Proofs

EC.1. Supplementary material

Proofs of Section 2

The proof of Theorem 1 requires the following lemma.

LEMMA EC.1. *Let $u:\mathbb{R}^N \to \mathbb{R}$ be continuously differentiable. Then $u \in \mathcal{U}_{\boldsymbol{\gamma}}$ if and only if*

$$\eta_2(u(\boldsymbol{x}_4) - u(\boldsymbol{x}_3)) \leq \eta_1(u(\boldsymbol{x}_2) - u(\boldsymbol{x}_1)) \tag{EC.1}$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ satisfying (2.5) *for some i and γ_i.*

If part: Assume that u fulfills (EC.1) for some i and γ_i. Then

$$\eta_2(\boldsymbol{x}_4 - \boldsymbol{x}_3) = \gamma_i \eta_1(\boldsymbol{x}_2 - \boldsymbol{x}_1) \implies \boldsymbol{x}_3 = \boldsymbol{x}_4 - \gamma_i \eta_1 \boldsymbol{e}_i$$

so (EC.1) implies

$$\gamma_i \frac{\partial}{\partial x_i} u(\boldsymbol{x}_4) = \gamma_i \lim_{\eta_1 \to 0} \frac{u(\boldsymbol{x}_4) - u(\boldsymbol{x}_3)}{\gamma_i \eta_1} \leq \lim_{\eta_2 \to 0} \frac{u(\boldsymbol{x}_2) - u(\boldsymbol{x}_1)}{\eta_2} = \frac{\partial}{\partial x_i} u(\boldsymbol{x}_1).$$

As this holds for arbitrary $\boldsymbol{x}_1, \boldsymbol{x}_4$ and the derivatives are assumed to be continuous, by (2.3) we get $u \in \mathcal{U}_{\boldsymbol{\gamma}}$.

Only if part: Now assume that $u \in \mathcal{U}_{\boldsymbol{\gamma}}$ is continuously differentiable. Let $\boldsymbol{h} \coloneqq \boldsymbol{x}_2 - \boldsymbol{x}_1$. For $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ satisfying (2.5) for some i and γ_i, from $\eta_2(\boldsymbol{x}_4 - \boldsymbol{x}_3) = \gamma_i \eta_1(\boldsymbol{x}_2 - \boldsymbol{x}_1)$, we get that

$$\boldsymbol{x}_4 - \boldsymbol{x}_3 = \frac{\gamma_i \eta_1}{\eta_2}(\boldsymbol{x}_2 - \boldsymbol{x}_1).$$

Thus, from (EC.1) we can deduce

$$\begin{aligned}
\eta_1(u(\boldsymbol{x}_2) - u(\boldsymbol{x}_1)) &= \int_0^1 \frac{\partial}{\partial x_i} u(\boldsymbol{x}_1 + t\boldsymbol{h})\,\mathrm{d}t \\
&\geq \eta_1 \gamma_i \int_0^1 \frac{\partial}{\partial x_i} u\left(\boldsymbol{x}_3 + t\frac{\gamma_i \eta_1}{\eta_2}\boldsymbol{h}\right) \mathrm{d}t \\
&= \eta_2 \frac{\gamma_i \eta_1}{\eta_2} \int_0^1 \frac{\partial}{\partial x_i} u\left(\boldsymbol{x}_3 + t\frac{\gamma_i \eta_1}{\eta_2}\boldsymbol{h}\right) \mathrm{d}t \\
&= \eta_2(u(\boldsymbol{x}_4) - u(\boldsymbol{x}_3)). \quad \square
\end{aligned}$$

Proof of Theorem 1 The proof is based on the duality theory for transfers. Lemma EC.1 shows that $\mathcal{U}_{\boldsymbol{\gamma}}$ can be described by a set of inequalities, as in Müller (2013, definition 2.2.1). Therefore it is induced by the corresponding set of transfers. The proof thus follows from Müller (2013, theorem 2.4.1).

