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Weak Dirichlet processes with jumps

Elena Bandini* [1] and Francesco Russo†[2]

[1]LUISS Roma, Dipartimento di Economia e Finanza, via Romania 32, I-00197 Roma, Italy

[2]ENSTA ParisTech, Université Paris-Saclay, Unité de Mathématiques appliquées, 828, Boulevard des Maréchaux, F-91120 Palaiseau, France

Abstract

This paper develops systematically the stochastic calculus via regularization in the case of jump processes. In particular one continues the analysis of real-valued càdlàg weak Dirichlet processes with respect to a given filtration. Such a process is the sum of a local martingale and an adapted process A such that $[N, A] = 0$, for any continuous local martingale N. Given a function $u : [0, T] \times \mathbb{R} \to \mathbb{R}$, which is of class $C^{0,1}$ (or sometimes less), we provide a chain rule type expansion for $u(t, X_t)$ which stands in applications for a chain Itô type rule.

Key words: Weak Dirichlet processes; Calculus via regularizations; Random measure; Stochastic integrals for jump processes; Orthogonality.

MSC 2010: 60J75; 60G57; 60H05

1 Introduction

The present paper extends the stochastic calculus via regularizations to the case of jump processes, and carries on the investigations on the so called weak Dirichlet processes in the discontinuous case. Applications of that calculus appear in the companion paper [5], where we provide the identification of the solution of a forward backward stochastic differential equation driven by a random measure.

Stochastic calculus via regularization was essentially known in the case of continuous integrators X, see e.g. [24, 25], with a survey in [29]. In this case a fairly complete theory was developed, see for instance Itô formulae for processes with finite quadratic (and more general) variation, stochastic differential equations, Itô-Wentzell type formulae [15], and generalizations to the case of Banach space type integrators, see e.g. [9]. The notion of covariation $[X, Y]$ (resp. quadratic variation $[X, X]$) for two processes X, Y (resp. a process X) has been introduced in the framework of regularizations (see [27]) and of discretizations as well (see [16]). Even if there is no direct theorem relating the two approaches, those coincide in all the examples considered in the literature. If X is a finite quadratic variation continuous process, an Itô formula was proved for the expansion of $F(X_t)$, when $F \in C^2$, see [27]; this constitutes the counterpart of the related result for discretizations, see [16]. Moreover, for F of class C^1 and X a reversible continuous semimartingale, an Itô expansion has been established in [28].

*ebandini@luiss.it

†francesco.russo@ensta-paristech.fr

A natural extension of the notion of continuous semimartingale is the one of Dirichlet (possibly càdlàg) process (with respect to a given filtration $(\mathcal{F}_t)$): it was introduced by [16] and [6] in the discretizations framework. A Dirichlet process X is the sum of a local martingale M and an adapted process A with zero quadratic variation: A is the generalization of a bounded variation process. If $A_0 = 0$, then the decomposition $M + A$ constitutes a the analogous of the Doob-Meyer decomposition for a semimartingale X. However, requiring A to have zero quadratic variation imposes that A is continuous, see Lemma 2.10; since a bounded variation process with jumps has a non zero finite quadratic variation, the generalization of the semimartingale for jump processes is not completely represented by the notion of Dirichlet process. A natural generalization should then at least include the possibility that A is a bounded variation process with jumps.

The concept of $(\mathcal{F}_t)$-weak Dirichlet process indeed constitutes a true generalization of the concept of (càdlàg) semimartingale. Such a process is defined as the sum of a local martingale M and an adapted process A such that $[A, N] = 0$ for every continuous local martingale N. For a continuous process X, that notion was introduced in [13]. In [18] a chain rule was established for $F(t, X_t)$ when F belongs to class $C^{0,1}$ and X is a weak Dirichlet process with finite quadratic variation. Such a process is indeed again a weak Dirichlet process (with possibly no finite quadratic variation). Relevant applications to stochastic control were considered in [17]. Contrarily to the continuous case, the decomposition $X = M + A$ is generally not unique. X is denominated *special weak Dirichlet process* if it admits a decomposition of the same type but where A is predictable. This concept is compatible with the one introduced in [8] using the discretization language. The authors of [8] were the first to introduce a notion of weak Dirichlet process in the framework of jump processes. The decomposition of a special weak Dirichlet process is now unique, see Proposition 5.9, at least fixing $A_0 = 0$. We remark that a continuous weak Dirichlet process is a special weak Dirichlet one. If the concept of (non necessarily special) weak Dirichlet process extends the notion of semimartingale, the one of special weak Dirichlet process appears to be a generalization of the one of special semimartingale.

Towards calculus via regularization in the jump case only a few steps were done in [27], [26], and several other authors, see Chapter 15 of [10] and references therein. For instance no Itô type formulae have been established in the framework of regularization and in the discretization framework only very few chain rule results are available for $F(X)$, when $F(X)$ is not a semimartingale. In that direction two peculiar results are available: the expansion of $F(X_t)$ when X is a reversible semimartingale and F is of class C^1 with some Hölder conditions on the derivatives (see [14]), and a chain rule for $F(X_t)$ when X is a (càdlàg special) weak Dirichlet process with finite quadratic variation and finite energy and F is of class C^1_b, see Corollary 3.2 in [8]. The work in [14] has been continued by several authors, see e.g. [12] and references therein, expanding the remainder making use of local time type processes. A systematic study of calculus via regularization was missing and this paper fills out this gap.

Let us now go through the description of the main results of the paper. As we have already mentioned, our first basic objective consists in developing calculus via regularization in the case of finite quadratic variation càdlàg processes. To this end, we revisit the definitions given by [27] concerning forward integrals (resp. covariations). Those objects are introduced as u.c.p. (uniform convergence in probability) limit of the expressions of the type (2.1) (resp. (2.2)). That convergence ensures that the limiting objects are càdlàg, since the approximating expressions have the same property. For instance a càdlàg process X will be called *finite quadratic variation process* whenever the limit (which will be denoted by $[X, X]$) of

$$[X, X]^{ucp}_{\varepsilon}(t) := \int_{]0,\, t]} \frac{(X((s+\varepsilon)\wedge t) - X(s))^2}{\varepsilon}\, ds, \tag{1.1}$$

exists u.c.p. In [27], the authors introduced a slightly different approximation of $[X, X]$ when X

is continuous, namely

$$C_\varepsilon(X,X)(t) := \int_{]0,\,t]} \frac{(X(s+\varepsilon)-X(s))^2}{\varepsilon}\,ds. \tag{1.2}$$

When the u.c.p. limit of $C_\varepsilon(X,X)$ exists, it is automatically a continuous process, since the approximating processes are continuous. For this reason, when X is a jump process, the choice of approximation (1.2) would not be suitable, since its quadratic variation is expected to be a jump process. In that case, however, the u.c.p. convergence of (1.1) can be shown to be equivalent to the a.s. pointwise convergence (up to subsequences) of $C_\varepsilon(X,X)$, see Remark A.7. Both formulations will be used in the development of the calculus.

For a càdlàg finite quadratic variation process X, we establish, via regularization techniques, a quasi pathwise Itô formula for $C^{1,2}$ functions of X. This is the object of Theorem 3.1, whose proof is based on an accurate separation between the neighborhood of "big" and "small" jumps, where specific tools are used, see for instance the preliminary results Lemma 2.11 and Lemma 2.12. Another significant instrument is Lemma 2.16, which is of Dini type for càdlàg functions. Finally, from Theorem 3.1 easily follows an Itô formula under weaker regularity conditions on F, see Proposition 3.2. We remark that a similar formula was stated in [14], using a discretization definition of the covariation, when F is time-homogeneous.

The second main task of the paper consists in investigating weak Dirichlet jump processes. Beyond some basic properties, two significant results are Theorem 5.15 and Theorem 5.31. They both concern expansions of $F(t,X_t)$ where F is of class $C^{0,1}$ and X is a weak Dirichlet process of finite quadratic variation. Theorem 5.15 states that $F(t,X_t)$ will be again a weak Dirichlet process, however not necessarily of finite quadratic variation. Theorem 5.31 concerns the cases when X and $F(t,X_t)$ are special weak Dirichlet processes. A first significant step in this sense was done in [8], where X belongs to a bit different class of special weak Dirichlet jump processes (of finite energy) and F does not depend on time and has bounded derivative. They show that $F(X)$ is again a special weak Dirichlet process. In [8] the underlying process has finite energy, which requires a control of the expectation of the approximating sequences of the quadratic variation. On the other hand, our techniques do not require that type of control. Moreover, the integrability condition (5.40) that we ask on $F(t,X_t)$ in order to get the chain rule in Theorem 5.31 is automatically verified under the hypothesis on the first order derivative considered in [8], see Remark 5.30. In particular, this allows us to recover the result in [8] by means of our techniques, see Corollary 5.33. Finally, in some cases a chain rule may hold even when F is not necessarily differentiable in the second variable, if we know a priori some information on the process $F(t,X_t)$. A result in this direction is provided by Proposition 5.37, and does not require even that X is a weak Dirichlet process.

In the present paper we also introduce a subclass of weak Dirichlet processes, called *particular*, see Definition 5.19. Those processes inherit some of the semimartingales features: as in the semimartingale case, the particular weak Dirichlet processes admit an integral representation (see Proposition 5.24) and a (unique) *canonical decomposition* holds when $x\,\mathbb{1}_{\{|x|>1\}} * \mu \in \mathcal{A}_{\mathrm{loc}}$. Under that condition, those particular processes are indeed special weak Dirichlet processes, see Proposition 5.22 and 5.24.

The paper is organized as follows. In Section 2 we introduce the notations and we give some preliminary results to the development of the calculus via regularization with jumps. Section 3 is devoted to the proof of the $C^{1,2}$ Itô's formula for càdlàg processes with finite quadratic variation. In Section 4 we recall some basic results on the stochastic integration with respect to integer-valued random measures, and we use them to reexpress the $C^{1,2}$ Itô's formula of Section 3 in terms of the (compensated) random measure associated to the càdlàg process. Section 5 concerns the study of weak Dirichlet processes, and presents the expansion of $F(t,X_t)$ for X weak Dirichlet,

when F is of class $C^{0,1}$. Finally, we report in the Appendix A some additional comments and technical results on calculus via regularizations.

2 Preliminaries, calculus via regularization with jumps and related technicalities

Let $T > 0$ be a finite horizon. We will indicate by $C^{1,2}$ (resp. $C^{0,1}$) the space of all functions

$$u : [0, T] \times \mathbb{R} \to \mathbb{R}, \quad (t, x) \mapsto u(t, x),$$

that are continuous together their derivatives $\partial_t u$, $\partial_x u$, $\partial_{xx} u$ (resp. $\partial_x u$). $C^{1,2}$ is equipped with the topology of uniform convergence on each compact of u, $\partial_x u$, $\partial_{xx} u$, $\partial_t u$, $C^{0,1}$ is equipped with the same topology on each compact of u and $\partial_x u$. Given a topological space E, in the sequel $\mathcal{B}(E)$ will denote the Borel σ-field associated with E.

In the whole article, we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$, fulfilling the usual conditions. The symbols $\mathbb{D}^{ucp}$ and $\mathbb{L}^{ucp}$ will denote the space of all adapted càdlàg and càglàd processes endowed with the u.c.p. (uniform convergence in probability) topology. By convention, any càdlàg process defined on $[0, T]$ is extended on $\mathbb{R}_+$ by continuity.

Let f and g be two functions defined on $\mathbb{R}$, and set

$$I^{-ucp}(\varepsilon, t, f, dg) = \int_{]0,t]} f(s) \frac{g((s+\varepsilon)\wedge t) - g(s)}{\varepsilon} ds, \tag{2.1}$$

$$[f, g]^{ucp}_{\varepsilon}(t) = \int_{]0,t]} \frac{(f((s+\varepsilon)\wedge t) - f(s))(g((s+\varepsilon)\wedge t) - g(s))}{\varepsilon} ds. \tag{2.2}$$

Notice that the function $I^{-ucp}(\varepsilon, t, f, dg)$ is càdlàg and admits the decomposition

$$I^{-ucp}(\varepsilon, t, f, dg) = \int_0^{(t-\varepsilon)_+} f(s) \frac{g(s+\varepsilon) - g(s)}{\varepsilon} ds + \int_{(t-\varepsilon)_+}^t f(s) \frac{g(t) - g(s)}{\varepsilon} ds. \tag{2.3}$$

Definition 2.1. *Let X be a càdlàg process and Y be a process belonging to $L^1([0, T])$ a.s. Suppose the existence of a process $(I(t))_{t\in[0,T]}$ such that $(I^{-ucp}(\varepsilon, t, Y, dX))_{t\in[0,T]}$ converges u.c.p. to $(I(t))_{t\in[0,T]}$, namely*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{0\le s\le t} |I^{-ucp}(\varepsilon, t, Y, dX) - I(t)| > \alpha\right) = 0 \qquad \text{for every } \alpha > 0.$$

*Then we will set $\int_{]0,t]} Y_s \, d^- X_s := I(t)$. That process will be called **the forward integral of** Y **with respect to** X.*

Remark 2.2. In [27] a very similar notion of forward integral is considered:

$$I^{-RV}(\varepsilon, t, f, dg) = \int_{\mathbb{R}} f_{t]}(s) \frac{g_{t]}(s+\varepsilon) - g_{t]}(s)}{\varepsilon} ds,$$

with

$$f_{t]} = \begin{cases} f(0_+) & \text{if } x \le 0, \\ f(x) & \text{if } 0 < x \le t, \\ f(t_+) & \text{if } x > t. \end{cases}$$

The u.c.p. limit of $I^{-RV}(\varepsilon, t, f, dg)$, when it exists, coincides with that of $I^{-ucp}(\varepsilon, t, f, dg)$. As a matter of fact, the process $I^{-RV}(\varepsilon, t, f, dg)$ is càdlàg and can be rewritten as

$$I^{-RV}(\varepsilon, t, f, dg) \quad = \quad I^{-ucp}(\varepsilon, t, f, dg) - f(0_+)\,\frac{1}{\varepsilon}\int_0^{\varepsilon}[g(s) - g(0_+)]\,ds. \tag{2.4}$$

In particular

$$\sup_{t\in[0,\,T]}[I^{-ucp}(\varepsilon, t, f, dg) - I^{-RV}(\varepsilon, t, f, dg)] = f(0_+)\,\frac{1}{\varepsilon}\int_0^{\varepsilon}[g(s) - g(0_+)]\,ds,$$

and therefore

$$\limsup_{\varepsilon\to 0}\ \sup_{t\in[0,\,T]}[I^{-RV}(\varepsilon, t, f, dg) - I^{-ucp}(\varepsilon, t, f, dg)] = 0.$$

Proposition 2.3. *Let A be a càdlàg predictable process and Y be a process belonging to $L^1([0,\,T])$ a.s. Then the forward integral*

$$\int_{]0,\,\cdot]} Y_s\, d^- A_s,$$

when it exists, is a predictable process.

