(u)$, and for every other vertex $y \in N(u)$, it holds that $xy \notin E$. For any 1-witness graph G^w of G, the neighborhood $N(u)$ is consecutive with respect to $\sigma(G^w)$.*

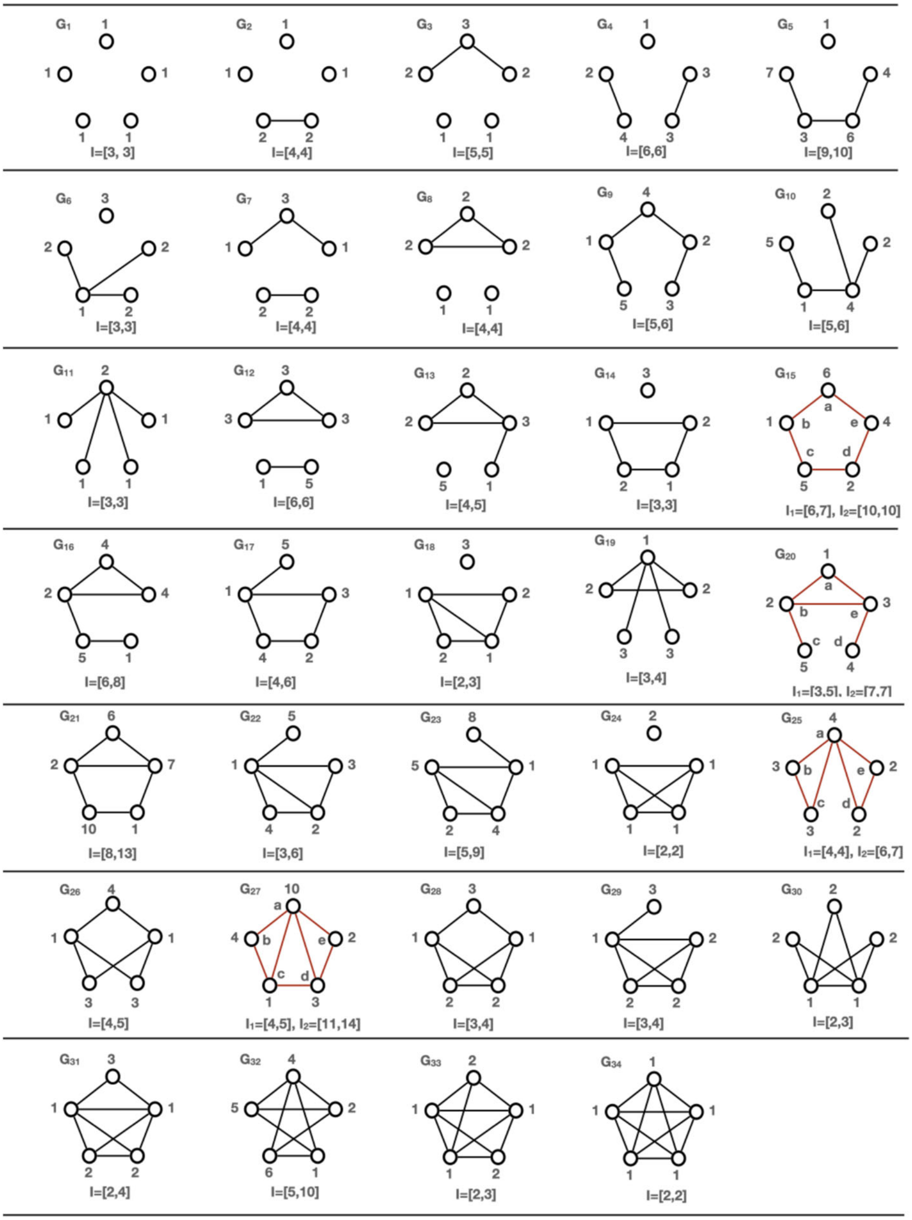


Fig. 4 The list for all non isomorphic graphs with at most 5 vertices. The graphs $G_{15}, G_{20}, G_{25}, G_{27}$ are star-2-PCGs. The rest of the graphs are all star-1-PCGs

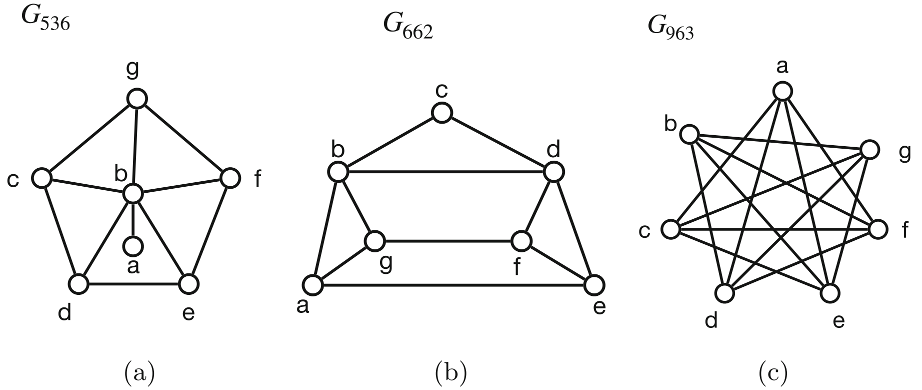


Fig. 5 The only three graphs on 7 vertices that are not star-2-PCG. **a** $\gamma(G_{536}) = 3$ by setting $w(a) = 7, w(b) = 1, w(c) = 6, w(d) = 4, w(e) = 5, w(f) = 9, w(g) = 8$ and $I_1 = [4, 10], I_2 = [14, 14], I_3 = [17, 17]$. **b** $\gamma(G_{662}) = 3$ by setting $w(a) = 5, w(b) = 3, w(c) = 4, w(d) = 1, w(e) = 7, w(f) = 6, w(g) = 2$ and $I_1 = [4, 5], I_2 = [7, 8], I_3 = [12, 13]$. **c** $\gamma(G_{963}) = 3$ by setting $w(a) = 5, w(b) = 4, w(c) = 7, w(d) = 2, w(e) = 1, w(f) = 3, w(g) = 4$ and $I_1 = [5, 8], I_2 = [10, 10], I_3 = [12, 13]$

Proof Let G^w be a 1-witness graph of G and let $u \in V$ with $|N(u)| = t$ and let $y_1, y_2, \dots y_t$ be the ordering of $N(u)$ according to the weight function w (see Fig. 3). Assume now that $N(u)$ is not consecutive in $\sigma(G^w)$ and thus there exists a vertex $z \in V - N(u)$ for which $w(y_1) \le w(z) \le w(y_t)$. Notice that if $z = u$ we have the 1-FSeq sequence x, u, y_t (if $w(x) < w(u)$) or y_1, u, x (if $w(x) > w(u)$) and from Lemma 4 this contradicts the fact G is a star-1-PCG. Hence, assume $z \neq u$ and notice that clearly, $z \neq x$. There are two cases to consider: either $w(x) \le w(z)$ or $w(x) \ge w(z)$. We consider only the first case as the second one follows using similar arguments. Thus, we have $w(x) \le w(z) \le w(y_t)$. Now, there are two cases to consider: if $w(u) \le w(x)$ we have $w(u) \le w(x) \le w(z) \le w(y_t)$, otherwise $w(u) \ge w(y_t)$ and we have $w(x) \le w(z) \le w(y_t) \le w(u)$. In both cases, ux, uz, uy_t will be a 1-FP and we reach a contradiction by Lemma 3. □

Lemma 6 $\gamma(G_{15}) \ge 2$.