Proofs of Section 3

The following lemma is the building block in the proofs of most of the subsequent results in our paper. The basic idea is that increments of functions $u \in \mathcal{U}_{\boldsymbol{\gamma}}$ can be bounded above and below by separable piecewise linear utility functions that depend on $\boldsymbol{\gamma}$. This fact allows us to find sufficient conditions for $\boldsymbol{\gamma}$-dominance that do not depend on the joint distributions of the random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, but only on the marginal distributions of their components.

Lemma EC.2. *Let*

$$v_U(x;\gamma) := \begin{cases} \gamma x & \text{if } x \le 0, \\ x & \text{if } x > 0, \end{cases}$$

$$v_L(x;\gamma) := \begin{cases} x & \text{if } x \le 0, \\ \gamma x & \text{if } x > 0. \end{cases}$$

For any $u \in \mathcal{U}_{\boldsymbol{\gamma}}$, *let* $b_i := \sup_{\boldsymbol{x} \in \mathbb{R}^N} u_i'(\boldsymbol{x})$ *and fix some* $\boldsymbol{c} \in \mathbb{R}^N$. *Then, for any* $\boldsymbol{x} \in \mathbb{R}^N$, *we have*

$$\sum_{i=1}^N b_i v_L(x_i - c_i; \gamma_i) \le u(\boldsymbol{x}) - u(\boldsymbol{c}) \le \sum_{i=1}^N b_i v_U(x_i - c_i; \gamma_i). \tag{EC.2}$$

An instance of functions v_L and v_U is shown in Figure EC.1.

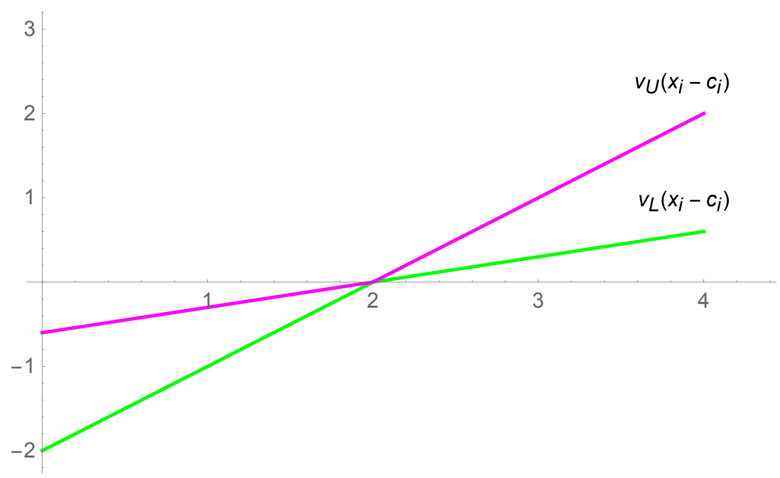

Figure EC.1 Functions v_L and v_U.

Proof of Lemma EC.2 Note that $u_i'(\boldsymbol{x}) \le \sup(u_i'(\boldsymbol{x})) = b_i$ and that by inequality (2.4) we have $u_i'(\boldsymbol{x}) \ge \gamma_i b_i$. By a multivariate first-order Taylor expansion, $u(\boldsymbol{x}) - u(\boldsymbol{c}) = \sum_{i=1}^N u_i'(\boldsymbol{y})(x_i - c_i)$, where y_i is between x_i and c_i. Then, using $u_i'(\boldsymbol{y}) \le b_i$ if $x_i > c_i$ and $u_i'(\boldsymbol{y}) \ge \gamma_i b_i$ if $x_i < c_i$ provides an upper bound, whereas using $u_i'(\boldsymbol{y}) \ge \gamma_i b_i$ if $x_i > c_i$ and $u_i'(\boldsymbol{y}) \le b_i$ if $x_i < c_i$ provides a lower bound.