Proof. From decomposition (2.3) it follows that $I^{-ucp}(\varepsilon, \cdot, Y, dA) = C^{\varepsilon} + D^{\varepsilon}$, where C^{ε} is a continuous adapted process, therefore predictable, and

$$D^{\varepsilon}_t = A_t\,\frac{1}{\varepsilon}\int_{(t-\varepsilon)+}^{t} Y_s\, ds, \quad t \in [0,\,T].$$

D^{ε} is the product of the predictable process A and another continuous adapted process, so it is itself predictable. By definition, the u.c.p. stochastic integral, when it exists, is the u.c.p. limit of $I^{-ucp}(\varepsilon, \cdot, Y, dA)$: since the u.c.p. convergence preserves the predictability, the claim follows. □

Definition 2.4. *Let X, Y be two càdlàg processes. Suppose the existence of a process $(\Gamma(t))_{t\geq 0}$ such that $[X, Y]^{ucp}_{\varepsilon}(t)$ converges u.c.p. to $(\Gamma(t))_{t\geq 0}$, namely*

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{0\leq s\leq t}|[X, Y]^{ucp}_{\varepsilon}(t) - \Gamma(t)| > \alpha\right) = 0 \qquad \text{for every } \alpha > 0,$$

Then we will set $[X, Y]_t := \Gamma(t)$. That process will be called ***the covariation between X and Y****. In that case we say that* ***the covariation between X and Y exists****.*

Definition 2.5. *We say that a càdlàg process X* ***is a finite quadratic variation*** *if $[X, X]$ exists. In that case $[X, X]$ is called the quadratic variation of X.*

Definition 2.6. *We say that a pair of càdlàg processes (X, Y)* ***admits all its mutual brackets*** *if $[X, X]$, $[X, Y]$, $[Y, Y]$ exist.*

Remark 2.7. Let X, Y be two càdlàg processes.

1. By definition $[X, Y]$ is necessarily a càdlàg process.
2. $[X, X]$ is an increasing process.

$[X, X]^c$ denotes the continuous part of $[X, X]$.

Forward integrals and covariations generalize Itô integrals and the classical square brackets of semimartingales.

Proposition 2.8. *Let X, Y be two càdlàg semimartingales, M^1, M^2 two càdlàg local martingales, H, K two càdlàg adapted process. Then the following properties hold.*

(i) $[X, Y]$ *exists and it is the usual bracket.*

(ii) $\int_{]0,\cdot]} H\, d^- X$ *is the usual stochastic integral* $\int_{]0,\cdot]} H_{s-} dX_s$.

(iii) $\left[\int_0^\cdot H_{s-}\, dM_s^1,\, \int_0^\cdot K_{s-}\, dM_s^2\right]$ *is the usual bracket and it equals* $\int_0^\cdot H_{s-}\, K_{s-}\, d[M^1, M^2]_s$.

Proof. Items (i) and (ii) are consequence of Proposition 1.1 in [27] and Remark 2.2. Item (iii) follows from (i) and the corresponding properties for classical brackets of local martingales, see Theorem 29, chapter 2 of [23]. □

Remark 2.9. In the sequel we will make often use of the following assumption on a càdlàg process X:

$$\sum_{s\in]0,\,T]} |\Delta X_s|^2 < \infty, \text{ a.s.} \tag{2.5}$$

Condition (2.5) holds for instance in the case of processes X of finite quadratic variation.

Lemma 2.10. *Suppose that X is a càdlàg, finite quadratic variation process. Then*

(i) $\forall s \in [0,\, T],\ \ \Delta[X, X]_s = (\Delta X_s)^2$;

(ii) $[X, X]_s = [X, X]_s^c + \sum_{t\le s}(\Delta X_t)^2, \quad \forall s \in [0,\, T],$ a.s.
In particular, (2.5) *holds.*

Proof. (i) Since X is a finite quadratic variation process, $[X, X]_\varepsilon^{ucp}$ converges u.c.p. to $[X, X]$. This implies the existence of a sequence (ε_n) such that $[X, X]_{\varepsilon_n}^{ucp}$ converges uniformly a.s. to $[X, X]$. We fix a realization ω outside a suitable null set, which will be omitted in the sequel. Let $\gamma > 0$. There is ε_0 such that

$$\varepsilon_n < \varepsilon_0 \Rightarrow |[X, X]_s - [X, X]_{\varepsilon_n}^{ucp}(s)| \le \gamma, \quad \forall s \in [0,\, T]. \tag{2.6}$$

We fix $s \in]0,\, T]$. Let $\varepsilon_n < \varepsilon_0$. For every $\delta \in [0,\, s[$, we have

$$|[X, X]_s - [X, X]_{\varepsilon_n}^{ucp}(s-\delta)| \le \gamma. \tag{2.7}$$

Since $[X, X]$ is càdlàg, we need to show that the quantity

$$|[X, X]_s - [X, X]_{s-\delta} - (\Delta X_s)^2| \tag{2.8}$$

goes to zero, when $\delta \to 0$. For $\varepsilon := \varepsilon_n < \varepsilon_0$, (2.8) is smaller or equal than

$$\begin{aligned}
&2\gamma + |[X, X]_\varepsilon^{ucp}(s) - [X, X]_\varepsilon^{ucp}(s-\delta) - (\Delta X_s)^2| \\
&= 2\gamma + \left|\frac{1}{\varepsilon}\int_{s-\varepsilon-\delta}^{s} (X_{(t+\varepsilon)\wedge s} - X_t)^2\, dt - \frac{1}{\varepsilon}\int_{s-\varepsilon-\delta}^{s-\delta} (X_{s-\delta} - X_t)^2\, dt - (\Delta X_s)^2\right| \\
&\le 2\gamma + \frac{1}{\varepsilon}\int_{s-\varepsilon-\delta}^{s-\delta} (X_{s-\delta} - X_t)^2\, dt + |I(\varepsilon,\, \delta,\, s)|, \quad \forall \delta \in [0,\, s[,
\end{aligned}$$

where

$$I(\varepsilon,\, \delta,\, s) = \frac{1}{\varepsilon}\int_{s-\varepsilon-\delta}^{s-\varepsilon} (X_{t+\varepsilon} - X_t)^2\, dt + \frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} [(X_s - X_t)^2 - (\Delta X_s)^2]\, dt.$$

At this point, we have

$$|[X,X]_s - [X,X]_{s-\delta} - (\Delta X_s)^2| \le 2\gamma + \frac{1}{\varepsilon}\int_{s-\varepsilon-\delta}^{s-\delta} (X_{s-\delta} - X_t)^2\, dt + |I(\varepsilon,\, \delta,\, s)|, \quad \forall s \in [0,\, T].$$

We take the $\limsup_{\delta\to 0}$ on both sides to get, since X has a left limit at s,

$$|\Delta[X,X]_s - (\Delta X_s)^2| \le 2\gamma + \frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} (X_{s-} - X_t)^2\, dt + \frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} |(X_s - X_t)^2 - (\Delta X_s)^2|\, dt, \quad \text{for } \varepsilon := \varepsilon_n < \varepsilon_0.$$

We take the limit when $n \to \infty$ and we get $|\Delta[X,X]_s - (\Delta X_s)^2| \le 2\gamma$, and this concludes the proof of (i).

(ii) We still work fixing a priori a realization ω. Set $Y_s = [X,X]_s,\ \ s \in [0,\, T]$. Since Y is an increasing càdlàg process, it can be decomposed as $Y_s = Y_s^c + \sum_{t\le s} \Delta Y_t$, for all $s \in [0,\, T]$, a.s., and the result follows from point (i). In particular, setting $s = T$, we get

$$\sum_{s\le T} (\Delta X_s)^2 \le \sum_{s\le T} (\Delta X_s)^2 + [X,X]_T^c = [X,X]_T < \infty, \text{ a.s.}$$

□

We now state and prove some fundamental preliminary results, that we will deeply use in the sequel.

Lemma 2.11. *Let Y_t be a càdlàg function with values in $\mathbb{R}^n$. Let $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an equicontinuous function on each compact, such that $\phi(y,y) = 0$ for every $y \in \mathbb{R}^n$. Let $0 \le t_1 \le t_2 \le ... \le t_N \le T$. We have*

$$\sum_{i=1}^{N} \frac{1}{\varepsilon}\int_{t_i-\varepsilon}^{t_i} \mathbb{1}_{]0,\, s]}(t)\, \phi(Y_{(t+\varepsilon)\wedge s}, Y_t)\, dt \overset{\varepsilon\to 0}{\longrightarrow} \sum_{i=1}^{N} \mathbb{1}_{]0,\, s]}(t_i)\, \phi(Y_{t_i}, Y_{t_i-}), \tag{2.9}$$

uniformly in $s \in [0,\, T]$.

Proof. Without restriction of generality, we consider the case $n = 1$. Let us fix $\gamma > 0$. Taking into account that ϕ is equicontinuous on compacts, by definition of left and right limits, there exists $\delta > 0$ such that, for every $i \in \{1, ..., N\}$,

$$\ell < t_i,\ u > t_i,\ |\ell - t_i| \le \delta,\ |u - t_i| \le \delta \Rightarrow |\phi(Y_u, Y_\ell) - \phi(Y_{t_i}, Y_{t_i-})| < \gamma, \tag{2.10}$$

$$\ell_2 < \ell_1 < t_i,\ |\ell_1 - t_i| \le \delta,\ |\ell_2 - t_i| \le \delta \Rightarrow |\phi(Y_{\ell_1}, Y_{\ell_2})| = |\phi(Y_{\ell_1}, Y_{\ell_2}) - \phi(Y_{t_i-}, Y_{t_i-})| < \gamma. \tag{2.11}$$

Since the sum in (2.9) is finite, it is enough to show the uniform convergence in s of the integrals on $]t_i - \varepsilon,\, t_i]$, for a fixed $t_i \in [0,\, T]$, namely that

$$I(\varepsilon, s) := \frac{1}{\varepsilon}\int_{t_i-\varepsilon}^{t_i} \mathbb{1}_{]0,\, s]}(t)\, \phi(Y_{(t+\varepsilon)\wedge s}, Y_t)\, dt - \mathbb{1}_{]0,\, s]}(t_i)\, \phi(Y_{t_i}, Y_{t_i-}) \tag{2.12}$$

converges to zero uniformly in s, when ε goes to zero. Let thus fix $t_i \in [0,\, T]$, and choose $\varepsilon < \delta$. We distinguish the cases (i), (ii), (iii), (iv) concerning the position of s with respect to t_i.

(i) $s < t_i - \varepsilon$. (2.12) vanishes.

(ii) $s \in [t_i - \varepsilon,\, t_i[$. By (2.11) we get

$$|I(\varepsilon, s)| \le \frac{1}{\varepsilon}\int_{t_i-\varepsilon}^{t_i} |\phi(Y_s, Y_t)|\, dt \le \gamma.$$

(iii) $s \in [t_i,\, t_i + \varepsilon[$. By (2.10) we get

$$|I(\varepsilon, s)| \leq \frac{1}{\varepsilon} \int_{t_i - \varepsilon}^{t_i} |\phi(Y_{(t+\varepsilon)\wedge s}, Y_t) - \phi(Y_{t_i}, Y_{t_i -})|\, dt \leq \gamma.$$

(iv) $s \geq t_i + \varepsilon$. By (2.10) we get

$$|I(\varepsilon, s)| \leq \frac{1}{\varepsilon} \int_{t_i - \varepsilon}^{t_i} |\phi(Y_{t+\varepsilon}, Y_t) - \phi(Y_{t_i}, Y_{t_i -})|\, dt \leq \gamma.$$

Collecting all the cases above, we see that $\limsup_{\varepsilon \to 0} \sup_{s \in [0,\, T]} |I(\varepsilon, s)| \leq \gamma$, and letting γ go to zero we get the uniform convergence. □

Lemma 2.12. *Let X be a càdlàg (càglàd) real process. Let $\gamma > 0$, t_0, $t_1 \in \mathbb{R}$ and $I = [t_0,\, t_1]$ be a subinterval of $[0,\, T]$ such that*

$$|\Delta X_t|^2 \leq \gamma^2, \quad \forall t \in I. \tag{2.13}$$

Then there is $\varepsilon_0 > 0$ such that

$$\sup_{\substack{a,\, t \in I \\ |a-t| \leq \varepsilon_0}} |X_a - X_t| \leq 3\gamma.$$

Proof. We only treat the càdlàg case, the càglàd one is a consequence of an obvious time reversal argument.

Also in this proof a realization ω will be fixed, but omitted. According to Lemma 1, Chapter 3, in [7], applied to $[t_0,\, t_1]$ replacing $[0,\, 1]$, there exist points

$$t_0 = s_0 < s_1 < ... < s_{l-1} < s_l = t_1$$

such that for every $j \in \{1, ..., l\}$

$$\sup_{d,\, u \in [s_{j-1},\, s_j[} |X_d - X_u| < \gamma. \tag{2.14}$$

Since X is càdlàg, we can choose ε_0 such that, $\forall j \in \{0,\, ...,\, l-1\}$,

$$|d - s_j| \leq \varepsilon_0 \quad \Rightarrow \quad |X_d - X_{s_j -}| \leq \gamma, \tag{2.15}$$

$$|u - s_j| \leq \varepsilon_0 \quad \Rightarrow \quad |X_u - X_{s_j}| \leq \gamma. \tag{2.16}$$

Let $t \in [s_{j-1},\, s_j[$ for some j and a such that $|t - a| \leq \varepsilon$ for $\varepsilon < \varepsilon_0$. Without restriction of generality we can take $t < a$. There are two cases.

(i) $a,\, t \in [s_{j-1},\, s_j[$. In this case, (2.14) gives $|X_a - X_t| < \gamma$.

(ii) $s_{j-1} \leq t < s_j \leq a$. Then,

$$|X_a - X_t| \leq |X_a - X_{s_j}| + |X_{s_j} - X_{s_j -}| + |X_{s_j -} - X_t| \leq 3\gamma,$$

where the first absolute value is bounded by (2.16), the second by (2.13) and the third by (2.15).