Proof Suppose on the contrary that there exists G^w which is a 1-witness of G_{15}. Let a, b, c, d, e be the sequence of the vertices in the cycle (as in Fig. 4). *W.l.o.g.* let a be the vertex of minimum weight in G^w (the same argument follows for any other vertex by symmetry). From Lemma 5 we have that each of the neighborhoods $N(b) = \{a, c\}$ and $N(e) = \{d, a\}$ must be consecutive in $\sigma(G^w)$. However, there is no possible weight function w for which $w(a)$ is the smallest and a appears consecutive with both c and d. Thus, we reach a contradiction and there exists no G^w which is a 1-witness of G_{15}. □

Lemma 7 $\gamma(G_{20}) \ge 2$.

Proof Let a, b, c, d, e be the vertices of G_{20} as in Fig. 4. Suppose on the contrary that there exists G^w which is a 1-witness of G_{20}. *W.l.o.g.* we can assume $w(b) < w(e)$. From Lemma 5 we have that each of the neighborhoods $N(b) = \{a, c, e\}$ and $N(e) = \{a, b, d\}$ must be consecutive in $\sigma(G^w)$. Thus, the total order $\sigma(G^w)$ must contain the partial order $\{b, d\}, \{a\}, \{c, e\}$. We distinguish two cases:

- $w(b) < w(d)$. In this case we have only two possibilities: either $w(b) \le w(d) \le w(a) \le w(c) \le w(e)$ or $w(b) \le w(d) \le w(a) \le w(e) \le w(c)$. In both cases we have that ba, da, ae is a 1-FP contradicting our initial hypothesis that G^w is a 1-witness.

- $w(d) < w(b)$. In this case we have only two possibilities: either $w(d) \leq w(b) \leq w(a) \leq w(c) \leq w(e)$ or $w(d) \leq w(b) \leq w(a) \leq w(e) \leq w(c)$. In the first case we have that bc, ac, ae is a 1-FP and in the second case, de, dc, bc is a 1-FP. Hence, in both cases we contradict our initial hypothesis that G^w is a 1-witness. □

Lemma 8 $\gamma(G_{25}) \geq 2$ *and* $\gamma(G_{27}) \geq 2$.

Proof Let a, b, c, d, e be the vertices of G_{25} as in Fig. 4. Suppose on the contrary that there exists G^w which is a 1-witness of G_{25}. The next observation follows by the fact that $G_{25} - \{a\}$ does contain neither a clique of three vertices nor an independent set of size three.

Observation 2 *Any set of three vertices in* G_{25} *that does not include vertex a will always induce a subgraph in* G_{25} *that contains at least one edge and at least one non-edge.*

Consider now the position of a in $\sigma(G^w)$. We consider three cases:

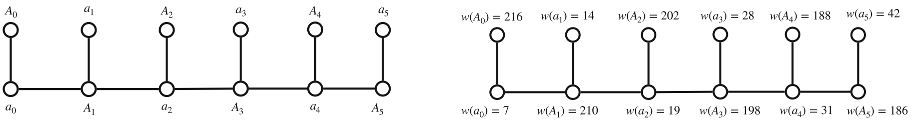
- the partial order $\{a\}, \{b, c, d, e\}$ is contained in the total order $\sigma(G^w)$. Let $\sigma(G^w) = a, x, y, z, t$. By Observation 2 we have that there is at least one non-edge among xy, xz, yz. *W.l.o.g.* let $xz \notin E(G_{25})$. Then, consider x, z, t and again by Observation 2 there is at least one edge among xt and zt. We reach a contradiction as if $xt \in E(G_{25})$, then ax, xz, xt is an 1-FP and if $zt \in E(G_{25})$, then ax, xz, zt is an 1-FP.
- the partial order $\{b, c, d, e\}, \{a\}$ is contained in $\sigma(G^w)$. Using similar argument as the previous item, let $\sigma(G^w) = x, y, z, t, a$. By Observation 2 we have that there is at least one edge among xy, xz, yz. *W.l.o.g.* let $xz \in E(G_{25})$. Then, consider x, z, t and again by Observation 2 there is at least one non-edge among xt and zt. We reach a contradiction as if $xt \in E(G_{25})$, then xz, xt, ta is an 1-FP and if $zt \in E(G_{25})$, then xz, zt, ta is an 1-FP.
- the partial order $S_1, \{a\}, S_2$, where S_1, S_2 form a partition of $\{b, c, d, e\}$, is contained in $\sigma(G^w)$. We show that there exists $x \in S_1$ and $y \in S_2$ such that $xy \notin E(G_{25})$. This concludes the proof as x, a, y is an 1-FSeq. *W.l.o.g.* we may assume that $1 \leq |S_1| \leq 2$. If $S_1 = \{x\}$ then for every vertex different from a there is at least one vertex y non adjacent to it that necessarily belongs to S_2 and thus we have found our $xy \notin E(G_{25})$. Otherwise, $S_1 = \{x, z\}$ and $S_2 = \{s, t\}$. Notice that one among xs, xt, zs and zt must be a non-edge as the graph induced by $G_{25} - \{a\}$ has strictly less then 4 edges. Thus, again we found our pair $xy \notin E(G_{25})$.

Finally, notice that Observation 2 holds also for the graph G_{27} and in the previous proof, we did not consider whether there is an edge between c and d or not. Therefore, the argument for the graph G_{27} follows identically as for the graph G_{25}. □

Theorem 1 *For all graphs with at most 5 vertices, the star number is 1, except for the four graphs* G_{15}, G_{20}, G_{25} *and* G_{27} *which have a star number equal to 2.*

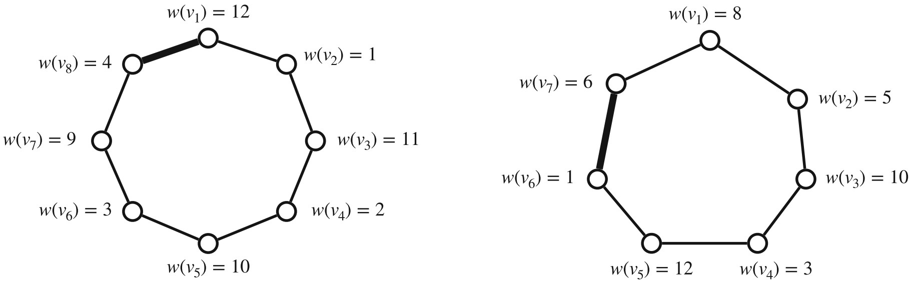
Proof From Lemmas 6, 7 and 8 we conclude that $\gamma(G_i) \geq 2$ for $i \in \{15, 20, 25, 27\}$. The witness graph and the corresponding interval depicted in Fig. 4 demonstrate that indeed $\gamma(G_i) = 2$ for these graphs. Next, it is straightforward to verify that the remaining graphs in Fig. 4 are star-1-PCGs by examining the witness graph along with the associated interval. The weights and the corresponding intervals are given in Fig. 4. This proves the claim for $n = 5$.