Proof of Proposition 1 We prove (a). The proof of (b) is similar. Let $u \in \mathcal{U}_{\boldsymbol{\gamma}}$ and let

$$b_i := \sup_{\boldsymbol{x} \in \mathbb{R}^N} u_i'(\boldsymbol{x}). \tag{EC.3}$$

Without any loss of generality, assume $u(\boldsymbol{c}) = 0$. By Lemma EC.2 we have

$$u(\boldsymbol{x}) \leq \sum_{i=1}^N b_i v_U(x_i - c_i; \gamma_i), \tag{EC.4}$$

where $v_U(x_i - c_i; \gamma_i) = -\gamma_i (c_i - x_i)_+ + (x_i - c_i)_+$. This implies

$$\mathbb{E}[u(\boldsymbol{X})] \leq \sum_{i=1}^N b_i(-\gamma_i \, \mathbb{E}[(c_i - X_i)_+] + \mathbb{E}[(X_i - c_i)_+]) \tag{EC.5}$$

Therefore, $\mathbb{E}[u(\boldsymbol{X})] \leq 0$ if $-\gamma_i \, \mathbb{E}[(c_i - X_i)_+] + \mathbb{E}[(X_i - c_i)_+] \leq 0$ for all $i = 1, \ldots, N$.

Notice that $-\gamma_i \, \mathbb{E}[(c_i - X_i)_+] + \mathbb{E}[(X_i - c_i)_+] \leq 0$ is equivalent to $X_i \leq_{\gamma_i} c_i$. This proves the if part.

Now we prove the only if part. Consider a sequence of utility functions

$$u_n(\boldsymbol{x}) = \sum_{i=1}^N b_{i,n} v_U(x_i - c_i; \gamma_i)_+ \in \mathcal{U}_{\boldsymbol{\gamma}} \tag{EC.6}$$

such that $\lim_{n \to \infty} b_{j,n} = 0$ for $j \neq i$ and $b_{i,n} \equiv 1$ for all n.

If $\boldsymbol{X} \leq_{\boldsymbol{\gamma}} \boldsymbol{c}$, then $\mathbb{E}[u_n(\boldsymbol{X})] \leq u_n(\boldsymbol{c}) = 0$. This implies $-\gamma_i \, \mathbb{E}[(c_i - X_i)_+] + \mathbb{E}[(X_i - c_i)_+] \leq 0$ for all $i = 1, \ldots, N$, i.e., $X_i \leq_{\gamma_i} c_i$, for all $i = 1, \ldots, N$.

Proof of Theorem 2 Given $u \in \mathcal{U}_{\boldsymbol{\gamma}}$, let $b_i = \sup(u_i'(\boldsymbol{x}))$, and without loss of generality, assume $u(\boldsymbol{\delta}) = 0$. By Lemma EC.2 we have

$$\sum_{i=1}^N b_i v_L(x_i - \delta_i; \gamma_i) \leq u(\boldsymbol{x}) \leq \sum_{i=1}^N b_i v_U(x_i - \delta_i; \gamma_i).$$

First, we show that, for $i = 1, \ldots, N$, for any δ_i we have

$$\mathbb{E}[v_L(Y_i - \delta_i; \gamma_i)] = \mathbb{E}[v_U(X_i - \delta_i; \gamma_i)]$$

for γ_i defined as in (3.6). This follows from

$$\mathbb{E}[v_L(Y_i - \delta_i; \gamma_i)] = -\,\mathbb{E}[(\delta_i - Y_i)_+] + \gamma_i \, \mathbb{E}[(Y_i - \delta_i)_+)],$$
$$\mathbb{E}[v_U(X_i - \delta_i; \gamma_i)] = -\gamma_i \, \mathbb{E}[(\delta_i - X_i)_+] + \mathbb{E}[(X_i - \delta_i)_+],$$

and the definition of γ_i.

Therefore, from inequality (EC.2) it follows that

$$\mathbb{E}[u(\boldsymbol{Y})] \geq \sum_{i=1}^N b_i \, \mathbb{E}[v_L(Y_i - \delta_i; \gamma_i)] = \sum_{i=1}^N b_i \, \mathbb{E}[v_U(X_i - \delta_i; \gamma_i)] \geq \mathbb{E}[u(\boldsymbol{X})]$$

holds for arbitrary δ_i. We want to choose δ_i such that γ_i is as small as possible. As