□

Remark 2.13. Let $I = [t_0,\, t_1] \subset [0,\, T]$, let $\varepsilon > 0$. Let $t \in]t_0,\, t_1 - \varepsilon]$ and $s > t$. We will apply Lemma 2.12 to the couple (a, t), where $a = (t + \varepsilon) \wedge s$. Indeed $a \in I$ because $a \leq t + \varepsilon \leq t_1$.

Proposition 2.14. *Let (Z_t) be a càdlàg process, (V_t) be a bounded variation process. Then $[Z, V]_s$ exists and equals*

$$\sum_{t \le s} \Delta Z_t \, \Delta V_t, \quad \forall s \in [0, T].$$

In particular, V is a finite quadratic variation process.

Proof. We need to prove the u.c.p. convergence to zero of

$$\frac{1}{\varepsilon} \int_{]0,\, s]} (Z_{(t+\varepsilon)\wedge s} - Z_t)(V_{(t+\varepsilon)\wedge s} - V_t) \, dt - \sum_{t \le s} \Delta Z_t \, \Delta V_t. \tag{2.17}$$

As usual the realization $\omega \in \Omega$ will be fixed, but often omitted. Let (t_i) be the enumeration of all the jumps of $Z(\omega)$ in $[0, T]$. We have

$$\lim_{i \to \infty} |\Delta Z_{t_i}(\omega)| = 0.$$

Indeed, if it were not the case, it would exists $a > 0$ and a subsequence (t_{i_l}) of (t_i) such that $|\Delta Z_{t_{i_l}}| \ge a$. This is not possible since a càdlàg function admits at most a finite number of jumps exceeding any $a > 0$, see considerations below Lemma 1, Chapter 2 of [7].

At this point, let $\gamma > 0$ and $N = N(\gamma)$ such that

$$n \ge N, \quad |\Delta Z_{t_n}| \le \gamma. \tag{2.18}$$

We introduce

$$A(\varepsilon, N) = \bigcup_{i=1}^{N}]t_i - \varepsilon, t_i], \tag{2.19}$$

$$B(\varepsilon, N) = \bigcup_{i=1}^{N}]t_{i-1}, t_i - \varepsilon] = [0, T] \setminus A(\varepsilon, N), \tag{2.20}$$

and we decompose (2.17) into

$$I_A(\varepsilon, N, s) + I_{B1}(\varepsilon, N, s) + I_{B2}(\varepsilon, N, s) \tag{2.21}$$

where

$$\begin{aligned}
I_A(\varepsilon, N, s) &= \frac{1}{\varepsilon} \int_{]0,\, s] \cap A(\varepsilon, N)} (Z_{(t+\varepsilon)\wedge s} - Z_t)(V_{(t+\varepsilon)\wedge s} - V_t) \, dt - \sum_{i=1}^{N} \mathbb{1}_{]0,\, s[}(t_i) \, \Delta Z_{t_i} \, \Delta V_{t_i}, \\
I_{B1}(\varepsilon, N, s) &= \frac{1}{\varepsilon} \int_{]0,\, s] \cap B(\varepsilon, N)} (Z_{(t+\varepsilon)\wedge s} - Z_t)(V_{(t+\varepsilon)\wedge s} - V_t) \, dt, \\
I_{B2}(N, s) &= - \sum_{i=N+1}^{\infty} \mathbb{1}_{]0,\, s[}(t_i) \, \Delta Z_{t_i} \, \Delta V_{t_i}.
\end{aligned}$$

Applying Lemma 2.11 to $Y = (Y^1, Y^2) = (Z, V)$ and $\phi(y_1, y_2) = (y_1^1 - y_2^1)(y_1^2 - y_2^2)$ we get

$$I_A(\varepsilon, N, s) \underset{\varepsilon \to 0}{\to} 0, \text{ uniformly in } s.$$

On the other hand, for $t \in]t_{i-1},\, t_i - \varepsilon[$ and $s > t$, by Remark 2.13 we know that $(t+\varepsilon)\wedge s \in [t_{i-1},\, t_i]$. Therefore Lemma 2.12 with $X = Z$, applied successively to the intervals $I = [t_{i-1},\, t_i]$ implies that

$$|I_{B1}(\varepsilon, N, s)| = \frac{1}{\varepsilon} \int_{]0,\, s] \cap B(\varepsilon, N)} |Z_{(t+\varepsilon)\wedge s} - Z_t| |V_{(t+\varepsilon)\wedge s} - V_t| \, dt$$

$$
\begin{aligned}
&\leq 3\,\gamma\,\frac{1}{\varepsilon}\int_{]0,\,s]\cap B(\varepsilon,N)} |V_{(t+\varepsilon)\wedge s} - V_t|\,dt \\
&\leq 3\,\gamma\int_{]0,\,s]} |V_{(t+\varepsilon)\wedge s} - V_t|\,\frac{dt}{\varepsilon} \\
&= 3\,\gamma\int_{]0,\,s]} \frac{dt}{\varepsilon}\int_{]t,\,(t+\varepsilon)\wedge s]} d\|V\|_r \\
&= 3\,\gamma\int_{]0,\,s]} d\|V\|_r \int_{[(r-\varepsilon)^+,\,r[} \frac{dt}{\varepsilon} \\
&\leq 3\,\gamma\,||V||_T,
\end{aligned}
$$

where $r \mapsto \|V\|_r$ denotes the total variation function of V. Finally, concerning $I_{B2}(N,s)$, by (2.18) we have

$$
|I_{B2}(N,s)| \leq \gamma \sum_{i=N+1}^{\infty} \mathbb{1}_{]0,\,s[}(t_i)\ |\Delta V_{t_i}| \leq \gamma\,||V||_T.
$$

Therefore, collecting the previous estimations we get

$$
\limsup_{\varepsilon\to 0}\ \sup_{s\in[0,\,T]} |I_A(\varepsilon,N,s) + I_{B1}(\varepsilon,N,s) + I_{B2}(N,s)| \leq 4\,\gamma\,||V||_T,
$$

and we conclude by the arbitrariness of $\gamma > 0$. □

Proposition 2.15. *Let X be a càdlàg process with finite quadratic variation and A be a process such that $[A,A]=0$. Then $[X,A]$ exists and equals zero.*

Proof. By Remark A.7, $[X,X]$ and $[A,A]$ exist in the pathwise sense. By Lemma 2.10, A is continuous. Consequently, by Proposition 1-6) of [29], $[X,A]=0$ in the pathwise sense. Finally, by Remark A.7, (X,A) admits all its mutual brackets and $[X,A]=0$. □

Finally we give a generalization of Dini type lemma in the càdlàg case.

Lemma 2.16. *Let $(G_n,\, n\in\mathbb{N})$ be a sequence of continuous increasing functions, let G (resp. F) from $[0,\,T]$ to $\mathbb{R}$ be a càdlàg (resp. continuous) function. We set $F_n = G_n + G$ and suppose that $F_n \to F$ pointwise. Then*

$$
\limsup_{n\to\infty}\ \sup_{s\in[0,\,T]} |F_n(s) - F(s)| \leq 2\ \sup_{s\in[0,\,T]} |G(s)|.
$$

Proof. Let us fix $m\in\mathbb{N}^*$. Let $0 = t_0 < t_1 < \ldots < t_m = T$ such that $t_i = \frac{i}{m}$, $i = 0,\ldots,m$. If $s\in[t_i,\,t_{i+1}]$, $0\leq i\leq m-1$, we have

$$
F_n(s) - F(s) \leq F_n(t_{i+1}) - F(s) + G(s) - G(t_{i+1}). \tag{2.22}
$$

Now

$$
\begin{aligned}
F_n(t_{i+1}) - F(s) &\leq F_n(t_{i+1}) - F(t_{i+1}) + F(t_{i+1}) - F(s) \\
&\leq \delta\left(F,\,\frac{1}{m}\right) + F_n(t_{i+1}) - F(t_{i+1}),
\end{aligned} \tag{2.23}
$$

where $\delta(F,\cdot)$ denotes the modulus of continuity of F. From (2.22) and (2.23) it follows

$$
F_n(s) - F(s) \leq F_n(t_{i+1}) - F(t_{i+1}) + G(s) - G(t_{i+1}) + \delta\left(F,\,\frac{1}{m}\right)
$$

$$\leq 2||G||_\infty + \delta\left(F,\, \frac{1}{m}\right) + |F_n(t_{i+1}) - F(t_{i+1})|, \tag{2.24}$$

where $||G||_\infty = \sup_{s\in[0,\,T]} |G(s)|$. Similarly,

$$F(s) - F_n(s) \geq -2||G||_\infty - \delta\left(F,\, \frac{1}{m}\right) - |F_n(t_i) - F(t_i)|. \tag{2.25}$$

So, collecting (2.24) and (2.25) we have, $\forall s \in [t_i,\, t_{i+1}]$,

$$|F_n(s) - F(s)| \leq 2||G||_\infty + \delta\left(F,\, \frac{1}{m}\right) + |F_n(t_i) - F(t_i)| + |F_n(t_{i+1}) - F(t_{i+1})|.$$

Consequently,

$$\sup_{s\in[0,\,T]} |F_n(s) - F(s)| \leq 2||G||_\infty + \delta\left(F,\, \frac{1}{m}\right) + \sum_{i=1}^{m} |F_n(t_i) - F(t_i)|. \tag{2.26}$$

Recalling that $F_n \to F$ pointwise, taking the lim sup in (2.26) we get

$$\limsup_{n\to\infty} \sup_{s\in[0,\,T]} |F_n(s) - F(s)| \leq 2||G||_\infty + \delta\left(F,\, \frac{1}{m}\right).$$

Since F is uniformly continuous, we let m go to infinity, so that the result follows. □

3 Itô's formula for $C^{1,2}$ functions: the basic formulae

We start with the Itô formula for finite quadratic variation processes in the sense of calculus via regularizations.

Theorem 3.1. *Let X be a finite quadratic variation càdlàg process and $F : [0,\, T] \times \mathbb{R} \to \mathbb{R}$ a function of class $C^{1,2}$. Then we have*

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t \partial_s F(s, X_s)\, ds + \int_0^t \partial_x F(s, X_s)\, d^- X_s + \frac{1}{2}\int_0^t \partial^2_{xx} F(s, X_{s-})\, d[X, X]^c_s \\ &+ \sum_{s\leq t} [F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-})\, \Delta X_s]. \end{aligned} \tag{3.1}$$

Proof. Since X is a finite quadratic variation process, by Lemma A.5, taking into account Definition A.2 and Corollary A.4-2), for a given càdlàg process (g_t) we have

$$\int_0^s g_t\, (X_{(t+\varepsilon)\wedge s} - X_t)^2\, \frac{dt}{\varepsilon} \overset{\varepsilon\to 0}{\longrightarrow} \int_0^s g_{t-}\, d[X, X]_t \quad \text{u.c.p.}$$

Setting $g_t = 1$ and $g_t = \frac{\partial^2_{xx} F(t, X_t)}{2}$, there exists a positive sequence ε_n such that

$$\lim_{n\to\infty} \int_0^s (X_{(t+\varepsilon_n)\wedge s} - X_t)^2\, \frac{dt}{\varepsilon_n} = [X, X]_s, \tag{3.2}$$

$$\lim_{n\to\infty} \int_0^s \frac{\partial^2_{xx} F(t,\, X_t)}{2}\, (X_{(t+\varepsilon_n)\wedge s} - X_t)^2\, \frac{dt}{\varepsilon_n} = \int_{]0,\, s]} \frac{\partial^2_{xx} F(t,\, X_{t-})}{2}\, d[X, X]_t, \tag{3.3}$$

uniformly in s, a.s. Let then $\mathcal{N}$ be a null set such that (3.2), (3.3) hold for every $\omega \notin \mathcal{N}$.

In the sequel we fix $\gamma > 0$, $\varepsilon > 0$, and $\omega \notin \mathcal{N}$, and we enumerate the jumps of $X(\omega)$ on $[0,\, T]$ by $(t_i)_{i\geq 0}$. Let $N = N(\omega)$ such that

$$\sum_{i=N+1}^{\infty} |\Delta X_{t_i}(\omega)|^2 \leq \gamma^2. \tag{3.4}$$

From now on the dependence on ω will be often neglected. The quantity

$$J_0(\varepsilon,\, s) = \frac{1}{\varepsilon}\int_0^s [F((t+\varepsilon)\wedge s,\, X_{(t+\varepsilon)\wedge s}) - F(t,\, X_t)]\, dt, \quad s \in [0,\, T], \tag{3.5}$$

converges to $F(s,\, X_s) - F(0,\, X_0)$ uniformly in s. As a matter of fact, setting $Y_t = (t,\, X_t)$, we have

$$\begin{aligned} J_0(\varepsilon,\, s) &= \frac{1}{\varepsilon}\int_{[0,\, s[} F(Y_{(t+\varepsilon)\wedge s})\, dt - \frac{1}{\varepsilon}\int_{[0,\, s[} F(Y_t)\, dt \\ &= \frac{1}{\varepsilon}\int_{[\varepsilon,\, s+\varepsilon[} F(Y_{t\wedge s})\, dt - \frac{1}{\varepsilon}\int_{[0,\, s[} F(Y_t)\, dt \\ &= \frac{1}{\varepsilon}\int_{[s,\, s+\varepsilon[} F(Y_{t\wedge s})\, dt - \frac{1}{\varepsilon}\int_{[0,\, \varepsilon[} F(Y_t)\, dt \\ &= F(Y_s) - \frac{1}{\varepsilon}\int_{[0,\, \varepsilon[} F(Y_t)\, dt \\ &\underset{\varepsilon\to 0}{\longrightarrow} F(Y_s) - F(Y_0), \text{ uniformly in } s. \end{aligned} \tag{3.6}$$

We define $A(\varepsilon, N)$ and $B(\varepsilon, N)$ as in (2.19)-(2.20). $J_0(\varepsilon,\, s)$ can be also rewritten as

$$J_0(\varepsilon,\, s) = J_A(\varepsilon,\, N,\, s) + J_B(\varepsilon,\, N,\, s), \tag{3.7}$$

where

$$J_A(\varepsilon,\, N,\, s) = \frac{1}{\varepsilon}\int_0^s [F((t+\varepsilon)\wedge s,\, X_{(t+\varepsilon)\wedge s}) - F(t,\, X_t)]\, \mathbb{1}_{A(\varepsilon,N)}(t)\, dt, \tag{3.8}$$

$$J_B(\varepsilon,\, N,\, s) = \frac{1}{\varepsilon}\int_0^s [F((t+\varepsilon)\wedge s,\, X_{(t+\varepsilon)\wedge s}) - F(t,\, X_t)]\, \mathbb{1}_{B(\varepsilon,N)}(t)\, dt. \tag{3.9}$$