Now, let G' be a graph of at most 4 vertices. We can obtain a graph G of exactly 5 vertices, by adding isolated vertices to G'. Thus, any graph of at most 4 vertices can be viewed as an induced subgraph of the following graphs depicted in Fig. 4: $G_1, G_2 \ldots, G_8$, $G_{13}, G_{14}, G_{18}, G_{24}$. These graphs are shown to be star-1-PCGs. Thus, for $n \leq 4$ the claim follows from Lemma 1. □



(a) (b)

Fig. 6 **a** A caterpillar with $n = 6$ and **b** the corresponding 1-witness graph. The interval is $I = [217, 229]$



(a) (b)

Fig. 7 **a** The 2-witness graph for C_8, the intervals are $I_1 = [12, 13]$ and $I_2 = [16, 16]$. **b** The 2-witness graph for C_7; the two intervals are $I_1 = [13, 15]$ and $I_2 = [7, 7]$. The normal and bold edges in the cycle correspond to edges for which the sum of the weights of their endpoints falls in I_1 and I_2, respectively

3.2 *n*-vertex graphs with $n = 6$ or $n = 7$

The number of non-isomorphic graphs with 6 and 7 vertices is 156 and 1044, respectively [23], which is significantly greater then that of graphs on 5 vertices. Hence, to determine the star number of these graphs we introduce two different algorithms that enable automatic verification. First we introduce a straightforward LP program, *Algorithm 1*, that takes as input a graph G, an integer k denoting the number of intervals required and an integer M denoting the maximum possible weight on the vertices of G. The program checks if there is a k-witness G^w where the highest vertex weight is at most M. We generated all the non-isomorphic graphs with at most 7 vertices and checked if there is a k-witness for $k \in \{1, 2, 3\}$. We set M empirically to the value of 10 and 20 for $n = 6$ and $n = 7$, respectively. For graphs with 6 vertices, *Algorithm 1*, produced either a 1-witness or 2-witness graph leading to the next theorem.

Theorem 2 *For any graph G with* 6 *vertices, it holds that* $\gamma(G) \leq 2$ *and there exist graphs on* 6 *vertices for which the equality holds.*

Proof In [24] we include the list of all the graphs with their respective constructions proving the membership in star-1-PCG or star-2-PCG.

We applied *Algorithm 1* to all the graphs with 7 vertices. For all these graphs, except for the three shown in Fig. 5, namely G_{536},G_{662} and G_{963},[1] the algorithm successfully generated either a 1-witness or a 2-witness graph. For the three graphs in Fig. 5, it managed to construct a 3-witness as shown in [24] and in the caption of the figure. The proofs that the star number

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[1] The index of the graphs refers to the index they have in the list in [24].

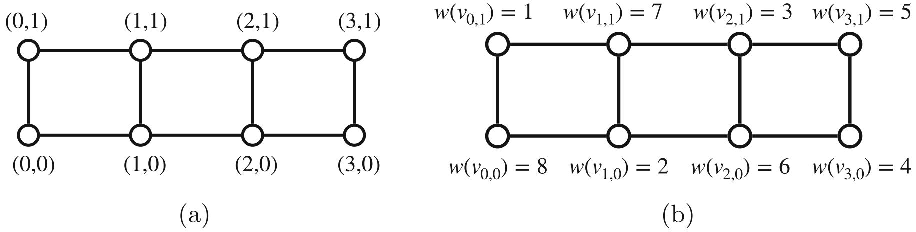

Fig. 8 **a** The grid $G_{4,2}$. **b** The 1-witness graph for $G_{4,2}$ with interval $I_1 = [8, 10]$

for these graphs is 3 is a case by case analysis that shows that for any graph G_i with $i \in \{536, 662, 963\}$, and any weight function w, the graph G_i^w contains a 2-FP. These proofs rely solely on the ordering of the vertices based on their weight and not on the actual weight values. The proofs are quite lengthy and technical and thus we use automatic verification to ensure that all cases were thoroughly covered. To this purpose we develop an algorithm, *Algorithm 2*, that given a graph G and an integer k, checks if there exists an ordering π of the vertices for which no k-FP appears. Notice that from Lemma 3, if for all possible orderings π an k-FP appears, then $\gamma(G) > k$. Notice that all three graphs in Fig. 5 do not contain two vertices u, v for which $N(u) - \{v\} = N(v) - \{u\}$ and hence by Lemma 2, in any k-witness graph G^w, there are no two vertices with the same weight. Thus, we can only focus on strict total orders of the vertices. For each one of the graphs in Fig. 5 and for all possible strict total orders π, *Algorithm 2* always found a 2-FP leading to the following result.

Theorem 3 *For all graphs with 7 vertices, the star number is at most 2, except for the three graphs depicted in Fig. 5 which have a star number equal to 3.*

4 The star number of caterpillars

Here we prove that caterpillars are star-1-PCGs. Notice that in [20] it is shown that caterpillars are 2-threshold and by Observation 1 they are also star-1-PCGs. However, we provide a new construction within the framework of star-k-PCGs, providing a different approach to understanding caterpillars in this context.

Theorem 4 *For any caterpillar T, it holds $\gamma(T) = 1$.*

Proof We consider first a caterpillar $T = (V, E)$ where every vertex of the central path is adjacent to *exactly one* leaf. We denote by n the number of vertices in the central path and the vertices are enumerated following an alternating pattern along the central path and its attached leaves. The vertices along the central path alternate between lowercase and uppercase labels: $a_0, A_1, a_2, A_3, \ldots, a_i, A_{i+1}, a_{i+2}, \ldots$. Each vertex in the central path is connected to a leaf labeled by the opposite case, hence a_0 is paired with A_0, A_1 with a_1, and so on. In Fig. 6a we show the labeling for a caterpillar with $n = 6$. Notice that we can assume $n \geq 3$ as for any $n < 3$ it is easy to see that a caterpillar is a star-1-PCG [24].

For each $0 \leq i < n$ we define the weight of a_i and A_i as follows:

$$w(a_i) = \begin{cases} (i+1)n + 1, & \text{if } i \text{ is even} \\ (i+1)n + 1 + i, & \text{if } i \text{ is odd} \end{cases}$$

$$w(A_i) = \begin{cases} n^3 - in - i, & \text{if } i \text{ is even} \\ n^3 - in, & \text{if } i \text{ is odd} \end{cases}$$

We define the interval:

$$I = [n^3 + 1, n^3 + 2n + 1]$$

See Fig. 6b for a construction of a 1-witness graph for caterpillar with $n = 6$.