$$\gamma_i = \frac{\mathbb{E}\big[(\delta_i - Y_i)_+\big] + \mathbb{E}\big[(X_i - \delta_i)_+\big]}{\mathbb{E}\big[(Y_i - \delta_i)_+\big] + \mathbb{E}\big[(\delta_i - X_i)_+\big]} = \frac{\mathbb{E}\big[(\delta_i - Y_i)_+\big] + \mathbb{E}\big[(X_i - \delta_i)_+\big]}{\mu_{Y_i} - \delta_i + \mathbb{E}\big[(\delta_i - Y_i)_+\big] + \delta_i - \mu_{X_i} + \mathbb{E}\big[(X_i - \delta_i)_+\big]},$$

we have to minimize $\mathbb{E}\big[(\delta_i - Y_i)_+\big] + \mathbb{E}\big[(X_i - \delta_i)_+\big]$ with respect to δ_i. The right derivative is

$$\frac{\partial^+}{\partial \delta_i}\Big(\mathbb{E}\big[(\delta_i - Y_i)_+\big] + \mathbb{E}\big[(X_i - \delta_i)_+\big]\Big) = \mathbb{E}\big[\mathbb{1}_{[\delta_i - Y_i \geq 0]}\big] - \mathbb{E}\big[\mathbb{1}_{[X_i - \delta_i \geq 0]}\big] = G_i(\delta_i) - 1 + F_i(\delta_i).$$

Therefore, δ_i is minimized for $\delta_i = \inf\{x \colon F_i(x) + G_i(x) \geq 1\}$.

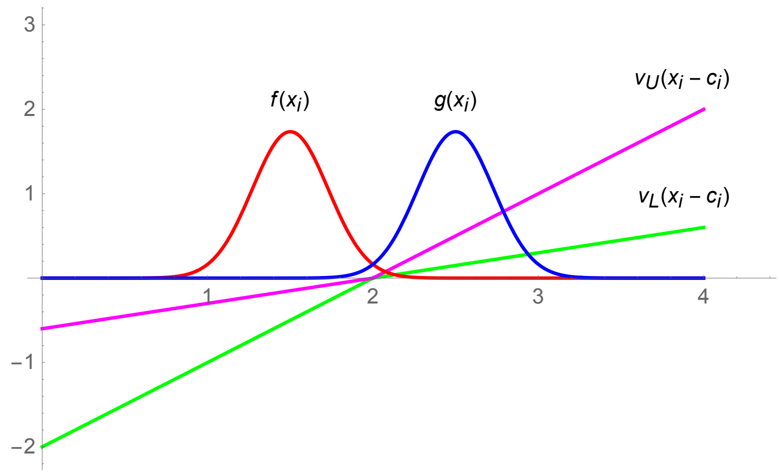

Figure EC.2 The variable Y_i $\boldsymbol{\gamma}$-dominates the constant c_i, which in turns dominates the variable X_i.

In Figure EC.2, for some $\boldsymbol{\gamma}$, the variable Y_i dominates c_i and c_i dominates X_i.

Proof of Proposition 2 In this case we can solve for δ_i from Theorem 2:

$$\begin{aligned} F_i(\delta_i) + G_i(\delta_i) = 1 &\iff H\left(\frac{\delta_i - \mu_{X_i}}{\sigma_{X_i}}\right) + H\left(\frac{\delta_i - \mu_{Y_i}}{\sigma_{Y_i}}\right) = 1 \\ &\iff H\left(\frac{\delta_i - \mu_{X_i}}{\sigma_{X_i}}\right) = H\left(\frac{\mu_{Y_i} - \delta_i}{\sigma_{Y_i}}\right) \\ &\iff \frac{\delta_i - \mu_{X_i}}{\sigma_{X_i}} = \frac{\mu_{Y_i} - \delta_i}{\sigma_{Y_i}} \\ &\iff \delta_i = \frac{\mu_{X_i}\sigma_{Y_i} + \mu_{Y_i}\sigma_{X_i}}{\sigma_{X_i} + \sigma_{Y_i}}. \end{aligned}$$

Hence

$$\gamma_i = \frac{\mathbb{E}\big[(Y_i - \delta_i)_+\big] + \mathbb{E}\big[(\delta_i - X_i)_+\big]}{\mathbb{E}\big[(\delta_i - Y_i)_+\big] + \mathbb{E}\big[(X_i - \delta_i)_+\big]} = \frac{\sigma_{Y_i}\,\mathbb{E}\big[(Z - \tau_i)_+\big] + \sigma_{X_i}\,\mathbb{E}\big[(Z - \tau_i)_+\big]}{\sigma_{Y_i}\,\mathbb{E}\big[(\tau_i - Z)_+\big] + \sigma_{X_i}\,\mathbb{E}\big[(\tau_i - Z)_+\big]} = \eta(\tau_i). \quad \square$$

The proof of Proposition 3 is along the lines of Müller et al. (2017, example 2.11).