Applying Lemma 2.11 with $n = 2$ to $Y = (Y^1, Y^2) = (t, X)$ and $\phi(y_1, y_2) = F(y_1^1, y_1^2) - F(y_2^1, y_2^2)$, we have

$$\begin{aligned} J_A(\varepsilon,\, N,\, s) &= \sum_{i=1}^{N} \frac{1}{\varepsilon}\int_{t_i-\varepsilon}^{t_i} [F((t+\varepsilon)\wedge s,\, X_{(t+\varepsilon)\wedge s}) - F(t,\, X_t)]\, dt \\ &\overset{\varepsilon\to 0}{\longrightarrow} \sum_{i=1}^{N} \mathbb{1}_{]0,\, s]}(t_i)\, [F(t_i,\, X_{t_i}) - F(t_i,\, X_{t_i-})], \text{ uniformly in } s. \end{aligned} \tag{3.10}$$

Concerning $J_B(\varepsilon,\, N,\, s)$, it can be decomposed into the sum of the two terms

$$\begin{aligned} J_{B1}(\varepsilon,\, N,\, s) &= \frac{1}{\varepsilon}\int_0^s [F((t+\varepsilon)\wedge s,\, X_{(t+\varepsilon)\wedge s}) - F(t,\, X_{(t+\varepsilon)\wedge s})]\, \mathbb{1}_{B(\varepsilon,N)}(t)\, dt, \\ J_{B2}(\varepsilon,\, N,\, s) &= \frac{1}{\varepsilon}\int_0^s [F(t,\, X_{(t+\varepsilon)\wedge s}) - F(t,\, X_t)]\, \mathbb{1}_{B(\varepsilon,N)}(t)\, dt. \end{aligned}$$

By finite increments theorem in time we get

$$J_{B1}(\varepsilon,\, N,\, s) = J_{B10}(\varepsilon,\, s) + J_{B11}(\varepsilon,\, N,\, s) + J_{B12}(\varepsilon,\, N,\, s) + J_{B13}(\varepsilon,\, N,\, s), \tag{3.11}$$

where

$$\begin{aligned}
J_{B10}(\varepsilon,\, s) &= \int_0^s \partial_t F(t,\, X_t)\, \frac{(t+\varepsilon)\wedge s - t}{\varepsilon}\, dt,\\
J_{B11}(\varepsilon,\, N,\, s) &= -\sum_{i=1}^N \int_{t_i-\varepsilon}^{t_i} \partial_t F(t,\, X_t)\, \frac{(t+\varepsilon)\wedge s - t}{\varepsilon}\, dt,\\
J_{B12}(\varepsilon,\, N,\, s) &= \int_0^s R_1(\varepsilon,\, t,\, s)\, \mathbb{1}_{B(\varepsilon,N)}(t)\, \frac{(t+\varepsilon)\wedge s - t}{\varepsilon}\, dt,\\
J_{B13}(\varepsilon,\, N,\, s) &= \int_0^s R_2(\varepsilon,\, t,\, s)\, \mathbb{1}_{B(\varepsilon,N)}(t)\, \frac{(t+\varepsilon)\wedge s - t}{\varepsilon}\, dt,
\end{aligned}$$

and

$$R_1(\varepsilon,\, t,\, s) = \int_0^1 [\partial_t F(t + a\,((t+\varepsilon)\wedge s - t),\, X_{(t+\varepsilon)\wedge s}) - \partial_t F(t,\, X_{(t+\varepsilon)\wedge s})]\, da, \tag{3.12}$$

$$R_2(\varepsilon,\, t,\, s) = \partial_t F(t,\, X_{(t+\varepsilon)\wedge s}) - \partial_t F(t,\, X_t). \tag{3.13}$$

A Taylor expansion in space up to second order gives

$$J_{B2}(\varepsilon,\, N,\, s) = J_{B20}(\varepsilon,\, s) + J_{B21}(\varepsilon,\, s) + J_{B22}(\varepsilon,\, N,\, s) + J_{B23}(\varepsilon,\, N,\, s), \tag{3.14}$$

where

$$J_{B20}(\varepsilon,\, s) = \frac{1}{\varepsilon}\int_0^s \partial_x F(t,\, X_t)\, (X_{(t+\varepsilon)\wedge s} - X_t)\, dt, \tag{3.15}$$

$$\begin{aligned}
J_{B21}(\varepsilon,\, s) &= \frac{1}{\varepsilon}\int_0^s \frac{\partial^2_{xx} F(t,\, X_t)}{2}\, (X_{(t+\varepsilon)\wedge s} - X_t)^2\, dt,\\
J_{B22}(\varepsilon,\, N,\, s) &= -\frac{1}{\varepsilon}\sum_{i=1}^N \int_{t_i-\varepsilon}^{t_i} \left[\partial_x F(t,\, X_t)\, (X_{(t+\varepsilon)\wedge s} - X_t) + \frac{\partial^2_{xx} F(t,\, X_t)}{2}\, (X_{(t+\varepsilon)\wedge s} - X_t)^2\right] dt,\\
J_{B23}(\varepsilon,\, N,\, s) &= \int_0^s R_3(\varepsilon,\, t,\, s)\, \mathbb{1}_{B(\varepsilon,N)}(t)\, \frac{(X_{(t+\varepsilon)\wedge s} - X_t)^2}{\varepsilon}\, dt,
\end{aligned}$$

and

$$R_3(\varepsilon,\, t,\, s) = \int_0^1 [\partial^2_{xx} F(t,\, X_t + a(X_{(t+\varepsilon)\wedge s} - X_t)) - \partial^2_{xx} F(t,\, X_t)]\, da. \tag{3.16}$$

Let us consider the term $J_{B22}(\varepsilon,\, N,\, s)$. Applying Lemma 2.11 with $n = 2$ to $Y = (Y^1, Y^2) = (t, X)$ and $\phi(y_1, y_2) = \partial_x F(y_2^1, y_2^2)(y_1^2 - y_2^2) + \partial^2_{xx} F(y_2^1, y_2^2)(y_1^2 - y_2^2)^2$, we get

$$\lim_{\varepsilon\to 0} J_{B22}(\varepsilon,\, N,\, s) = -\sum_{i=1}^N \mathbb{1}_{]0,\, s]}(t_i)\, \left[\partial_x F(t_i,\, X_{t_i-})\, (X_{t_i} - X_{t_i-}) + \frac{\partial^2_{xx} F(t_i,\, X_{t_i-})}{2}(X_{t_i} - X_{t_i-})^2\right] \tag{3.17}$$

uniformly in s. Moreover, the term $J_{B10}(\varepsilon,\, N,\, s)$ can be decomposed into

$$J_{B10}(\varepsilon,\, s) = \int_0^s \partial_t F(t,\, X_t)\, dt + J_{B10'}(\varepsilon,\, s) + J_{B10''}(\varepsilon,\, s), \tag{3.18}$$

with

$$J_{B10'}(\varepsilon,\, s) = \int_{s-\varepsilon}^{s} \partial_t F(t, X_t)\, \frac{s-t}{\varepsilon}\, dt, \tag{3.19}$$

$$J_{B10''}(\varepsilon,\, s) = -\int_{s-\varepsilon}^{s} \partial_t F(t, X_t)\, dt. \tag{3.20}$$

At this point we remark that identity (3.7) can be rewritten as

$$\begin{aligned} J_0(\varepsilon, s) = J_A(\varepsilon, N, s) + \int_0^s \partial_t F(t,\, X_t)\, dt + J_{B10'}(\varepsilon, s) + J_{B10''}(\varepsilon, s) + J_{B11}(\varepsilon, N, s) + J_{B12}(\varepsilon, N, s) \\ + J_{B13}(\varepsilon, N, s) + J_{B20}(\varepsilon, s) + J_{B21}(\varepsilon, s) + J_{B22}(\varepsilon, N, s) + J_{B23}(\varepsilon, N, s). \end{aligned} \tag{3.21}$$

Passing to the limit in (3.21) on both the left-hand and right-hand sides, uniformly in s, as ε goes to zero, taking into account convergences (3.6), (3.10), (3.17). we get

$$\begin{aligned} F(s,\, X_s) - F(0,\, X_0) = \int_0^s \partial_t F(t,\, X_t)\, dt + \sum_{i=1}^{N} \mathbb{1}_{]0,\, s]}(t_i)\, \left[F(t_i,\, X_{t_i}) - F(t_i,\, X_{t_i-})\right] \\ - \sum_{i=1}^{N} \mathbb{1}_{]0,\, s]}(t_i)\, \left[\partial_x F(t_i,\, X_{t_i-})\, (X_{t_i} - X_{t_i-}) - \frac{\partial^2_{xx} F(t_i,\, X_{t_i-})}{2} (X_{t_i} - X_{t_i-})^2\right] \\ + \lim_{\varepsilon \to 0} \left(J_{B20}(\varepsilon,\, N,\, s) + J_{B21}(\varepsilon,\, s) + L(\varepsilon, N, s)\right) \end{aligned} \tag{3.22}$$

where the previous limit is intended uniformly in s, and we have set

$$\begin{aligned} L(\varepsilon, N, s) := J_{B10'}(\varepsilon,\, s) + J_{B10''}(\varepsilon,\, s) + J_{B11}(\varepsilon,\, N,\, s) + J_{B12}(\varepsilon,\, N,\, s) \\ + J_{B13}(\varepsilon,\, N,\, s) + J_{B23}(\varepsilon,\, N,\, s). \end{aligned}$$

We evaluate previous limit uniformly in s, for every $\omega \notin \mathcal{N}$. Without restriction of generality it is enough to show the uniform convergence in s for the subsequence ε_n introduced in (3.2)-(3.3), when $n \to \infty$.

According to (3.3), we get

$$\lim_{n\to\infty} J_{B21}(\varepsilon_n,\, s) = \int_{]0,\, s]} \frac{\partial^2_{xx} F(t,\, X_{t-})}{2}\, d[X, X]_t, \tag{3.23}$$

uniformly in s.

We now should discuss $J_{B12}(\varepsilon_n,\, N,\, s)$, $J_{B13}(\varepsilon_n,\, N,\, s)$ and $J_{B23}(\varepsilon_n,\, N,\, s)$. In the sequel, $\delta(f, \cdot)$ will denote the modulus of continuity of a function f, and by I_l the interval $[t_{l-1},\, t_l]$, $l \geq 0$. Since $\frac{(t+\varepsilon)\wedge s - t}{\varepsilon} \leq 1$ for every t, s, by Remark 2.13 we get

$$\begin{aligned} \mathbb{1}_{B(\varepsilon,N)}(t)\, |R_1(\varepsilon,\, t,\, s)| \leq & \delta\left(\partial_t F,\, \varepsilon\right), \\ \mathbb{1}_{B(\varepsilon,N)}(t)\, |R_2(\varepsilon,\, t,\, s)| \leq & \delta\Big(\partial_t F,\, \sup_{l} \sup_{\substack{t,a\in I_l \\ |t-a|\leq\varepsilon}} |X_a - X_t|)\Big), \\ \mathbb{1}_{B(\varepsilon,N)}(t)\, |R_3(\varepsilon,\, t,\, s)| \leq & \delta\Big(\partial^2_{xx} F,\, \sup_{l} \sup_{\substack{t,a\in I_l \\ |t-a|\leq\varepsilon}} |X_a - X_t|)\Big). \end{aligned}$$

Considering the two latter inequalities, Lemma 2.12 applied successively to the intervals I_l implies

$$\mathbb{1}_{B(\varepsilon,N)}(t)\, |R_2(\varepsilon,\, t,\, s)| \leq \delta(\partial_t F,\, 3\gamma),$$

$$\mathbb{1}_{B(\varepsilon,N)}(t)\,|R_3(\varepsilon,\,t,\,s)| \le \delta(\partial^2_{xx}F,\,3\gamma).$$

Then, using again $\frac{(t+\varepsilon_n)\wedge s-t}{\varepsilon} \le 1$, we get

$$\begin{aligned}
&\sup_{s\in[0,\,T]} |J_{B12}(\varepsilon_n,\,N,\,s)| \le \delta(\partial_t F,\,\varepsilon_n)\cdot T,\\
&\sup_{s\in[0,\,T]} |J_{B13}(\varepsilon_n,\,N,\,s)| \le \delta(\partial_t F,\,3\gamma)\cdot T,\\
&\sup_{s\in[0,\,T]} |J_{B23}(\varepsilon_n,\,N,\,s)| \le \delta(\partial^2_{xx}F,\,3\gamma)\cdot \sup_{n\in\mathbb{N},s\in[0,\,T]} [X,X]^{ucp}_{\varepsilon_n}(s),
\end{aligned} \tag{3.24}$$

where we remark that the supremum in the right-hand side of (3.24) is finite taking into account (3.2). Therefore

$$\limsup_{n\to\infty}\sup_{s\in[0,\,T]} |J_{B23}(\varepsilon_n,\,N,\,s)| = \delta(\partial^2_{xx}F,\,3\gamma)\cdot \sup_{n\in\mathbb{N},s\in[0,\,T]} [X,X]^{ucp}_{\varepsilon_n}(s), \tag{3.25}$$

$$\limsup_{n\to\infty}\sup_{s\in[0,\,T]} |J_{B13}(\varepsilon_n,\,N,\,s)| = \delta(\partial_t F,\,3\gamma)\cdot T, \tag{3.26}$$

while

$$\lim_{n\to\infty}\sup_{s\in[0,\,T]} |J_{B12}(\varepsilon_n,\,N,\,s)| = 0. \tag{3.27}$$

Let now consider the terms $J_{B10'}(\varepsilon_n,\,s)$, $J_{B10''}(\varepsilon_n,\,s)$ and $J_{B11}(\varepsilon_n,\,N,\,s)$.

$$\begin{aligned}
\sup_{s\in[0,\,T]} |J_{B10'}(\varepsilon_n,\,s)| &\le \sup_{y\,\in\,\mathbb{K}^X(\omega)\times[0,\,T]} |\partial_t F(y)|\cdot\varepsilon_n,\\
\sup_{s\in[0,\,T]} |J_{B10''}(\varepsilon_n,\,s)| &\le \sup_{y\,\in\,\mathbb{K}^X(\omega)\times[0,\,T]} |\partial_t F(y)|\cdot\varepsilon_n,\\
\sup_{s\in[0,\,T]} |J_{B11}(\varepsilon_n,\,N,\,s)| &\le \sup_{y\,\in\,\mathbb{K}^X(\omega)\times[0,\,T]} |\partial_t F(y)|\,N\cdot\varepsilon_n,
\end{aligned}$$

where $\mathbb{K}^X(\omega)$ is the (compact) set $\{X_t(\omega),\,t\in[0,\,T]\}$. So, it follows

$$\lim_{n\to\infty}\sup_{s\in[0,\,T]} |J_{B10'}(\varepsilon_n,\,s)| = \lim_{n\to\infty}\sup_{s\in[0,\,T]} |J_{B10''}(\varepsilon_n,\,s)| = \lim_{n\to\infty}\sup_{s\in[0,\,T]} |J_{B11}(\varepsilon_n,\,N,\,s)| = 0. \tag{3.28}$$