Notice that the only edges are of type $a_i A_i$ and $a_{2r} A_{2r+1}$ and $A_{2r+1} a_{2r+2}$ for some integer $r \geq 0$. Consider two vertices x, y in the caterpillar. We need to consider the following cases:

Case I. $x = a_i$ and $y = a_j$. Notice that in this case $a_i a_j \notin E$. Indeed, $w(a_i) + w(a_j) < 2 \max_r w(a_r) = 2w(a_{n-1}) \leq 2n^2 + 2n \leq n^3$ as $n \geq 3$. Thus, $w(a_i) + w(a_j) \notin I$.

Case II. $x = A_i$ and $y = A_j$. Notice that in this case $A_i A_j \notin E$. Indeed, $w(A_i) + w(A_j) > 2 \min_r w(A_r) = 2w(A_0) = 2n^3 > n^3 + 2n + 1$ as $n \geq 3$. Thus, $w(A_i) + w(A_j) \notin I$.

Case III. $x = a_i$ and $y = A_j$. We have to consider the following cases:

- i and j are both odd: Notice that in this case $a_i A_j \in E$ if and only if $i = j$. Indeed, consider $w(a_i) + w(A_j) = n^3 + (i - j + 1)n + i + 1$. If $i = j$ then $w(a_i) + w(A_j) = n^3 + n + i + 1 \in I$ since $0 \leq i < n$. If $i < j$ then $i - j \leq -2$ and thus $w(a_i) + w(A_j) \leq n^3 - n + i + 1 \leq n^3 \notin I$. If $i > j$ then $i - j \geq 2$ and thus $w(a_i) + w(A_j) \geq n^3 + 3n + i + 1 \notin I$.
- i and j are both even: Notice that in this case $a_i A_j \in E$ if and only if $i = j$. Indeed, consider $w(a_i) + w(A_j) = n^3 + (i - j + 1)n - j + 1$. If $i = j$ then $w(a_i) + w(A_j) = n^3 + n - j + 1 \in I$ since $0 \leq j < n$. If $i < j$ then $i - j \leq -2$ and thus $w(a_i) + w(A_j) \leq n^3 - n - j + 1 \leq n^3 \notin I$. If $i > j$ then $i - j \geq 2$ and thus $w(a_i) + w(A_j) \geq n^3 + 3n - j + 1 \geq n^3 + 2n + 2 \notin I$ (where the last inequality follows as $j \leq n - 1$).
- i odd and j even: Notice that in this case $a_i A_j \notin E$. Indeed, $w(a_i) + w(A_j) = 1 + i + (1 + i)n + n^3 - j(n + 1) = n^3 + (i - j + 1)(n + 1)$. If $i > j$ then $i - j \geq 1$ and we have $w(a_i) + w(A_j) \geq n^3 + 2(n + 1) \geq n^3 + 2n + 2 \notin I$. If $i < j$ then $i - j \leq -1$ and we have $w(a_i) + w(A_j) \leq n^3 \notin I$.
- i even and j odd: Notice that in this case $a_i A_j \in E$ if and only if $|i - j| = 1$. Indeed, consider $w(a_i) + w(A_j) = n^3 + (i - j + 1)n + 1$. If $|i - j| = 1$ we have that either $w(a_i) + w(A_j) = n^3 + 2n + 1 \in I$ (when $i = j + 1$) or $w(a_i) + w(A_j) = n^3 + 1 \in I$ (when $j = i + 1$). If $|i - j| \geq 2$ then either $w(a_i) + w(A_j) \geq n^3 + 3n + 1 \notin I$ (when $i - j \geq 2$) or $w(a_i) + w(A_j) \leq n^3 - n + 1 \notin I$ (when $i - j \leq -2$).

It remains to show that the construction we provided generalizes to any caterpillar. Indeed if a vertex of the central path is connected to more than one leaf, we assign the same weight to all of its leaves. The construction is still valid as from Case I and Case II we have that for any $0 \leq i < n$, $2w(a_i) \notin I$ and $2w(A_i) \notin I$. Finally, if a vertex on the central path is not connected to any leaf, then the original construction trivially holds. □

5 The star number of cycle graphs

We already showed that $\gamma(C_5) = 2$, here we show that this holds for every cycle C_n with $n \geq 5$.

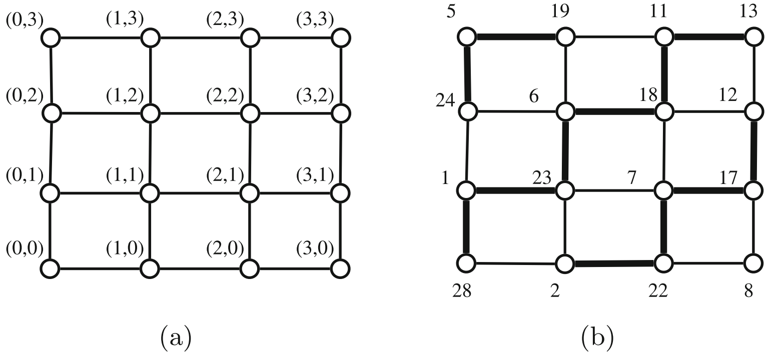

Fig. 9 **a** The grid $G_{4,4}$. **b** The corresponding 2-witness graph graph. The two intervals are $I_1 = [24, 25]$ and $I_2 = [29, 30]$. The normal and bold edges correspond to edges for which the sum of their endpoints falls in I_1 and I_2, respectively

Theorem 5 *For any $n \geq 5$ it holds $\gamma(C_n) = 2$.*

Proof Note that the proof of Lemma 6 can be extended to every cycle C_n with $n \geq 5$ and thus $\gamma(C_n) > 1$. We provide now a construction to show that cycle graphs are star-2-PCGs. To this purpose we extend a construction for the 1-witness of a path. The construction depends on the parity of n. Let $C_n = v_1, v_2, \ldots, v_n, v_1$ and to simplify the notation we set for every $1 \leq i \leq n$, $w(v_i) = w_i$.

Case n even. For every $1 \leq i \leq n$, we define w_i as follows:

$$w_i = \begin{cases} n + \dfrac{n}{2} - \dfrac{i-1}{2}, & \text{if } i \text{ is odd} \\ \dfrac{i}{2}, & \text{if } i \text{ is even} \end{cases}$$

We define two intervals:

$$I_1 = \left[n + \frac{n}{2}, n + \frac{n}{2} + 1\right] \quad I_2 = [2n, 2n]$$

In Fig. 7a we depict an example of the 2-witness graph for C_8. Consider any two arbitrary vertices v_i, v_j, with $v_i \neq v_j$. There are three cases to consider:

(a) Both i and j are odd. In this case $w_i + w_j = 3n - \frac{i+j}{2} + 1 > 2n + 1$ (where the last inequality follows from $\frac{i+j}{2} < n$). Thus, $w_i + w_j \notin I_1$ and $w_i + w_j \notin I_2$.

(b) Both i and j are even. In this case $w_i + w_j = \frac{i+j}{2} < n$. Thus, $w_i + w_j \notin I_1$ and $w_i + w_j \notin I_2$.