Proof of Proposition 3 The following condition for γ_i^{M}-dominance in location-scale models can be found in Müller et al. (2017, bottom of page 2940):

$$\gamma_i^{\mathrm{M}} = \frac{\int_{-\infty}^{\infty} (G_i(x) - F_i(x))_+ \, \mathrm{d}x}{\int_{-\infty}^{\infty} (F_i(x) - G_i(x))_+ \, \mathrm{d}x} = \frac{\int_{-\infty}^{\infty} \left(H\left(\frac{x-\mu_{Y_i}}{\sigma_{Y_i}}\right) - H\left(\frac{x-\mu_{X_i}}{\sigma_{X_i}}\right)\right)_+ \mathrm{d}x}{\int_{-\infty}^{\infty} \left(H\left(\frac{x-\mu_{X_i}}{\sigma_{X_i}}\right) - H\left(\frac{x-\mu_{Y_i}}{\sigma_{Y_i}}\right)\right)_+ \mathrm{d}x}. \tag{EC.7}$$

The two distribution functions F_i and G_i single-cross at a point δ_i such that

$$\frac{\delta_i - \mu_{X_i}}{\sigma_{X_i}} = \frac{\delta_i - \mu_{Y_i}}{\sigma_{Y_i}}, \tag{EC.8}$$

which implies

$$\delta_i = \frac{\mu_{Y_i}\sigma_{X_i} - \mu_{X_i}\sigma_{Y_i}}{\sigma_{X_i} - \sigma_{Y_i}}. \tag{EC.9}$$

Notice that, for $x < \delta_i$, the distribution with a larger variance takes larger values than the other one. Moreover, integrating by parts, we get the well-known equalities:

$$\int_{\infty}^{\delta_i} F_i(x)\,\mathrm{d}x = \mathbb{E}\big[(\delta_i - X_i)_+\big], \qquad \int_{\delta_i}^{\infty} F_i(x)\,\mathrm{d}x = \mathbb{E}\big[(X_i - \delta_i)_+\big]. \tag{EC.10}$$

Therefore, when $\sigma_{Y_i} > \sigma_{X_i}$, (EC.7) becomes

$$\gamma_i^{\mathrm{M}} = \frac{\int_{-\infty}^{\delta_i} \left(H\left(\frac{x-\mu_{Y_i}}{\sigma_{Y_i}}\right) - H\left(\frac{x-\mu_{X_i}}{\sigma_{X_i}}\right)\right) \mathrm{d}x}{\int_{\delta_i}^{\infty} \left(H\left(\frac{x-\mu_{X_i}}{\sigma_{X_i}}\right) - H\left(\frac{x-\mu_{Y_i}}{\sigma_{Y_i}}\right)\right) \mathrm{d}x} = \frac{\mathbb{E}\big[(\delta_i - Y_i)_+\big] - \mathbb{E}\big[(\delta_i - X_i)_+\big]}{\mathbb{E}\big[(Y_i - \delta_i)_+\big] - \mathbb{E}\big[(X_i - \delta_i)_+\big]}. \tag{EC.11}$$