Taking into account (3.28), (3.26), (3.25), and (3.23), we see that

$$\limsup_{n\to\infty}\sup_{s\in[0,\,T]} |L(\varepsilon_n,N,s)| = \delta(\partial^2_{xx}F,\,3\gamma)\cdot \sup_{n\in\mathbb{N},s\in[0,\,T]} [X,X]^{ucp}_{\varepsilon_n}(s) + \delta(\partial_t F,\,3\gamma)\cdot T. \tag{3.29}$$

Recalling that $J_{B20}(\varepsilon,s)$ in (3.15) is the ε-approximation of the forward integral $\int_0^t \partial_x F(s,X_s)\,d^-X_s$, to conclude it remains to show that

$$\sup_{s\in[0,\,T]} \big|J_{B20}(\varepsilon_n,\,s) - J(s)\big| \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.}, \tag{3.30}$$

where

$$\begin{aligned}
J(s) = F(s,\,X_s) - F(0,\,X_0) - \int_{]0,\,s]} \partial_t F(t,\,X_t)\,dt - \sum_{t\le s}\left[F(t,\,X_t) - F(t,\,X_{t-})\right]\\
+ \sum_{0<t\le s}\left[\partial_x F(t,\,X_{t-})\,(X_t - X_{t-}) + \frac{\partial^2_{xx}F(t,\,X_{t-})}{2}\,(X_t - X_{t-})^2\right]
\end{aligned}$$

$$- \frac{1}{2} \int_{]0,\, s]} \partial^2_{xx} F(t,\, X_{t-})\, d[X, X]_t. \tag{3.31}$$

In particular this would imply that $\int_{]0,\, s]} \partial_x F(t,\, X_t)\, d^- X_t$ exists and equals $J(s)$. Taking into account (3.21), we have

$$J_{B20}(\varepsilon_n,\, s) = J_0(\varepsilon_n, s) - J_A(\varepsilon_n,\, N,\, s) - \int_0^s \partial_t F(t,\, X_t)\, dt - L(\varepsilon_n, N, s) - J_{B21}(\varepsilon_n,\, s) - J_{B22}(\varepsilon_n,\, N,\, s). \tag{3.32}$$

Taking into account (3.31) and (3.32), we see that the term inside the absolute value in (3.30) equals

$$\begin{aligned}
&J_0(\varepsilon_n,\, s) - (F(s,\, X_s) - F(0,\, X_0)) \\
&- J_A(\varepsilon_n,\, N,\, s) + \sum_{i=1}^{N} \mathbb{1}_{]0,\, s]}(t_i)[F(t_i,\, X_{t_i}) - F(t_i,\, X_{t_i-})] \\
&- J_{B22}(\varepsilon_n,\, N,\, s) - \sum_{i=1}^{N} \mathbb{1}_{]0,\, s]}(t_i) \left[\partial_x F(t_i,\, X_{t_i-})\, (X_{t_i} - X_{t_i-}) + \frac{\partial^2_{xx} F(t_i,\, X_{t_i-})}{2}\, (X_{t_i} - X_{t_i-})^2\right] \\
&- J_{B21}(\varepsilon_n,\, s) + \frac{1}{2} \int_{]0,\, s]} \partial^2_{xx} F(t,\, X_{t-})\, d[X, X]_t \\
&- L(\varepsilon_n, N, s) \\
&+ \sum_{i=N+1}^{\infty} \mathbb{1}_{]0,\, s]}(t_i) \left[F(t_i,\, X_{t_i}) - F(t_i,\, X_{t_i-}) - \partial_x F(t_i,\, X_{t_i-})\, (X_{t_i} - X_{t_i-}) - \frac{\partial^2_{xx} F(t_i,\, X_{t_i-})}{2}\, (X_{t_i} - X_{t_i-})^2\right].
\end{aligned}$$

Taking into account (3.6), (3.10), (3.17), (3.27), (3.29),we have

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{s\in[0,\, T]} \left| J_{B20}(\varepsilon_n,\, s) - J(s) \right| \\
&\le \limsup_{n\to\infty} \sup_{s\in[0,\, T]} |L(\varepsilon_n,\, N,\, s)| \\
&+ \sup_{s\in[0,\, T]} \sum_{i=N+1}^{\infty} \mathbb{1}_{]0,\, s]}(t_i) \left| F(t_i,\, X_{t_i}) - F(t_i,\, X_{t_i-}) - \partial_x F(t_i,\, X_{t_i-})\, \Delta X_{t_i} - \frac{\partial^2_{xx} F(t_i,\, X_{t_i-})}{2}\, (\Delta X_{t_i})^2 \right| \\
&= \limsup_{n\to\infty} \sup_{s\in[0,\, T]} |L(\varepsilon_n,\, N,\, s)| \\
&+ \sup_{s\in[0,\, T]} \sum_{i=N+1}^{\infty} (\Delta X_s)^2\, \mathbb{1}_{]0,\, s]}(t_i)\, \frac{1}{2} \left| \int_0^1 \partial^2_{xx} F(t_i,\, X_{t_i-} + a(\Delta X_{t_i}))\, da - \partial^2_{xx} F(t_i,\, X_{t_i-}) \right| \\
&\le \delta(\partial_t F,\, 3\gamma) \cdot T + \delta(\partial^2_{xx} F,\, 3\gamma) \cdot \sup_{n\in\mathbb{N}, s\in[0,\, T]} [X, X]^{ucp}_{\varepsilon_n}(s) + \gamma^2 \sup_{y\in\mathbb{K}^X(\omega)\times[0,\, T]} |\partial^2_{xx} F(y)|,
\end{aligned} \tag{3.33}$$

where the last term on the right-hand side of (3.33) is obtained using (3.4). Since γ is arbitrarily small, we conclude that

$$\lim_{n\to\infty} \sup_{s\in[0,\, T]} \left| J_{B20}(\varepsilon_n,\, s) - J(s) \right| = 0, \quad \forall \omega \notin \mathcal{N}.$$

This concludes the proof of the Itô formula. □

From Theorem 3.1, Proposition 2.8-ii), and by classical Banach-Steinhaus theory (see, e.g., [11], Theorem 1.18 p. 55) for F-type spaces, we have the following.

Proposition 3.2. *Let $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that $\partial_x F$ is Hölder continuous with respect to the second variable for some $\lambda \in [0, 1[$. Let $(X_t)_{t\in[0,T]}$ be a reversible semimartingale, satisfying moreover*

$$\sum_{0<s\leq t} |\Delta X_s|^{1+\lambda} < \infty \quad \text{a.s.}$$

Then

$$F(t, X_t) = F(0, X_0) + \int_0^t \partial_s F(s, X_s)\, ds + \int_0^t \partial_x F(s, X_{s-})\, dX_s + \frac{1}{2}\,[\partial_x F(\cdot, X),\, X]_t + J(F, X)(t),$$

where

$$J(F, X)(t) = \sum_{0<s\leq t} \left[F(s, X_s) - F(s, X_{s-}) - \frac{\partial_x F(s, X_s) + \partial_x F(s, X_{s-})}{2}\, \Delta X_s \right]$$

Remark 3.3. (i) Previous result can be easily extended to the case when X is multidimensional.

(ii) When F does not depend on time, previous statement was the object of [14], Theorem 3.8, example 3.3.1. In that case however, stochastic integrals and covariations were defined by discretizations means.

(iii) The proof of Proposition 3.2 follows the same lines as the one of Theorem 3.8. in [14].

4 Itô's formula for $C^{1,2}$ functions related to random measures

The concept of random measure allows a very tractable description of the jumps of a càdlàg process. The object of the present section is to reexpress the statement of Theorem 3.1 making use of the jump measure associated with a càdlàg process X.

We start by recalling the main definitions and some properties that we will extensively use in the sequel; for a complete discussion on this topic and the unexplained notations we refer to Chapter II, Section 1, in [21], Chapter XI, Section 1, in [19], and also to the Appendix in [5]. We set $\tilde{\Omega} = \Omega \times [0, T] \times \mathbb{R}$, and $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$, where $\mathcal{P}$ is the predictable σ-field on $\Omega \times [0, T]$. A function W defined on $\tilde{\Omega}$ which is $\tilde{\mathcal{P}}$-measurable will be called predictable. We will also indicate by $\mathcal{A}$ (resp $\mathcal{A}_{\text{loc}}$) the collection of all adapted processes with integrable variation (resp. with locally integrable variation), and by $\mathcal{A}^+$ (resp $\mathcal{A}^+_{\text{loc}}$) the collection of all adapted integrable increasing (resp. adapted locally integrable) processes. The significance of locally is the usual one, which refers to localization by stopping times, see e.g. (0.39) of [20]. We only remark that our stopping times will take values in $[0, T] \cup \{+\infty\}$ instead of $[0, \infty]$. For instance, adapting the definition of locally bounded process stated before Theorem 15, Chapter IV, in [23], to the processes indexed by $[0, T]$, we can state the following.

Definition 4.1. *A process $(X_t)_{t\in[0,T]}$ is locally bounded if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ in $[0, T] \cup \{+\infty\}$ increasing to ∞ a.s., such that $(X_{\tau_n \wedge t}\, \mathbb{1}_{\{\tau_n>0\}})_{t\in[0,T]}$ is bounded.*

Remark 4.2. (i) Any càglàd process is locally bounded, see the lines above Theorem 15, Chapter IV, in [23].

(ii) Let X be a càdlàg process satisfying condition (2.5). Set $(Y_t)_{t\in[0,\,T]} = (X_{t-}, \sum_{s<t} |\Delta X_s|^2)_{t\in[0,\,T]}$. The process Y is càglàd, therefore locally bounded by item (i). In particular, we can fix a sequence of stopping times $(\tau_n)_{n\geq 1}$ in $[0,\,T]\cup\{+\infty\}$ increasing to ∞ a.s., such that $(Y_{\tau_n\wedge t}\,\mathbb{1}_{\{\tau_n>0\}})_{t\in[0,\,T]}$ is bounded.

For any $X=(X_t)$ adapted real valued càdlàg process on $[0,\,T]$, we call **jump measure** of X the integer-valued random measure on $\mathbb{R}_+\times\mathbb{R}$ defined as

$$\mu^X(\omega; dt\,dx) := \sum_{s\in]0,\,T]} \mathbb{1}_{\{\Delta X_s(\omega)\neq 0\}}\,\delta_{(s,\,\Delta X_s(\omega))}(dt\,dx). \tag{4.1}$$

The compensator of $\mu^X(ds\,dy)$ is called the Lévy system of X, and will be denoted by $\nu^X(ds\,dy)$; we also set

$$\hat{\nu}^X_t = \nu^X(\{t\},\,dy) \quad \text{for every } t\in[0,\,T]. \tag{4.2}$$

Remark 4.3. The jump measure μ^X acts in the following way: for any positive function $W:\Omega\times[0,\,T]\times\mathbb{R}\to\mathbb{R}$, measurable with respect to the product sigma-field, we have

$$\sum_{s\in]0,\,T]} \mathbb{1}_{\{\Delta X_s\neq 0\}} W_s(\cdot,\Delta X_s) = \int_{]0,T]\times\mathbb{R}} W_s(\cdot,x)\,\mu^X(\cdot,ds\,dx). \tag{4.3}$$

Clearly, the same result holds if the left or the right hand-side of (4.3) is finite a.s., replacing the process W by the process $|W|$.

Proposition 4.4. *Let X be a càdlàg process on $[0,\,T]$ satisfying condition* (2.5)*, and let F be a function of class $C^{1,2}$. Then*

$$|(F(s,X_{s-}+x)-F(s,X_{s-})-x\,\partial_x F(s,X_{s-})|\,\mathbb{1}_{\{|x|\leq 1\}} * \mu^X \in \mathcal{A}_{\mathrm{loc}}.$$

Proof. Let $(\tau_n)_{n\geq 1}$ be the sequence of stopping times introduced in Remark 4.2-(ii) for the process $Y_t = (X_{t-}, \sum_{s<t}|\Delta X_s|^2)$. Fix $\tau=\tau_n$, and let M such that $\sup_{t\in[0,\,T]}|Y_{t\wedge\tau}\,\mathbb{1}_{\{\tau>0\}}|\leq M$. So, by an obvious Taylor expansion, taking into account Remark 4.3, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{]0,\,t\wedge\tau]\times\mathbb{R}} |(F(s,X_{s-}+x)-F(s,X_{s-})-x\,\partial_x F(s,X_{s-})|\,\mathbb{1}_{\{|x|\leq 1\}}\,\mu^X(ds,\,dx)\right]\\
&= \mathbb{E}\left[\sum_{0<s\leq t\wedge\tau} [F(s,X_s)-F(s,X_{s-})-\partial_x F(s,X_{s-})\,\Delta X_s]\right]\\
&= \mathbb{E}\left[\sum_{0<s\leq t\wedge\tau} (\Delta X_s)^2\,\mathbb{1}_{\{\tau>0\}}\,\frac{1}{2}\int_0^1 \partial^2_{xx}F(s,X_{s-}+a\,\Delta X_s)\,da\right]\\
&\leq \frac{1}{2}\sup_{\substack{y\in[-M,\,M]\\ t\in[0,\,T]}} |\partial^2_{xx}F|(t,\,y)\,\mathbb{E}\left[\sum_{0<s<t\wedge\tau} |\Delta X_s|^2\,\mathbb{1}_{\{|\Delta X_s|\leq 1\}}\,\mathbb{1}_{\{\tau>0\}} + |\Delta X_\tau|^2\,\mathbb{1}_{\{|\Delta X_\tau|\leq 1\}}\,\mathbb{1}_{\{\tau>0\}}\right]\\
&\leq \frac{1}{2}\sup_{\substack{y\in[-M,\,M]\\ t\in[0,\,T]}} |\partial^2_{xx}F|(t,\,y)\cdot(M+1),
\end{aligned}$$

which concludes the proof. □

Proposition 4.5. *Let X be a càdlàg process on $[0, T]$ satisfying condition (2.5), and let F be a function of class $C^{0,1}$. Then*

$$|(F(s, X_{s-} + x) - F(s, X_{s-})|^2 \, \mathbb{1}_{\{|x|\leq 1\}} * \mu^X \in \mathcal{A}^+_{\text{loc}}, \tag{4.4}$$

$$|x \, \partial_x F(s, X_{s-})|^2 \, \mathbb{1}_{\{|x|\leq 1\}} * \mu^X \in \mathcal{A}^+_{\text{loc}}. \tag{4.5}$$

In particular, the integrands in (4.4)-(4.5) belong to $\mathcal{G}^2_{\text{loc}}(\mu^X)$, so that the stochastic integrals

$$((F(s, X_{s-} + x) - F(s, X_{s-})) \, \mathbb{1}_{\{|x|\leq 1\}} * (\mu^X - \nu^X), \tag{4.6}$$

$$x \, \partial_x F(s, X_{s-}) \, \mathbb{1}_{\{|x|\leq 1\}} * (\mu^X - \nu^X) \tag{4.7}$$

are well-defined and are two square integrable purely discontinuous martingales.