(c) i and j have different parity. *W.l.o.g.* let i be odd and j be even. In this case $w_i + w_j = n + \frac{n}{2} - \frac{i-j-1}{2}$. Clearly, $w_i + w_j \in I_1$ if and only $\frac{i-j-1}{2} \in \{-1, 0\}$. If $\frac{i-j-1}{2} = -1$ then $i = j - 1$. Otherwise if $\frac{i-j-1}{2} = 0$ we have $i = j + 1$. Thus $w_i + w_j \in I_1$ if and only if $|i - j| = 1$ which corresponds to an edge in C_n, more precisely in $P_n = v_1, v_2, \ldots, v_n$. Moreover, $w_i + w_j \in I_2$ if and only if $j - i = n - 1$. The latter holds only for $j = n$ and $i = 1$ as $1 \leq i, j \leq n$, which again corresponds to the edge $v_n v_1$ in C_n.

Thus, from points (a)-(c) we conclude that there is an edge among v_i and v_j if and only if $v_i v_j$ is an edge in C_n.

Case n odd. For every $1 \leq i \leq n$ we define w_i so that:

$$w_i = \begin{cases} n-i, & \text{if } i \text{ is even} \\ n+i, & \text{if } i \text{ is odd and } i \neq n \\ n-1, & \text{if} i = n \end{cases}$$

We define two intervals:

$$I_1 = [2n-1, 2n+1] \quad I_2 = [n, n]$$

In Fig. 7a we depict an example of the 2-witness graph for C_7. Consider any two arbitrary vertices v_i, v_j, with $v_i \neq v_j$. There are three cases to consider:

(a) Both i and j are even. In this case $w_i + w_j = 2n - (i+j) < 2n - 1$ (where the last inequality follows from $i + j \geq 6$). Thus $w_i + w_j \notin I_1$. Moreover $i + j$ is even thus $w_i + w_j = 2n - (i+j)$ is even too. Hence $w_i + w_j \notin I_2$.

(b) Both i and j are odd. First assume i and j different from n. In this case $w_i + w_j = 2n + (i+j) > 2n+1$ (where the last inequality follows from $i+j \geq 4$). Thus $w_i+w_j \notin I_1 \cup I_2$. Assume now that $i = n$, in this case we have $w_n + w_j = n - 1 + n + j = 2n + j - 1$. If $j = 1$ (i.e. we are considering the edge $v_n v_1$) then $w_n + w_j = 2n \in I_1$. Otherwise, for $j \geq 3$, it holds that $w_n + w_j \geq 2n + 2$. Thus, $w_n + w_j \notin I_1 \cup I_2$.

(c) The vertices i and j have different parity and *w.l.o.g.* let i be even and j be odd. Assume first that $j \neq n$. In this case $w_i + w_j = 2n - i + j$. Clearly, $w_i + w_j \in I_1$ if and only if $-1 \leq j - i \leq 1$ and since $i \neq j$ it must be $i = j + 1$ or $i = j - 1$ which correspond to edges of $P_{n-1} = v_1, v_2, \ldots, v_{n-1}$. Consider now the case $j = n$. We have $w_i + w_j = 2n - i - 1$. Clearly, as $i \geq 2$, $w_i + w_j \leq 2n - 3 \notin I_1$. Finally, $w_i + w_j \in I_2$ if and only if $i = n - 1$. This corresponds to the edge $v_{n-1}v_n$ of C_n.

Thus there is an edge among i and j if and only if $v_i v_j$ is an edge in C_n. This concludes the proof. □

6 The star number of grid graphs

In [20] it has been shown that every graph with minimum degree δ in which any two distinct vertices have at most c neighbors in common is not a $[2(\delta - c - 1) - 1]$-threshold graph. Since any d-dimensional grid has minimum degree $\delta = d$ and any two vertices can share at most $c = 2$ neighbors by Observation 1 we have that the star number of a d-dimensional grid is at least $\delta - 3$. Thus, the following theorem holds.

Theorem 6 [20] *Given an integer $d \geq 5$, for any d-dimensional grid G it holds $\gamma(G) \geq d-3$.*

In this section we analyse the cases $d = 2$ and $d = 4$. To simplify the notation we introduce the following convention: for any vertex u described by its coordinates $(i_1, \ldots, i_d)$, we will write $w(i_1, \ldots, i_d) = w(u)$.

Theorem 7 *For any 2-dimensional grid G_{n_1,n_2} with $\min\{n_1, n_2\} \leq 2$ it holds $\gamma(G_{n_1,n_2}) = 1$.*

Proof If $\min\{n_1, n_2\} = 1$ then the graph is a path and it is already known that it is a star-1-PCG [10, 25]. Assume now $\min\{n_1, n_2\} = 2$ and *w.l.o.g.* let $n_2 = 2$. It is worth mentioning that it is already known that $G_{n_1,2}$ is a PCG but the witness tree is a caterpillar [26]. We define a 1-witness graph for $G_{n_1,2}$ as follows: For each vertex (i, j) with $0 \leq i \leq n_1 - 1$ and $0 \leq j \leq 1$, we define its weight $w(i, j)$, as follows:

$$w(i, j) = \begin{cases} 2n_1 - i & \text{if } i + j \text{ is even} \\ i + 1 & \text{if } i + j \text{ is odd} \end{cases}$$

We define

$$I_1 = [2n_1, 2n_1 + 2]$$

See Fig. 8 for a construction of a 1-witness graph for $G_{4,2}$. We now show that G^w is a 1-witness graph for $G_{n_1,2}$. For this let (i, j) and (i', j') be two vertices and we consider the following three cases:

Case $j = j' = 0$.
Notice that we must have $i \neq i'$ and $i + i' \leq n_1 - 1 + n_1 - 3 = 2n_1 - 4$. We consider the following three subcases:

- *Both i and i′ are odd.* We have $w(i, 0) + w(i', 0) = i + i' + 2 \leq 2n_1 - 2 \notin I_1$.
- *Both i and i′ are even.* We have $w(i, 0) + w(i', 0) = 4n_1 - (i + i') \geq 2n_1 + 4 \notin I_1$.
- *i and i′ have different parity. W.l.o.g.* assume i odd and i' even. Then $w(i, 0) + w(i', 0) = 2n_1 + i - i' + 1$.Thus, as $I_1 = [2n_1, 2n_1 + 2]$ it must hold that either $2n_1 + i - i' + 1 = 2n_1$ or $2n_1 + i - i' + 1 = 2n_1 + 2$. hus, $w(i, 1) + w(i', 1) \in I_1$ if and only if $|i - i'| = 1$, which corresponds to the edges of $G_{n_1,2}$ for which $|i - i'| = 1$ and $j = j' = 0$.