Because

$$\mathbb{E}\big[(\delta_i - Y_i)_+\big] = \mathbb{E}\big[(\delta_i - \mu_{Y_i} - \sigma_{Y_i} Z)_+\big] = \sigma_{Y_i}\,\mathbb{E}\left[\left(\frac{\delta_i - \mu_{Y_i}}{\sigma_{Y_i}} - Z\right)_+\right], \tag{EC.12}$$

we have

$$\begin{aligned} \frac{\delta_i - \mu_{Y_i}}{\sigma_{Y_i}} &= \frac{1}{\sigma_{Y_i}}\left(\frac{\mu_{X_i}\sigma_{Y_i} - \mu_{Y_i}\sigma_{X_i}}{\sigma_{Y_i} - \sigma_{X_i}} - \mu_{Y_i}\right) \\ &= \frac{1}{\sigma_{Y_i}}\left(\frac{\mu_{X_i}\sigma_{Y_i} - \mu_{Y_i}\sigma_{X_i} - \mu_{Y_i}\sigma_{Y_i} + \mu_{Y_i}\sigma_{X_i}}{\sigma_{Y_i} - \sigma_{X_i}}\right) \\ &= \frac{1}{\sigma_{Y_i}}\left(\frac{\mu_{X_i}\sigma_{Y_i} - \mu_{Y_i}\sigma_{Y_i}}{\sigma_{Y_i} - \sigma_{X_i}}\right) \\ &= \frac{\mu_{X_i} - \mu_{Y_i}}{\sigma_{Y_i} - \sigma_{X_i}}. \end{aligned} \tag{EC.13}$$

This implies that

$$\mathbb{E}\big[(\delta_i - Y_i)_+\big] = \sigma_{Y_i}\,\mathbb{E}\left[\left(\frac{\mu_{X_i} - \mu_{Y_i}}{\sigma_{Y_i} - \sigma_{X_i}} - Z\right)_+\right]. \tag{EC.14}$$

Applying a similar argument to the other components in (EC.11), we obtain

$$\gamma_i^{\mathrm{M}} = \frac{\mathbb{E}\left[\left(\frac{\mu_{X_i}-\mu_{Y_i}}{\sigma_{Y_i}-\sigma_{X_i}} - Z\right)_+\right]}{\mathbb{E}\left[Z - \left(\frac{\mu_{X_i}-\mu_{Y_i}}{\sigma_{Y_i}-\sigma_{X_i}}\right)_+\right]}. \tag{EC.15}$$

A similar derivation holds for $\sigma_{Y_i} > \sigma_{X_i}$.

Proof of Theorem 3 The proof uses ideas that are similar to the ones in the proof of theorem 3 in Müller et al. (2021). Fix arbitrary $\boldsymbol{\delta}$, consider $u \in \mathcal{U}_{\boldsymbol{\gamma}}$, and let $b_i = \sup(u_i'(\boldsymbol{x}))$. Without loss of generality assume $u(\boldsymbol{\delta}) = 0$. By Lemma EC.2,

$$\sum_{i=1}^N b_i v_L(x_i - \delta_i; \gamma_i) \le u(\boldsymbol{x}) \le \sum_{i=1}^N b_i v_U(x_i - \delta_i; \gamma_i).$$

We need to show that, for some appropriate δ_i and γ_i, $\mathbb{E}[v_L(Y_i - \delta_i; \gamma_i)] \ge \mathbb{E}[v_U(X_i - \delta_i; \gamma_i)]$ for $i = 1, \ldots, N$. With the same tedious but straightforward calculation as in the proof of theorem 3 in Müller et al. (2021), we can establish that the smallest possible choice for γ_i is obtained by choosing

$$\delta_i = \frac{\mu_{X_i}\sigma_{Y_i} + \mu_{Y_i}\sigma_{X_i}}{\sigma_{X_i} + \sigma_{Y_i}}$$

and

$$\gamma_i = \frac{1}{1 + 2t\left(t + \sqrt{t^2+1}\right)}$$

for

$$t = \frac{\mu_{Y_i} - \mu_{X_i}}{\sigma_{X_i} + \sigma_{Y_i}}. \quad \square$$

Proof of Theorem 4 The proof is similar to the proof of Theorem 2. We get

$$\sum_{i=1}^N b_i v_L(x_i - \delta_i; \gamma_i) \le u(\boldsymbol{x}, \boldsymbol{z}) - u(\boldsymbol{\delta}, \boldsymbol{z}) \le \sum_{i=1}^N b_i v_U(x_i - \delta_i; \gamma_i),$$