Proof. Proceeding as in the proof of Proposition 4.4, we consider the sequence of stopping times $(\tau_n)_{n\geq 1}$ defined in Remark 4.2-(ii) for the process $Y_t = (X_{t-}, \sum_{s<t} |\Delta X_s|^2)$. Fix $\tau = \tau_n$, and let M such that $\sup_{t\in[0,\,T]} |Y_{t\wedge\tau} \, \mathbb{1}_{\{\tau>0\}}| \leq M$. For any $t \in [0, T]$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{]0,\,t\wedge\tau]\times\mathbb{R}} |(F(s, X_{s-} + x) - F(s, X_{s-})|^2 \, \mathbb{1}_{\{|x|\leq 1\}} \, \mu^X(ds,\, dx)\right] \\
&\leq \sup_{\substack{y\in[-M,\,M]\\ t\in[0,\,T]}} |\partial_x F|^2(t,\, y) \, \mathbb{E}\left[\sum_{0<s<t\wedge\tau} |\Delta X_s|^2 \, \mathbb{1}_{\{|\Delta X_s|\leq 1\}} \mathbb{1}_{\{\tau>0\}} + |\Delta X_\tau|^2 \, \mathbb{1}_{\{|\Delta X_\tau|\leq 1\}} \, \mathbb{1}_{\{\tau>0\}}\right] \\
&\leq \sup_{\substack{y\in[-M,M]\\ t\in[0,\,T]}} |\partial_x F|^2(t,\, y) \cdot (M+1),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\int_{]0,\,t\wedge\tau]\times\mathbb{R}} |x \, \partial_x F(s, X_{s-})|^2 \, \mathbb{1}_{\{|x|\leq 1\}} \, \mu^X(ds,\, dx)\right] \\
&= \mathbb{E}\left[\int_{]0,\,t\wedge\tau]\times\mathbb{R}} |x|^2 \, |\partial_x F|^2(t,\, X_{s-}) \, \mathbb{1}_{\{|x|\leq 1\}} \, \mu^X(ds,\, dx)\right] \\
&\leq \sup_{\substack{y\in[-M,\,M]\\ t\in[0,\,T]}} |\partial_x F|^2(t,\, y) \, \mathbb{E}\left[\sum_{0<s<t\wedge\tau} |\Delta X_s|^2 \, \mathbb{1}_{\{|\Delta X_s|\leq 1\}} \, \mathbb{1}_{\{\tau>0\}} + |\Delta X_\tau|^2 \, \mathbb{1}_{\{|\Delta X_\tau|\leq 1\}} \, \mathbb{1}_{\{\tau>0\}}\right] \\
&\leq \sup_{\substack{y\in[-M,\,M]\\ t\in[0,\,T]}} |\partial_x F|^2(t,\, y) \cdot (M+1).
\end{aligned}$$

By Lemma 2.4-2. in [5], the integrands in (4.6) and (4.7) belong to $\mathcal{G}^2_{\text{loc}}(\mu^X)$. Then the conclusion follows by Theorem 11.21, point 3), in [19]. □

Taking $F(t, x) = x$, we have the following.

Corollary 4.6. *Let X be a càdlàg process satisfying condition (2.5). Then*

$$x \, \mathbb{1}_{\{|x|\leq 1\}} \in \mathcal{G}^2_{\text{loc}}(\mu^X), \tag{4.8}$$

and the stochastic integral

$$\int_{]0,\,t]\times\mathbb{R}} x \, \mathbb{1}_{\{|x|\leq 1\}} \, (\mu^X - \nu^X)(ds\, dx) \tag{4.9}$$

is well-defined and defines a purely discontinuous square integrable local martingale.

We state now the main result of the section. Even if the following decomposition is not canonical, it will serve as a significant tool for the case when F is of class $C^{0,1}$.

Proposition 4.7. *Let X be a finite quadratic variation càdlàg process and $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ a function of class $C^{1,2}$. Then we have*

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t \partial_s F(s, X_s)\, ds + \int_0^t \partial_x F(s, X_s)\, d^- X_s + \frac{1}{2} \int_0^t \partial^2_{xx} F(s, X_s)\, d[X, X]^c_s \\ &+ \int_{]0,\, t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, \mathbb{1}_{\{x\le 1\}}\, (\mu^X - \nu^X)(ds\, dx) \\ &- \int_{]0,\, t]\times\mathbb{R}} x\, \partial_x F(s, X_{s-})\, \mathbb{1}_{\{x\le 1\}}\, (\mu^X - \nu^X)(ds\, dx) \\ &+ \int_{]0,\, t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\, \mathbb{1}_{\{x>1\}}\, \mu^X(ds\, dx) \\ &+ \int_{]0,\, t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\, \mathbb{1}_{\{x\le 1\}}\, \nu^X(ds\, dx). \end{aligned} \tag{4.10}$$

Proof. This result is a consequence of Theorem 3.1. Indeed, by Remark 4.3, the term

$$\sum_{s\le t} [F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-})\, \Delta X_s]$$

equals

$$\begin{aligned} &\int_{]0,\, t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\, \mu^X(ds\, dx) \\ &= \int_{]0,\, t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\, \mathbb{1}_{\{x>1\}}\, \mu^X(ds\, dx) \\ &+ \int_{]0,\, t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\, \mathbb{1}_{\{x\le 1\}}\, \mu^X(ds\, dx). \end{aligned} \tag{4.11}$$

We set

$$W_s(x) = (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\, \mathbb{1}_{\{|x|\le 1\}}.$$

By Propositions 4.4, $|W| * \mu^X$ belongs to $\mathcal{A}^+_{\text{loc}}$, and consequently $|W| * \nu^X$ belongs to $\mathcal{A}^+_{\text{loc}}$ so that the last integral in (4.10) is well-defined. Setting

$$\begin{aligned} K_s(x) &= (F(s, X_{s-} + x) - F(s, X_{s-}))\, \mathbb{1}_{\{|x|\le 1\}}, \\ Y_s(x) &= x\, \partial_x F(s, X_{s-}) \mathbb{1}_{\{|x|\le 1\}}, \end{aligned}$$

Proposition 4.5 insures that $K * (\mu^X - \nu^X)$, $Y * (\mu^X - \nu^X)$ are well-defined and are purely discontinuous square integrable local martingales. For those reasons, the second integral of the right-hand side of (4.11) gives us the contributions of second, third and fifth lines of (4.10). □

5 About weak Dirichlet processes

5.1 Basic definitions and properties

We consider again the filtration $(\mathcal{F}_t)_{t\ge 0}$ introduced at Section 2. Without further mention, the underlying filtration will be indeed $(\mathcal{F}_t)_{t\ge 0}$.

Definition 5.1. *Let X be an $(\mathcal{F}_t)$-adapted process. We say that X is $(\mathcal{F}_t)$-orthogonal if $[X, N] = 0$ for every N continuous local $(\mathcal{F}_t)$-martingale.*

Remark 5.2. Basic examples of $(\mathcal{F}_t)$-orthogonal processes are purely discontinuous $(\mathcal{F}_t)$-local martingales. Indeed, according to Theorem 7.34 in [19] and the comments above, any $(\mathcal{F}_t)$-local martingale, null at zero, is a purely discontinuous local martingale if and only if it is $(\mathcal{F}_t)$-orthogonal.

Proposition 5.3. *If M is a purely discontinuous $(\mathcal{F}_t)$-local martingale, then*

$$[M, M]_t = \sum_{s \leq t} (\Delta M_s)^2.$$

Proof. The result follows from Theorem 5.2, Chapter I, in [21], and Proposition 2.8-(i). □

Definition 5.4. *We say that an $(\mathcal{F}_t)$-adapted process X is a* ***Dirichlet process*** *if it admits a decomposition $X = M + A$, where M is a local martingale and A is a finite quadratic variation process with $[A, A] = 0$.*

Definition 5.5. *We say that X is an $(\mathcal{F}_t)$-adapted* ***weak Dirichlet process*** *if it admits a decomposition $X = M + A$, where M is a local martingale and the process A is $(\mathcal{F}_t)$-orthogonal.*

Definition 5.6. *We say that an $(\mathcal{F}_t)$-adapted process X is a* ***special weak Dirichlet process*** *if it admits a decomposition of the type above such that, in addition, A is predictable.*

Proposition 5.7. *A Dirichlet process X given in Definition 5.4 is a special weak Dirichlet process, and $[X, X] = [M, M]$.*

Proof. Let N be a continuous local martingale. By Proposition 2.15, $[N, A] = 0$. This shows in particular that X is a weak Dirichlet process. It is even special since A is continuous, see Lemma 2.10. By the bilinearity of the covariation, $[X, X] = [M, M]$. □

Corollary 5.8. *Let X be an $(\mathcal{F}_t)$-Dirichlet process. Then*

(i) $[X, X] = [M^c, M^c] + \sum_{s \leq \cdot} (\Delta X_s)^2$;

(ii) $[X, X]^c = [M^c, M^c]$.

Proof. (ii) follows by (i) and Lemma 2.10-(ii). Concerning (i), by the bilinearity of the covariation, and by the definitions of purely discontinuous local martingale (see Remark 5.2) and of Dirichlet process, we have

$$\begin{aligned} [X, X]_t &= [M^c, M^c]_t + [M^d, M^d]_t \\ &= [M^c, M^c]_t + \sum_{s \leq t} (\Delta M^d_s)^2 \\ &= [M^c, M^c]_t + \sum_{s \leq t} (\Delta X_s)^2, \end{aligned}$$

where the second equality holds because of Proposition 5.3. □

Proposition 5.9. *Let X be a special weak Dirichlet process of the type*

$$X = M^c + M^d + A, \tag{5.1}$$

where M^c is a continuous local martingale, and M^d is a purely discontinuous local martingale. Supposing that $A_0 = M^d_0 = 0$, the decomposition (5.1) *is unique. In that case the decomposition $X = M^c + M^d + A$ will be called the* **canonical decomposition** *of X.*

Proof. Assume that we have two decompositions $X = M^c + M^d + A = (M')^c + (M')^d + A'$, with A and A' predictable, verifying $[A, N] = [A', N] = 0$ for every continuous local martingale N. We set $\tilde{A} = A - A'$, $\tilde{M}^c = M^c - (M')^c$ and $\tilde{M}^d = M^d - (M')^d$. By linearity,

$$\tilde{M}^c + \tilde{M}^d + \tilde{A} = 0. \tag{5.2}$$

We have

$$\begin{aligned} 0 &= [\tilde{M}^c + \tilde{M}^d + \tilde{A}, \tilde{M}^c] = [\tilde{M}^c, \tilde{M}^c] + [\tilde{M}^d, \tilde{M}^c] + [\tilde{A}, \tilde{M}^c] \\ &= [\tilde{M}^c, \tilde{M}^c], \end{aligned}$$

therefore $\tilde{M}^c = 0$ since $\tilde{M}^c$ is a continuous martingale vanishing at zero. It follows in particular that $\tilde{A}$ is a predictable local martingale, hence a continuous local martingale, see e.g., Corollary 2.24 and Corollary 2.31 in [21]. In particular, calculating the bracket of both sides of (5.2) against $\tilde{M}^d$,

$$0 = [\tilde{M}^d, \tilde{M}^d] + [\tilde{A}, \tilde{M}^d] = [\tilde{M}^d, \tilde{M}^d]$$

and, since $\tilde{M}^d_0 = 0$, we deduce that $\tilde{M}^d = 0$ and therefore $\tilde{A} = 0$. □

Remark 5.10. Every $(\mathcal{F}_t)$-special weak Dirichlet process is of the type (5.1). Indeed, every local martingale M can be decomposed as the sum of a continuous local martingale M^c and a purely discontinuous local martingale M^d, see Theorem 4.18, Chapter I, in [21].

A very simple example of (special) $(\mathcal{F}_t)$-weak Dirichlet process is given below. This result extends Remark 2.4-(ii) in [8], valid if f is continuous.