Case $j = j' = 1$.
This case is symmetrical to the previous one. Thus, $w(i, 1) + w(i', 1) \in I_1$ if and only if $|i - i'| = 1$, which corresponds to the edges of $G_{n_1,2}$ for which $|i - i'| = 1$ and $j = j' = 1$.
Case j and j' of different parity.
W.l.o.g. assume $j = 0$ and $j' = 1$. Then we consider the following three subcases:

- *Both i and i′ are odd.* We have $w(i, 0) + w(i', 1) = 2n_1 + i - i' + 1$. If $i \neq i'$ then $|i - i'| \geq 2$ and thus either $w(i, 0) + w(i', 1) \geq 2n_1 + 3 \notin I_1$ or $w(i, 0) + w(i', 1) \leq 2n_1 - 1 \notin I_1$. Otherwise, if $i = i'$ then $w(i, 0) + w(i', 1) = 2n_1 + 1 \in I_1$ which corresponds to the edges of $G_{n_1,2}$ for which $i = i'$ and $|j - j'| = 1$.
- *Both i and i′ are even.* We have $w(i, 0) + w(i', 1) = 2n_1 - i + i' + 1$ and the case follows identical to the previous one. Thus, $w(i, 1) + w(i', 1) \in I_1$ if and only if $i = i'$, which corresponds to the edges of $G_{n_1,2}$ for which $i = i'$ and $|j - j'| = 1$.
- *i and i′ have different parity.* Notice that we must have $i \neq i'$ and thus $i + i' \leq 2n_1 - 4$. Assume first i odd and i' even. Then $w(i, 0) + w(i', 1) = i + i' + 2 \leq 2n_1 - 2 \notin I_1$. Otherwise, if i even and i' odd. Then $w(i, 0) + w(i', 1) = 4n_1 - (i + i') \geq 2n_1 + 4 \notin I_1$.

Thus, $G^w_{n_1,2}$ is a 1-witness graph for $G_{n_1,2}$. □

Theorem 8 *For any 2-dimensional grid G_{n_1,n_2} with $n_1, n_2 \geq 3$ it holds $\gamma(G_{n_1,n_2}) = 2$.*

Proof Notice that the cycle graph C_8 is an induced subgraph of G_{n_1,n_2} with $n_1, n_2 \geq 3$. By Lemma 1 and Theorem 5 we have $\gamma(G_{n_1,n_2}) \geq 2$. We now prove that any 2-dimensional grid graph is a star-2-PCG. Notice that if a graph G is a star-2-PCG, then so is any vertex induced subgraph of G. Hence, it is sufficient to focus on the case where $n_1 = n_2 = h$. Indeed the construction for any G_{n_1,n_2} can be obtained by the one of $G_{h,h}$ where $h = \max\{n_1, n_2\}$. Let $G = G_{h,h}$ and for each vertex (i, j), we define its weight $w(i, j)$, as follows:

$$w(i, j) = \begin{cases} \frac{(i + j - 1)h}{2} + i + 1, & \text{if } i + j \text{ is odd} \\ (2h - 1)h - \frac{(i + j)h}{2} - i, & \text{if } i + j \text{ is even} \end{cases}$$

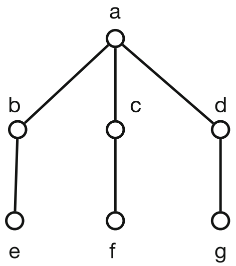

Fig. 10 The smallest tree known not to be a star-1-PCG

We define two intervals:

$$I_1 = [2h(h-1), 2h(h-1)+1]$$
$$I_2 = [2h(h-1)+h+1, 2h(h-1)+h+2]$$

See Fig. 9 for a construction of a 2-witness graph for $G_{4,4}$. By construction two vertices (i, j) and (i', j') are adjacent if the sum of their weights is one of the 4 integer values in $S = \{2h(h-1), 2h(h-1)+1, 2h(h-1)+h+1, 2h(h-1)+h+2\}$. Consider two vertices (i, j), (i', j') in the grid with $0 \leq i, j, i', j' \leq h-1$. There are two cases to consider. Case $i+j$ and $i'+j'$ have the same parity. By definition of a 2-dimensional grid these vertices are not adjacent. Consider first the case where $i+j$ and $i'+j'$ are both odd. Notice that $w(i, j) = \frac{(i+j-1)h}{2} + i + 1$ is maximized for $i = h-1$ and $j = h-2$ (notice that as $i+j$ is odd we cannot have $i = j = h-1$. Hence, $w(i, j) + w(i', j') < (2h-4)h + 2h = (2h-2)h$ and thus is not in S where the last inequality holds as $(i, j) \neq (i', j')$.

Consider now the case $i+j$ and $i'+j'$ are both even. Notice that $w(i, j) = (2h-1)h - \dfrac{(i+j)h}{2} - i$ is minimized for $i = h-1$ and $j = h-1$. Hence, $w(i, j) + w(i', j') > 2h(h-1)+2$ and thus is not in S. Case $i+j$ and $i'+j'$ have the different parity. *W.l.o.g.* assume $i+j$ odd and $i'+j'$ even and thus:

$$w(i, j) + w(i', j') = (2h-1)h + (i - i' + j - j' - 1)\frac{h}{2} + i - i' + 1$$

We consider now for what values of i, j, i', j' we have $w(i, j) + w(i', j') = s \in S$. To this purpose we solve the following equations for each possible value of s.

- In the case $s = 2h(h-1)$ we obtain the equation $(i - i' + j - j' + 1)h + 2(i - i' + 1) = 0$. Let $c = i - i' + j - j' + 1$ and we consider for which values of c the equation has solutions. Notice first that for $c \leq -3$ and $c \geq 3$ there are no solutions as $-2h \leq 2(i - i' + 1) \leq 2h$ (recall that $0 \leq i, j, i', j' \leq h-1$). Moreover, as $i+j$ is odd and $i'+j'$ is even we have that c must be even. Thus, the only possible cases that remain to consider are $c \in \{-2, 0, 2\}$. If $c = 2$ then $2h + 2(i - i' + 1) = 0$ and thus $i - i' + 1 = -h$. From this and $i - i' + j - j' + 1 = 2$ we have $j - j' = h + 2$ which is not possible since $j - j' \leq h - 1$. If $c = -2$ then $-2h + 2(i - i' + 1) = 0$ and thus $i - i' + 1 = h$. From this and $i - i' + j - j' + 1 = -2$ we have $j - j' = -(h+2)$ which is not possible since $j - j' \geq -(h+1)$. The only possibility is $c = 0$ and as a consequence $2(i - i' + 1) = 0$ from which we have $i = i' - 1$. Then as $i - i' + j - j' + 1 = 0$ we have $j = j'$.
- In the case $s = 2h(h-1)+1$ following a similar argument as in the previous point we have the only possibility is $i = i'$ and $j = j' - 1$.
- In the case $s = 2h(h-1)+h+1$ following a similar argument as in the previous point we have the only possibility is $i = i'$ and $j = j' + 1$.

- In the case $s = 2h(h-1) + h + 2$ following a similar argument as in the previous point we have the only possibility is $i = i' + 1$ and $j = j'$.

From the previous four items we have that two vertices (i, j), (i', j') are adjacent if and only if $i = i'$ and $|j - j'| = 1$ or $j = j'$ and $|i - i'| = 1$ and $j = j'$. This concludes the proof.