and thus

$$\begin{aligned}
\mathbb{E}\left[u(\boldsymbol{Y}, \boldsymbol{Z})\right] &\ge \sum_{i=1}^N b_i \mathbb{E}\left[v_L(Y_i - \delta_i; \gamma_i)\right] + \mathbb{E}\left[u(\boldsymbol{\delta}, \boldsymbol{Z})\right] \\
&= \sum_{i=1}^N b_i \mathbb{E}\left[v_U(X_i - \delta_i; \gamma_i)\right] + \mathbb{E}\left[u(\boldsymbol{\delta}, \boldsymbol{Z})\right] \\
&\ge \mathbb{E}\left[u(\boldsymbol{X}, \boldsymbol{Z})\right]. \quad \square
\end{aligned}$$

Proofs of Section 4

Proof of Theorem 7 As in Lemma EC.2, we get for $\mathcal{U}_{\gamma,\boldsymbol{\beta}}$

$$\sum_{i=1}^{N} \beta_i v_L(x_i - \delta_i; \gamma) \le u(\boldsymbol{x}) - u(\boldsymbol{\delta}) \le \sum_{i=1}^{N} \beta_i v_U(x_i - \delta_i; \gamma).$$

Therefore, as in Theorem 2, a sufficient condition for $\mathbb{E}[u(\boldsymbol{Y})] \ge \mathbb{E}[u(\boldsymbol{X})]$ is

$$\sum_{i=1}^{N} \beta_i \mathbb{E}[v_L(Y_i - \delta_i; \gamma)] \ge \sum_{i=1}^{N} \beta_i \mathbb{E}[v_U(X_i - \delta_i; \gamma)],$$

which is equivalent to

$$\gamma \ge \frac{\sum_{i=1}^{N} \beta_i \left(\mathbb{E}\left[(X_i - \delta_i)_+\right] + \mathbb{E}\left[(\delta_i - Y_i)_+\right]\right)}{\sum_{i=1}^{N} \beta_i \left(\mathbb{E}\left[(\delta_i - X_i)_+\right] + \mathbb{E}\left[(Y_i - \delta_i)_+\right]\right)}. \quad \square$$

Proof of Theorem 8 Assume that (4.4) holds. Fix arbitrary $\boldsymbol{\delta}$, consider $u \in \mathcal{U}_{\gamma,\boldsymbol{\beta}}$, and without loss of generality set $u(\boldsymbol{\delta}) = 0$. As in Lemma EC.2, it follows that

$$\sum_{i=1}^{N} \beta_i v_L(x_i - \delta_i; \gamma) \le u(\boldsymbol{x}) \le \sum_{i=1}^{N} \beta_i v_U(x_i - \delta_i; \gamma).$$

It is sufficient to show that for some $\boldsymbol{\delta}$ we have

$$\sum_{i=1}^{N} \beta_i \mathbb{E}[v_L(Y_i - \delta_i; \gamma)] \ge \sum_{i=1}^{N} \beta_i \mathbb{E}[v_U(X_i - \delta_i; \gamma)]$$

for any $\boldsymbol{X}$ and $\boldsymbol{Y}$ such that (3.1) holds. As in the proof of theorem 3 in Müller et al. (2021), we get

$$\mathbb{E}[v_L(Y_i - \delta_i; \gamma)] \ge \gamma(\mu_{Y_i} - \delta_i) - (1-\gamma)\frac{1}{2}\left(\delta_i - \mu_{Y_i} + \sqrt{\sigma_{Y_i}^2 + (\mu_{Y_i} - \delta_i)^2}\right)$$

and

$$\mathbb{E}[v_U(X_i - \delta_i; \gamma)] \le \gamma(\mu_{X_i} - \delta_i) + (1-\gamma)\frac{1}{2}\left(\mu_{X_i} - \delta_i + \sqrt{\sigma_{X_i}^2 + (\mu_{X_i} - \delta_i)^2}\right).$$