Proposition 5.11. *Let $f : [0, T] \to \mathbb{R}$ be a càdlàg (deterministic) function. Then it is $(\mathcal{F}_t)$-orthogonal; consequently it is a (special) $(\mathcal{F}_t)$-weak Dirichlet process.*

Proof. Let N be an $(\mathcal{F}_t)$-continuous local martingale. We need to show that

$$[f, N]^{ucp}_\varepsilon(t) = \int_0^t \frac{ds}{\varepsilon} \left(f((s+\varepsilon)\wedge t) - f(s)\right)\left(N((s+\varepsilon)\wedge t) - N(s)\right)$$

converges u.c.p. to zero. Previous expression is the difference

$$I^{+ucp}(\varepsilon, t, f, dN) - I^{-ucp}(\varepsilon, t, f, dN),$$

where $I^{-ucp}(\varepsilon, t, f, dN)$ is given by (2.1), and

$$I^{+ucp}(\varepsilon, t, f, dN) := \int_0^t \frac{ds}{\varepsilon} f((s+\varepsilon)\wedge t)\left(N((s+\varepsilon)\wedge t) - N(s)\right).$$

By Proposition 2.8-(ii), $I^{-ucp}(\varepsilon, t, f, dN)$ converges u.c.p. to the Itô-Wiener integral $\int_{]0,t]} f(s-)\, dN_s$. It remains to prove that $I^{+ucp}(\varepsilon, t, f, dN)$ converges u.c.p. to the same quantity. We have

$$\begin{aligned} I^{+ucp}(\varepsilon, t, f, dN) &= \int_0^t \frac{ds}{\varepsilon} f((s+\varepsilon)\wedge t) \int_s^{(s+\varepsilon)\wedge t} dN(u) \\ &= \int_0^t \frac{ds}{\varepsilon} f(s+\varepsilon) \int_s^{(s+\varepsilon)\wedge t} dN(u) + J(\varepsilon, t), \end{aligned}$$

where

$$|J(\varepsilon, t)| := \left| \int_0^t \frac{ds}{\varepsilon} \left(f((s+\varepsilon)\wedge t) - f(s+\varepsilon)\right) \int_s^{(s+\varepsilon)\wedge t} dN(u) \right|$$

$$\leq \int_{t-\varepsilon}^{t} \frac{ds}{\varepsilon} \left| f(t) - f(s+\varepsilon) \right| \delta(N,\varepsilon),$$

and $\delta(N,\varepsilon)$ is the modulus of continuity of N. So

$$|J(\varepsilon,t)| \leq 2\, ||f||_\infty\, \delta(N,\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0, \quad \text{u.c.p.}$$

On the other hand, by stochastic Fubini's theorem,

$$\int_0^t \frac{ds}{\varepsilon} f(s+\varepsilon) \int_s^{(s+\varepsilon)\wedge t} dN(u) = \int_0^t dN(u) \int_{(u-\varepsilon)-}^{u} \frac{ds}{\varepsilon} f(s+\varepsilon). \tag{5.3}$$

Since

$$\int_0^T d[N,N]_u \left(\int_{(u-\varepsilon)-}^{u} \frac{ds}{\varepsilon} \left(f(s+\varepsilon) - f(u) \right) \right)^2 \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in probability,}$$

by [22], Problem 2.27, Chapter 3, (5.3) goes u.c.p. to $\int_0^t dN(u)\, f(u) = \int_0^t dN(u)\, f(u-)$, which concludes the proof. □

Remark 5.12. A weak Dirichlet processes may be of finite quadratic variation or not.

1. Assume that f in Proposition 5.11 has no quadratic variation. This constitutes a simple example of weak Dirichlet process without quadratic variation. For instance, when f is a single path of a fractional Brownian motion of Hurst index $H < 1/2$, f is a deterministic continuous function, whose quadratic variation is infinite.

2. Let F be a function of class $C^{0,1}$, and X an $(\mathcal{F}_t)$-semimartingale. By the subsequent Theorem 5.15, and since a semimartingale is a weak Dirichlet process with finite quadratic variation (see Remark 5.20-2.), $F(t, X_t)$ is an $(\mathcal{F}_t)$-weak Dirichlet process. When X is continuous, typical examples arise in the framework of SDEs with time-dependent distributional drift, see for instance [3]. If F depends roughly on time, then $F(t, X_t)$ is expected not to have finite quadratic variation.

3. Let us consider two independent $(\mathcal{F}_t)$-Brownian motions W and B. By Proposition 2.10 of [2], $\int_0^t B_{t-s}\, dW_s$ is an $(\mathcal{F}_t)$-martingale orthogonal process, therefore an $(\mathcal{F}_t)$-weak Dirichlet process. By Remark 2.16-2) in [2], it is also a finite quadratic variation process, and $[X,X] = t^2/2$. Similar examples appear in Section 4 of [8].

4. At Section 5.3 we have introduced the notion of $(\mathcal{F}_t)$-particular weak Dirichlet, see Definition 5.21. If A' is of zero quadratic variation, then X is of finite quadratic variation. By Proposition 5.3, $[X,X] = [M,M] + \sum_{s\leq \cdot} (\Delta V_s)^2 + 2 \sum_{s\leq \cdot} \Delta V_s\, \Delta M_s$. For instance, this happens when A' is a fractional Brownian motion with Hurst index $H > 1/2$, independent of M.

Proposition 5.13. *Let X be an $(\mathcal{F}_t)$-special weak Dirichlet, and τ an $(\mathcal{F}_t)$-stopping time. Then $X^\tau = M^\tau + A^\tau$ is still an $(\mathcal{F}_t)$-special weak Dirichlet process.*

Proof. Obviously M^τ is an $(\mathcal{F}_t)$-local martingale. On the other hand, A^τ is an $(\mathcal{F}_t)$-predictable process, see Proposition 2.4-b), Chapter I, in [21]. Let now N be a continuous $(\mathcal{F}_t)$-local martingale. It remains to prove that $[A^\tau, N] = 0$. By Proposition A.3, it will be enough to show that

$$C_\varepsilon(A^\tau, N) \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{u.c.p.}$$

We have

$$C_\varepsilon(A^\tau, N) := \frac{1}{\varepsilon}\int_0^t (N_{s+\varepsilon} - N_s)\,(A_{(s+\varepsilon)\wedge\tau} - A_s)\,ds = C_\varepsilon(A, N) + J(\varepsilon, t),$$

where

$$|J(\varepsilon, t)| \leq 2\,\delta(N, \varepsilon)\,||A||_\infty\,T \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{u.c.p.}$$

Since $[A, N] = 0$, again by Proposition A.3, $C_\varepsilon(A, N) \underset{\varepsilon\to 0}{\longrightarrow} 0$ u.c.p., which concludes the proof. □

Proposition 5.14. *Let S be an $(\mathcal{F}_t)$-semimartingale which is a special weak Dirichlet process. Then it is a special semimartingale.*

Proof. Let $S = M^1 + V$ such that M^1 is a local martingale and V is a bounded variation process. Let moreover $S = M^2 + A$, where a predictable $(\mathcal{F}_t)$-orthogonal process. Then $0 = V - A + M$, where $M = M^2 - M^1$. So A is a predictable semimartingale. By Corollary 8.7 in [19], A is a special semimartingale, and so by additivity S is a special semimartingale as well. □

5.2 Stability of weak Dirichlet processes under $C^{0,1}$ transformation

Theorem 5.15. *Let $X = M + A$ be a càdlàg weak Dirichlet process of finite quadratic variation, and $F : [0, T] \times \mathbb{R} \to \mathbb{R}$.*

(i) If F is of class $C^{0,1}$, we have

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \partial_x F(s, X_{s-})\,dM_s \\ &+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\,\mathbb{1}_{\{|x|\leq 1\}}\,(\mu^X - \nu^X)(ds\,dx) \\ &- \int_{]0,\,t]\times\mathbb{R}} x\,\partial_x F(s, X_{s-})\,\mathbb{1}_{\{|x|\leq 1\}}\,(\mu^X - \nu^X)(ds\,dx) \\ &+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\,\partial_x F(s, X_{s-}))\,\mathbb{1}_{\{|x|>1\}}\,\mu^X(ds\,dx) + \Gamma^F(t), \end{aligned} \tag{5.4}$$

where $\Gamma^F : C^{0,1} \to \mathbb{D}^{ucp}$ is a continuous linear map such that, for every $F \in C^{0,1}$, it fulfills the following properties.

(a) $[\Gamma^F, N] = 0$ for every N continuous local martingale.

(b) If A is predictable, then Γ^F is predictable.

(ii) If F is of class $C^{1,2}$, (5.4) holds with

$$\begin{aligned} \Gamma^F(t) := &\int_0^t \partial_s F(s, X_s)\,ds + \int_0^t \partial_x F(s, X_s)\,d^- A_s + \int_0^t \partial^2_{xx} F(s, X_s)\,d[X, X]^c_s \\ &+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\,\partial_x F(s, X_{s-}))\mathbb{1}_{\{|x|\leq 1\}}\,\nu^X(ds\,dx). \end{aligned} \tag{5.5}$$

Remark 5.16. Point (a) in Theorem 5.15-(i) implies in particular that $F(s, X_s)$ is a weak Dirichlet process when X is a weak Dirichlet process.

Remark 5.17. Proposition 2.3 implies that, when A is predictable, Γ^F in (5.5) is a predictable process for any $F \in C^{1,2}$.

Proof. We start by proving item (ii). Expressions (5.4)-(5.5) follow by Proposition 4.7, in particular by (4.10). We remark that, since M is a local martingale and $\partial_x F(s, X_s)$ is a càdlàg process, by Proposition 2.8-(ii) we have

$$\begin{aligned}\int_0^t \partial_x F(s, X_s)\, d^- X_s &= \int_0^t \partial_x F(s, X_s)\, d^- M_s + \int_0^t \partial_x F(s, X_s)\, d^- A_s \\ &= \int_0^t \partial_x F(s, X_{s-})\, dM_s + \int_0^t \partial_x F(s, X_s)\, d^- A_s.\end{aligned}$$

Let us now prove item (i). Let F be a function of class $C^{0,1}$. Setting

$$\begin{aligned}K_s(x) &= (F(s, X_{s-} + x) - F(s, X_{s-}))\, \mathbb{1}_{\{|x|\le 1\}}, \\ Y_s(x) &= x\, \partial_x F(s, X_{s-}) \mathbb{1}_{\{|x|\le 1\}},\end{aligned}$$

Proposition 4.5 insures that $K * (\mu^X - \nu^X)$, $Y * (\mu^X - \nu^X)$ are well-defined and are purely discontinuous square integrable local martingales. On the other hand, the fourth integral of the right hand-side of (5.4) is well-defined since, by Remark 4.3, it equals

$$\sum_{s\le t} [F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-})\, \Delta X_s]\, \mathbb{1}_{\{|\Delta X_s|>1\}}.$$

The sum is finite since there are at most a finite number of jumps whose size is larger than one. Previous considerations, in agreement with (5.4), allow us to set

$$\begin{aligned}\Gamma^F(t) :=& F(t, X_t) - F(0, X_0) - \int_0^t \partial_x F(s, X_{s-})\, dM_s \\ &- \int_{]0,t]\times\mathbb{R}} \{F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-})\}\, \mathbb{1}_{\{|x|>1\}}\, \mu^X(ds\, dx) \\ &- \int_{]0,t]\times\mathbb{R}} \{F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-})\}\, \mathbb{1}_{\{|x|\le 1\}}\, (\mu^X - \nu^X)(ds\, dx).\end{aligned} \tag{5.6}$$

The second step consists in proving that $C^{0,1} \ni F \mapsto \Gamma^F(t)$ is continuous with respect to the u.c.p. topology. For this we first observe that the map $F \mapsto F(t, X_t) - F(0, X_0)$ fulfills the mentioned continuity. Moreover, if $F^n \to F$ in $C^{0,1}$, then $\int_0^t (\partial_x F^n - \partial_x F)(s, X_{s-})\, dM_s$ converges to zero u.c.p. since $\partial_x F^n(s, X_{s-})$ converges to $\partial_x F(s, X_{s-})$ in $\mathbb{L}^{ucp}$, see Chapter II Section 4 in [23].

Let us consider the second line of (5.6). For almost all fixed ω, the process X has a finite number of jumps, $s_i = s_i(\omega), 1 \le i \le N(\omega)$, larger than one. Let $F^n \to F$ in $C^{0,1}$. Since the map is linear we can suppose that $F = 0$.

$$\begin{aligned}&\sup_{0<t\le T} \left| \int_{]0,t]\times\mathbb{R}} \{F^n(s, X_{s-}(\omega) + x) - F^n(s, X_{s-}(\omega)) - x\, \partial_x F^n(s, X_{s-}(\omega))\}\, \mathbb{1}_{\{|x|>1\}}\, \mu^X(\omega, ds\, dx) \right| \\ &\le \int_{]0,T]\times\mathbb{R}} |F^n(s, X_{s-}(\omega) + x) - F^n(s, X_{s-}(\omega)) - x\, \partial_x F^n(s, X_{s-}(\omega))|\, \mathbb{1}_{\{|x|>1\}}\, \mu^X(\omega, ds\, dx) \\ &= \sum_{i=1}^{N(\omega)} |F^n(s_i, X_{s_i}(\omega)) - F^n(s_i, X_{s_i-}(\omega)) - \Delta X_{s_i}(\omega)\, \partial_x F^n(s_i, X_{s_i-}(\omega))|\, \mathbb{1}_{\{|\Delta X_{s_i}(\omega)|>1\}} \underset{n\to\infty}{\to} 0.\end{aligned}$$

This shows in particular that

$$\int_{]0,\cdot]\times\mathbb{R}} \{F^n(s,X_{s-}(\omega)+x)-F^n(s,X_{s-}(\omega))-x\,\partial_x F^n(s,X_{s-}(\omega))\}\,\mathbb{1}_{\{|x|>1\}}\,\mu^X(\omega,ds\,dx) \underset{n\to\infty}{\to} 0 \quad \text{u.c.p.}$$

and so the map defined by the second line in (5.6) is continuous.

The following proposition shows the continuity properties of the last term in (5.6) with respect to F, which finally allows to conclude the continuity of the map $\Gamma^F : C^{0,1} \to \mathbb{D}^{ucp}$.

Proposition 5.18. *The map*

$$\begin{aligned} I : C^{0,1} &\to \mathbb{D}^{ucp} \\ g &\mapsto \int_{]0,\cdot]\times\mathbb{R}} G^g\,(s,\,X_{s-},\,x)\,\mathbb{1}_{\{|x|\le 1\}}\,(\mu^X-\nu^X)(ds\,dx), \end{aligned}$$

where

$$G^g\,(s,\,\xi,\,x) = g(s,\,\xi+x) - g(s,\xi) - x\,\partial_\xi g(s,\,\xi), \tag{5.7}$$

is continuous.