The next theorem states the result for the 4-dimensional grids. We will make use of the following definitions. Let a be a vertex of degree 8 in a given 4-dimensional grid. Two neighbors b and b' of vertex a are called *opposed* if there exists a dimension s for which $|b_s - b'_s| = 2$ and for all $t \neq s$, $b_t = b'_t$. Notice that the 8 neighbors of a are partitioned in 4 pairs of opposed vertices. Indeed, suppose on the contrary there exist three neighbors b, c, c' of a such that (b, c), (b, c') are both opposed pairs. This means that b and c differ by 2 in exactly one coordinate, b and c' differ by 2 in exactly one coordinate and the three of them differ by 1 from a in exactly one coordinate. It is easy to check that this cannot happen.

Consider now any two neighbors b_s and b_t of a that are not opposed. Then there exists exactly one vertex y different from a, such that $N(a) \cap N(y) = \{b_s, b_t\}$. We denote this vertex as $Q^a_{b_s,b_t}$.

Theorem 9 *Given four positive integers* $n_1, n_2, n_3, n_4 \geq 3$, *for any* 4-*dimensional grid* G_{n_1,n_2,n_3,n_4}*it holds* $\gamma(G_{n_1,n_2,n_3,n_4}) \geq 3$.

Proof Consider $G = G_{n_1,n_2,n_3,n_4}$, and let G^ω be any of its k-witness graphs. We will show that in G^ω there exists a vertex x with v_1, v_2, v_3 in $N(x)$ and u_1, u_2 not in $N(x) \cup \{x\}$ such that $w(v_1) \leq w(u_1) \leq w(v_2) \leq w(u_2) \leq w(v_3)$ and the proof will follow from Lemma 3 since the existence of the vertex x implies the presence in G^ω of the 2-FP $xv_1, xu_1, xv_2, xu_2, xv_3$.

Consider the vertex $a = (1, 1, 1, 1)$. Notice that a has exactly 8 neighbors and by Lemma 2 all these vertices have different weights, *w.l.o.g.* let $b_1, \ldots, b_8$ be these neighbors and $w(b_1) < w(b_2) < \ldots < w(b_8)$. We consider now two cases:

Case I. There exist two vertices $u_1, u_2 \notin N(a)$ and two integers i and j, $1 \leq i < j < 7$ such that it holds $w(b_i) < w(u_1) < w(b_{i+1})$ and $w(b_j) < w(u_2) < w(b_{j+1})$. In this case we have $x = a$ and $v_1 = b_1$, $v_2 = b_{i+1}$, $v_3 = b_8$.

Case II. If Case I does not hold, then there exists an integer i, $1 \leq i \leq 7$ such that for any $u \notin N(a)$ one of the followings holds: (i) $w(u) < w(b_1)$, (ii) $w(b_i) < w(u) < w(b_{i+1})$, (iii) $w(u) > w(b_8)$. Notice that if every $u \notin N(a)$ satisfies (i) or (iii) then i can be any value between 1 and 7. We will consider only the cases $1 \leq i \leq 4$ as the reasoning in the cases for $i > 4$ is identical to the cases $8 - i$.

Case II.a. $1 \leq i \leq 2$. Notice that as the neighbors of a are partitioned into opposed pairs, at least one between the pairs (b_4, b_6) and (b_4, b_7) is not an opposed pair. *W.l.o.g.* let (b_4, b_6) be such a pair. Let $y = Q^a_{b_4,b_6}$ and consider an arbitrary vertex $v \in N(y) - \{b_4, b_6\}$ (notice that the minimum degree of a vertex in a 4-dimensional grid is 4). Since $v \notin N(a)$ we have that one of the followings must hold: (i) $w(v) < w(b_1)$, (ii) $w(b_i) < w(v) < w(b_{i+1})$, (iii) $w(v) > w(b_8)$. We consider each case separately and prove that in all the cases we have $x = y$.

(i) If $w(v) < w(b_1)$ we can set $v_1 = v$, $v_2 = b_4$, $v_3 = b_6$ and $u_1 = b_1$, $u_2 = b_5$.
(ii) If $w(b_i) < w(v) < w(b_{i+1})$ we can set $v_1 = v$, $v_2 = b_4$, $v_3 = b_6$ and $u_1 = b_3$, $u_2 = b_5$.
(iii) If $w(v) > w(b_8)$ we can set $v_1 = b_4$, $v_2 = b_6$, $v_3 = v$ and $u_1 = b_5$, $u_2 = b_8$.

Case II.b. $3 \le i \le 4$. Notice that as the neighbors of a are partitioned into opposed pairs, at least one between the pairs, at least one between the pairs (b_2, b_6) and (b_2, b_7) is not an opposed pair. *W.l.o.g.* let (b_2, b_6) be such a pair. Let $y = Q^a_{b_2,b_6}$, similarly to the previous case let $v \in N(y) - \{b_2, b_6\}$. Since $v \notin N(a)$ we have that one of the followings must hold: (i) $w(v) < w(b_1)$, (ii) $w(b_i) < w(v) < w(b_{i+1})$, (iii) $w(v) > w(b_8)$. We consider each case separately and prove that in all the cases we have $x = y$.

(i) If $w(v) < w(b_1)$ we can set $v_1 = v$, $v_2 = b_2$, $v_3 = b_6$ and $u_1 = b_1$, $u_2 = b_5$.
(ii) If $w(b_i) < w(v) < w(b_{i+1})$ we can set $v_1 = b_2$, $v_2 = v$, $v_3 = b_6$ and $u_1 = b_3$, $u_2 = b_5$.
(iii) If $w(v) > w(b_8)$ we can set $v_1 = b_2$, $v_2 = b_6$, $v_3 = v$ and $u_1 = b_5$, $u_2 = b_8$.

This concludes the proof. □

7 Conclusions and open problems

In this paper we consider the problem of characterizing star-k-PCGs. This is particularly interesting as this class connects two important graph classes: the PCGs and multithreshold graphs for both of which a complete characterization is not yet known. Here we investigate the star number, γ, of simple graph classes, such as graphs of small size, caterpillars, cycles and grids. Specifically, we determine the exact value of $\gamma(G)$ for all the graphs with at most 7 vertices. By doing so we show that the smallest graphs with star number 2 are only 4 and have exactly 5 vertices; the smallest graphs with star number 3 are only 3 and have exactly 7 vertices. Next, we provide a construction showing that the star number of caterpillars is one. Moreover, we show that the star number of cycles and two dimensional grid graphs is 2 and that for 4-dimensional grids the star number is at least 3. Many problems remain open.

Problem 1 *For a d-dimensional grid G, with $d \neq 2$, determine the value of $\gamma(G)$.*

Given a graph G, from Lemma 3 it follows that if for any ordering π of the vertices, the presence of a k-FP can be deduced, then $\gamma(G) > k$. However, even if we find an ordering π in which no k-FP appears, we currently lack a method to obtain a k-witness graph G^w. Thus we have

Problem 2 *Given a graph G, is it true that the existence of an ordering of the vertices π such that no k-FP can be deduced, implies the existence of a weight function w, for which G^w is a k-witness graph and $\sigma(G^w) = \pi$?*

For $k = 1$ a similar result has been proved in [10] so a positive answer to the previous problem can be seen as a generalization of the result in [10] to an arbitrary k. Notice that this is can be interesting as in [10] it lead to a polynomial time algorithm for the recognition of star-1-PCGs. Thus, this may be an important step toward the solution of the following problem.