Thus, we need to find some γ such that

$$\sum_{i=1}^{N} \beta_i \left(\gamma(\mu_{Y_i} - \delta_i) - (1-\gamma)\frac{1}{2}\left(\delta_i - \mu_{Y_i} + \sqrt{\sigma_{Y_i}^2 + (\mu_{Y_i} - \delta_i)^2}\right)\right)$$
$$\ge \sum_{i=1}^{N} \beta_i \left(\gamma(\mu_{X_i} - \delta_i) + (1-\gamma)\frac{1}{2}\left(\mu_{X_i} - \delta_i + \sqrt{\sigma_{X_i}^2 + (\mu_{X_i} - \delta_i)^2}\right)\right)$$

for some $\boldsymbol{\delta}$. Following Müller et al. (2021, Theorem 3), we choose

$$\delta_i = \frac{\mu_{X_i}\sigma_{Y_i} + \mu_{Y_i}\sigma_{X_i}}{\sigma_{Y_i} + \sigma_{X_i}},$$

so that

$$\frac{\mu_{Y_i}-\delta_i}{\sigma_{Y_i}} = t_i \quad \text{and} \quad \frac{\mu_{X_i}-\delta_i}{\sigma_{X_i}} = -t_i, \quad \text{where} \quad t_i = \frac{\mu_{Y_i}-\mu_{X_i}}{\sigma_{X_i}+\sigma_{Y_i}}.$$

Then the equation for γ becomes

$$\sum_{i=1}^{N} \beta_i \left(\gamma \sigma_{Y_i} t_i - (1-\gamma)\frac{1}{2}\left(-\sigma_{Y_i} t_i + \sigma_{Y_i}\sqrt{1+t_i^2}\right)\right) \\ = \sum_{i=1}^{N} \beta_i \left(\gamma\left(-\sigma_{X_i} t_i\right) + (1-\gamma)\frac{1}{2}\left(-\sigma_{X_i} t_i + \sigma_{X_i}\sqrt{1+t_i^2}\right)\right),$$

which is equivalent to

$$\gamma \sum_{i=1}^{N} \beta_i t_i \left(\sigma_{Y_i}+\sigma_{X_i}\right) = (1-\gamma)\frac{1}{2}\sum_{i=1}^{N} \beta_i \left(-\sigma_{X_i} t_i - \sigma_{Y_i} t_i + \left(\sigma_{X_i}+\sigma_{Y_i}\right)\sqrt{1+t_i^2}\right).$$

Define

$$\Delta = \sum_{i=1}^{N} \beta_i t_i \left(\sigma_{Y_i}+\sigma_{X_i}\right) = \sum_{i=1}^{N} \beta_i (\mu_{Y_i}-\mu_{X_i}).$$

Then

$$\left(\gamma + (1-\gamma)\frac{1}{2}\right)\Delta = (1-\gamma)\frac{1}{2}\sum_{i=1}^{N} \beta_i \left(\sigma_{X_i}+\sigma_{Y_i}\right)\sqrt{1+t_i^2},$$

or equivalently,

$$(1+\gamma)\,\Delta = (1-\gamma)\sum_{i=1}^{N} \beta_i \left(\sigma_{X_i}+\sigma_{Y_i}\right)\sqrt{1+t_i^2}.$$

This yields

$$\gamma = \frac{\sum_{i=1}^{N} \beta_i \left(\sigma_{X_i}+\sigma_{Y_i}\right)\sqrt{1+t_i^2} - \Delta}{\Delta + \sum_{i=1}^{N} \beta_i \left(\sigma_{X_i}+\sigma_{Y_i}\right)\sqrt{1+t_i^2}}.$$

Alternatively, we can express γ as

$$\gamma = \frac{\sum_{i=1}^{N} \beta_i \left(\sqrt{\left(\sigma_{X_i}+\sigma_{Y_i}\right)^2 + \left(\mu_{Y_i}-\mu_{X_i}\right)^2} - \left(\mu_{Y_i}-\mu_{X_i}\right)\right)}{\sum_{i=1}^{N} \beta_i \left(\sqrt{\left(\sigma_{X_i}+\sigma_{Y_i}\right)^2 + \left(\mu_{Y_i}-\mu_{X_i}\right)^2} + \left(\mu_{Y_i}-\mu_{X_i}\right)\right)}. \quad \square$$