Problem 3 *Determine the computational complexity of recognizing star-k-PCGs for $k \ge 2$.*

From the results of Sect. 3 we have that for all graphs with at most 7 vertices, the graph in Fig. 10 is the only acyclic graph that is not a star-1-PCG. To the best of our knowledge, there are no acyclic graphs with a star number of at least 3 documented in the literature.

Problem 4 *Is there an acyclic graph G for which $\gamma(G) > 2$?*

More generally,

Problem 5 *What is the star number of acyclic graphs?*

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Declarations

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References

1. Monti, A., Sinaimeri, B.: On star-multi-interval pairwise compatibility graphs. In: WALCOM: Algorithms and Computation: 17th International Conference and Workshops, WALCOM 2023, Hsinchu, Taiwan, March 22-24, 2023, Proceedings, pp. 267–278. Springer, Berlin, Heidelberg (2023). https://doi.org/10.1007/978-3-031-27051-2_23
2. Ahmed, S., Rahman, M.S.: Multi-interval pairwise compatibility graphs. In: Gopal, T.V., Jäger, G., Steila, S. (eds.) Theory and Applications of Models of Computation, pp. 71–84. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-55911-7_6
3. Kearney, P., Munro, J.I., Phillips, D.: Efficient generation of uniform samples from phylogenetic trees. In: Benson, G., Page, R.D.M. (eds.) Algorithms in Bioinformatics, pp. 177–189. Springer, Berlin, Heidelberg (2003). https://doi.org/10.1007/978-3-540-39763-2_14
4. Nishimura, N., Ragde, P., Thilikos, D.M.: On graph powers for leaf-labeled trees. J. Algorithms **42**(1), 69–108 (2002). https://doi.org/10.1006/jagm.2001.1195
5. Long, Y., Stadler, P.F.: Exact-2-relation graphs. Discrete Appl. Math. **285**, 212–226 (2020). https://doi.org/10.1016/j.dam.2020.05.015
6. Brandstädt, A., Hundt, C.: Ptolemaic Graphs and Interval Graphs Are Leaf Powers, pp. 479–491. Springer, Berlin (2008). https://doi.org/10.1007/978-3-540-78773-0_42
7. Calamoneri, T., Petreschi, R., Sinaimeri, B.: On the pairwise compatibility property of some superclasses of threshold graphs. Discrete Math. Algorithms Appl. (2013). https://doi.org/10.1142/s1793830913600021
8. Benjamin, B., Høgemo, S., Telle, J.A., Vatshelle, M.: Recognition of linear and star variants of leaf powers is in P. In: Bekos, M.A., Kaufmann, M. (eds.) Graph-Theoretic Concepts in Computer Science, pp. 70–83. Springer, Cham (2022). https://doi.org/10.1007/978-3-031-15914-5_6
9. Monti, A., Sinaimeri, B.: On graphs that are not Star-k-PCGs (short paper). In: Castiglione, G., Sciortino, M. (eds.) Proceedings of the 24th Italian Conference on Theoretical Computer Science, Palermo, Italy, September 13-15, 2023. CEUR Workshop Proceedings, vol. 3587, pp. 92–97. CEUR-WS.org, Aachen, Germany (2023). https://ceur-ws.org/Vol-3587/0910.pdf
10. Xiao, M., Nagamochi, H.: Characterizing Star-PCGs. Algorithmica **82**(10), 3066–3090 (2020). https://doi.org/10.1007/s00453-020-00712-8
11. Kobayashi, Y., Okamoto, Y., Otachi, Y., Uno, Y.: Linear-time recognition of double-threshold graphs. Algorithmica **84**(4), 1163–1181 (2022). https://doi.org/10.1007/s00453-021-00921-9

12. Jamison, R.E., Sprague, A.P.: Double-threshold permutation graphs. J. Algebraic Comb. (2021). https://doi.org/10.1007/s10801-021-01029-7
13. Chvátal, V., Hammer, P.L.: Aggregation of inequalities in integer programming. Ann. Discrete Math. **1**, 145–162 (1977). https://doi.org/10.1016/S0167-5060(08)70731-3
14. Mahadev, N.V.R.: Threshold Graphs and Related Topics. Annals of Discrete Mathematics, vol. 543. Elsevier, Amsterdam, New York (1995)
15. Puleo, G.J.: Some results on multithreshold graphs. Gr. Comb. **36**(3), 913–919 (2020). https://doi.org/10.1007/s00373-020-02168-7
16. Chen, G., Hao, Y.: Multithreshold multipartite graphs. J. Gr. Theory (2022). https://doi.org/10.1002/jgt.22805
17. Calamoneri, T., Frascaria, D., Sinaimeri, B.: All graphs with at most seven vertices are pairwise compatibility graphs. Comput. J. **56**(7), 882–886 (2012). https://doi.org/10.1093/comjnl/bxs087
18. Durocher, S., Mondal, D., Rahman, M.S.: On graphs that are not PCGs. Theor. Comput. Sci. **571**, 78–87 (2015). https://doi.org/10.1016/j.tcs.2015.01.011
19. Calamoneri, T., Monti, A., Petroni, F.: All Graphs with at most 8 nodes are 2-interval-PCGs. arXiv (2022). https://doi.org/10.48550/ARXIV.2202.13844
20. Jamison, R.E., Sprague, A.P.: Multithreshold graphs. J. Gr. Theory **94**(4), 518–530 (2020). https://doi.org/10.1002/jgt.22541
21. Yanhaona, M.N., Hossain, K.S.M.T., Rahman, M.S.: Pairwise compatibility graphs. J. Appl. Math. Comput. **30**, 479–503 (2009). https://doi.org/10.1007/s12190-008-0204-7
22. Hakim, S.A., Papan, B.B., Rahman, M.S.: New results on pairwise compatibility graphs. Inf. Process. Lett. **178**, 106284 (2022). https://doi.org/10.1016/j.ipl.2022.106284
23. OEIS Foundation Inc.: Number of graphs on n unlabeled nodes. Entry A000088, The On-Line Encyclopedia of Integer Sequences. https://oeis.org/A000088 (n.b.)
24. https://github.com/sinaimeri/star-k-PCGs (2024)
25. Papan, B.B., Pranto, P.B., Rahman, M.S.: On 2-interval pairwise compatibility properties of two classes of grid graphs. Comput. J. (2022). https://doi.org/10.1093/comjnl/bxac011
26. Salma, S., Rahman, M., Hossain, M.: Triangle-free outerplanar 3-graphs are pairwise compatibility graphs. J. Gr. Algorithms Appl. **17**(2), 81–102 (2013). https://doi.org/10.7155/jgaa.00286

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