Proof. We consider the sequence $(\tau_l)_{l\ge1}$ of increasing stopping times introduced in Remark 4.2-(ii) for the process $Y_t = (X_{t-}, \sum_{s<t}|\Delta X_s|^2)$. Since $\Omega = \cup_l\,\{\omega : \tau_l(\omega) > T\}$ a.s., the result is proved if we show that, for every fixed $\tau = \tau_l$,

$$g \mapsto \mathbb{1}_{\{\tau>T\}}(\omega)\int_{]0,\,\cdot]\times\mathbb{R}} G^g(s,\,X_{s-},\,x)\,\mathbb{1}_{\{|x|\le1\}}\,(\mu^X-\nu^X)(ds\,dx)$$

is continuous. Let $g^n \to g$ in $C^{0,1}$. Then $G^{g^n} \to G^g$ in $C^0([0,\,T]\times\mathbb{R}^2)$. Since the map is linear we can suppose that $g=0$. Let $\varepsilon_0>0$. We aim at showing that

$$\mathbb{P}\Big(\sup_{t\in[0,\,T]}\Big|\mathbb{1}_{\{\tau>T\}}(\omega)\int_{]0,\,t]\times\mathbb{R}} G^{g^n}(s,\,X_{s-},\,x)\,\mathbb{1}_{\{|x|\le1\}}\,(\mu^X-\nu^X)(ds\,dx)\Big| > \varepsilon_0\Big) \underset{n\to\infty}{\longrightarrow} 0. \tag{5.8}$$

Let $W^n_s(x)$ (resp. by $\hat{W}^n_s$) denote the random field $G^{g^n}(s,\,X_{s-},\,x)\,\mathbb{1}_{\{|x|\le1\}}$ (resp. the process $\int_{\mathbb{R}} G^{g^n}(s,\,X_{s-},\,x)\,\mathbb{1}_{\{|x|\le1\}}\,\hat{\nu}^X(dx)$), and define

$$I^n_t := \int_{]0,\,t]\times\mathbb{R}} W^n_s(x)\,(\mu^X-\nu^X)(ds\,dx).$$

The convergence (5.8) will follow if we show that

$$\mathbb{P}\big(\sup_{t\in[0,\,T]}|I^n_{t\wedge\tau}| > \varepsilon_0\big) \underset{n\to\infty}{\longrightarrow} 0. \tag{5.9}$$

For every process $\phi = (\phi_t)$, we indicate the stopped process at τ by $\phi^\tau_t(\omega) := \phi_{t\wedge\tau(\omega)}(\omega)$. We prove below that

$$(|W^n|^2 * \mu^X)^\tau \in \mathcal{A}^+. \tag{5.10}$$

As a matter of fact, let M such that $\sup_{t\in[0,\,T]}|Y_{t\wedge\tau}\,\mathbb{1}_{\{\tau>0\}}| \le M$. Recalling Remark 4.3, an obvious Taylor expansion yields

$$\mathbb{E}\left[\int_{]0,\,t\wedge\tau]\times\mathbb{R}} |W^n_s(x)|^2\,\mu^X(ds,\,dx)\right]$$

$$\leq 2 \sup_{\substack{y\in[-M,\,M]\\ t\in[0,\,T]}} |\partial_x g^n|^2(t,\,y)\,\mathbb{E}\left[\sum_{0<s<\tau} |\Delta X_s|^2\,\mathbb{1}_{\{|\Delta X_s|\leq 1\}}\,\mathbb{1}_{\{\tau>0\}} + |\Delta X_\tau|^2\,\mathbb{1}_{\{|\Delta X_\tau|\leq 1\}}\,\mathbb{1}_{\{\tau>0\}}\right]$$

$$\leq 2 \sup_{\substack{y\in[-M,\,M]\\ t\in[0,\,T]}} |\partial_x g^n|^2(t,\,y)\,\cdot(M+1), \tag{5.11}$$

which shows (5.10). By Lemma 2.4-1. in [5], it follows that $W^n\,\mathbb{1}_{[0,\,\tau]} \in \mathcal{G}^2(\mu^X)$. Consequently, by Proposition 3.66 of [20],

$$I^n_{t\wedge\tau} \text{ is a purely discontinuous square integrable martingale.} \tag{5.12}$$

On the other hand, $W^n \in \mathcal{G}^2_{\text{loc}}(\mu^X)$, and by Theorem 11.12, point 3), in [19] , it follows that

$$\langle I^n,\,I^n\rangle_t = \int_{]0,\,t]\times\mathbb{R}} |W^n_s(x)|^2\,\nu^X(ds\,dx) - \sum_{0<s\leq t} |\hat{W}^n_s|^2 \leq \int_{]0,\,t]\times\mathbb{R}} |W^n_s(x)|^2\,\nu^X(ds\,dx). \tag{5.13}$$

Taking into account (5.12), we can apply Doob's inequality. Using estimates (5.11), (5.13) and (5.12), we get

$$\begin{aligned}
\mathbb{P}\left[\sup_{t\in[0,\,T]} |I^n_{t\wedge\tau}| > \varepsilon_0\right] &\leq \frac{1}{\varepsilon_0^2}\,\mathbb{E}\left[|I^n_{T\wedge\tau}|^2\right]\\
&= \frac{1}{\varepsilon_0^2}\,\mathbb{E}\left[\langle I^n,\,I^n\rangle_{T\wedge\tau}\right]\\
&\leq \frac{2\,(M+1)}{\varepsilon_0^2} \sup_{\substack{y\in[-M,\,M]\\ t\in[0,\,T]}} |\partial_x g^n|^2(t,\,y).
\end{aligned}$$

Therefore, since $\partial_x g^n \to 0$ in C^0 as n goes to infinity,

$$\lim_{n\to\infty} \mathbb{P}\left[\sup_{t\in[0,\,T]} |I^n_{t\wedge\tau}| > \varepsilon_0\right] = 0.$$

□

We continue the proof of item (i) of Theorem 5.15. It remains to prove items (a) and (b).
(a) We have to prove that, for any continuous local martingale N, we have

$$\begin{aligned}
&\Bigg[F(\cdot,X) - \int_0^{\cdot} \partial_x F(s,X_{s-})\,dM_s\\
&- \int_{]0,\,\cdot]\times\mathbb{R}} \{F(s,X_{s-}+x) - F(s,X_{s-}) - x\,\partial_x F(s,X_{s-})\}\,\mathbb{1}_{\{|x|>1\}}\,\mu^X(ds\,dx)\\
&- \int_{]0,\,\cdot]\times\mathbb{R}} \{F(s,X_{s-}+x) - F(s,X_{s-}) - x\,\partial_x F(s,X_{s-})\}\,\mathbb{1}_{\{|x|\leq 1\}}\,(\mu^X-\nu^X)(ds\,dx), N\Bigg] = 0.
\end{aligned}$$

We set

$$\begin{aligned}
Y_t &= \int_{]0,\,t]\times\mathbb{R}} W_s(x)\,\mathbb{1}_{\{|x|\leq 1\}}\,(\mu^X-\nu^X)(ds\,dx),\\
Z_t &= \int_{]0,\,t]\times\mathbb{R}} W_s(x)\,\mathbb{1}_{\{|x|>1\}}\,\mu^X(ds\,dx),
\end{aligned}$$

with $W_s(x) = F(s, X_{s-} + x) - F(s, X_{s-}) - x\,\partial_x F(s, X_{s-})$.

Since X has almost surely a finite number of jumps larger than one, by Remark 4.3 Z is a bounded variation process. Being N continuous, Proposition 2.14 insures that $[Z, N] = 0$. By Proposition 4.5 $W^2 \mathbb{1}_{\{|x|\le 1\}} * \mu^X \in \mathcal{A}^+_{\text{loc}}$, therefore $W\mathbb{1}_{\{|x|\le 1\}}$ belongs to $\mathcal{G}^2_{\text{loc}}(\mu^X)$ as well, see Lemma 2.4-2. in [5]. In particular, by Theorem 11.21, point 3), in [19], Y is a purely discontinuous (square integrable) local martingale. Recalling that a local $(\mathcal{F}_t)$-martingale, null at zero, is a purely discontinuous martingale if and only if it is $(\mathcal{F}_t)$-orthogonal (see Remark 5.2), from Proposition 2.8-(i) we have $[Y, N] = 0$. From Proposition 2.8-(iii), and the fact that $[M, N]$ is continuous, it follows that

$$\left[\int_0^{\cdot} \partial_x F(s, X_{s-})\, dM_s, N\right] = \int_0^{\cdot} \partial_x F(s, X_{s-})\, d\,[M, N]_s\,.$$

Therefore it remains to check that

$$[F(\cdot, X), N]_t = \int_0^{\cdot} \partial_x F(s, X_{s-})\, d\,[M, N]_s\,. \tag{5.14}$$

To this end, we evaluate the limit of

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t (F((s+\varepsilon)\wedge t, X_{(s+\varepsilon)\wedge t}) - F(s, X_s))\,(N_{(s+\varepsilon)\wedge t} - N_s)\,ds\\
&= \frac{1}{\varepsilon}\int_0^t (F((s+\varepsilon)\wedge t, X_{(s+\varepsilon)\wedge t}) - F((s+\varepsilon)\wedge t, X_s))\,(N_{(s+\varepsilon)\wedge t} - N_s)\,ds\\
&\quad + \frac{1}{\varepsilon}\int_0^t (F((s+\varepsilon)\wedge t, X_s) - F(s, X_s))\,(N_{(s+\varepsilon)\wedge t} - N_s)\,ds\\
&=: I_1(\varepsilon,\, t) + I_2(\varepsilon,\, t).
\end{aligned}$$

Concerning the term $I_1(\varepsilon,\, t)$, it can be decomposed as

$$I_1(\varepsilon,\, t) = I_{11}(\varepsilon,\, t) + I_{12}(\varepsilon,\, t) + I_{13}(\varepsilon,\, t),$$

where

$$\begin{aligned}
I_{11}(\varepsilon,\, t) &= \frac{1}{\varepsilon}\int_0^t \partial_x F(s, X_s)\,(N_{(s+\varepsilon)\wedge t} - N_s)(X_{(s+\varepsilon)\wedge t} - X_s)\,ds,\\
I_{12}(\varepsilon,\, t) &= \frac{1}{\varepsilon}\int_0^t (\partial_x F((s+\varepsilon)\wedge t, X_s) - \partial_x F(s, X_s))\,(N_{(s+\varepsilon)\wedge t} - N_s)(X_{(s+\varepsilon)\wedge t} - X_s)\;ds,\\
I_{13}(\varepsilon,\, t) &= \frac{1}{\varepsilon}\int_0^t \left(\int_0^1 (\partial_x F((s+\varepsilon)\wedge t, X_s + a(X_{(s+\varepsilon)\wedge t} - X_s)) - \partial_x F((s+\varepsilon)\wedge t, X_s))\,da\right)\cdot\\
&\qquad \cdot (N_{(s+\varepsilon)\wedge t} - N_s)(X_{(s+\varepsilon)\wedge t} - X_s)\,ds.
\end{aligned}$$

Notice that the brackets $[X, X]$, $[X, N]$ and $[N, N]$ exist. Indeed, $[X, X]$ exists by definition, $[N, N]$ exists by Proposition 2.8-(i). Concerning $[X, N]$, it can be decomposed as $[X, N] = [M, N] + [A, N]$, where $[M, N]$ exists by Proposition 2.8-(i) and $[A, N] = 0$ by hypothesis, since A comes from the weak Dirichlet decomposition of X.

Then, from Corollary A.4-2) and Proposition A.6 we have

$$I_{11}(\varepsilon,\, t) \underset{\varepsilon\to 0}{\longrightarrow} \int_0^t \partial_x F(s, X_{s-})\, d[M, N]_s \quad \text{u.c.p.} \tag{5.15}$$

At this point, we have to prove the u.c.p. convergence to zero of the remaining terms $I_{12}(\varepsilon,t)$, $I_{13}(\varepsilon,t)$, $I_2(\varepsilon,t)$. First, since $\partial_x F$ is uniformly continuous on each compact, we have

$$|I_{12}(\varepsilon,\, t)| \leq \delta\Big(\partial_x F\Big|_{[0,\,T]\times\mathbb{K}^X};\, \varepsilon\Big)\sqrt{[X,X]^{ucp}_{\varepsilon}[N,N]^{ucp}_{\varepsilon}}, \tag{5.16}$$

where $\mathbb{K}^X$ is the (compact) set $\{X_t(\omega) : t \in [0,\, T]\}$. When ε goes to zero, the modulus of continuity component in (5.16) converges to zero a.s., while the remaining term u.c.p. converges to $\sqrt{[X,X]_t[N,N]_t}$ by definition. Therefore,

$$I_{12}(\varepsilon,\, t) \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{u.c.p.} \tag{5.17}$$

Let us then evaluate $I_{13}(t,\,\varepsilon)$. Since $[X,X]^{ucp}_{\varepsilon}$, $[N,N]^{ucp}_{\varepsilon}$ u.c.p. converge, there exists of a sequence (ε_n) such that $[X,X]^{ucp}_{\varepsilon_n}$, $[N,N]^{ucp}_{\varepsilon_n}$ converges uniformly a.s. respectively to $[X,X]$, $[N,N]$. We fix a realization ω outside a null set. Let $\gamma > 0$. We enumerate the jumps of $X(\omega)$ on $[0,\, T]$ by $(t_i)_{i\geq 0}$. Let $M = M(\omega)$ such that

$$\sum_{i=M+1}^{\infty} |\Delta X_{t_i}|^2 \leq \gamma^2.$$

We define $A(\varepsilon_n, M)$ and $B(\varepsilon_n, M)$ as in (2.19)-(2.20). The term $I_{13}(\varepsilon_n,\, t)$ can be decomposed as the sum of two terms:

$$\begin{aligned}
I^A_{13}(\varepsilon_n,\, t) &= \sum_{i=1}^{M}\int_{t_i-\varepsilon_n}^{t_i}\frac{ds}{\varepsilon_n}\,\mathbb{1}_{]0,\,t]}(s)\,(X_{(s+\varepsilon_n)\wedge t}-X_s)(N_{(s+\varepsilon_n)\wedge t}-N_s)\cdot\\
&\quad\cdot\int_0^1\big(\partial_x F((s+\varepsilon_n)\wedge t,\, X_s+a(X_{(s+\varepsilon_n)\wedge t}-X_s))-\partial_x F((s+\varepsilon_n)\wedge t,\, X_s)\big)\,da,\\
I^B_{13}(\varepsilon_n,\, t) &= \frac{1}{\varepsilon_n}\int_{]0,\,t]}(X_{(s+\varepsilon_n)\wedge t}-X_s)(N_{(s+\varepsilon_n)\wedge t}-N_s)\,R^B(\varepsilon_n,s,t,M)\,ds,
\end{aligned}$$

with

$$R^B(\varepsilon_n,s,t,M) = \mathbb{1}_{B(\varepsilon_n,M)}(s)\int_0^1[\partial_x F((s+\varepsilon_n)\wedge t,\, X_s+a(X_{(s+\varepsilon_n)\wedge t}-X_s))-\partial_x F((s+\varepsilon_n)\wedge t,\, X_s)]\,da.$$

By Remark 2.13, we have for every s, t,

$$R^B(\varepsilon_n,s,t,M) \leq \delta\Bigg(\partial_x F\Big|_{[0,\,T]\times\mathbb{K}^X},\, \sup_{l}\sup_{\substack{r,a\in[t_{l-1},\,t_l]\\ |r-a|\leq\varepsilon_n}}|X_a-X_r|\Bigg),$$

so that Lemma 2.12 applied successively to the intervals $[t_{l-1},\, t_l]$ implies

$$R^B(\varepsilon_n,s,t,M) \leq \delta\big(\partial_x F\big|_{[0,\,T]\times\mathbb{K}^X},3\gamma\big).$$

Then

$$|I^B_{13}(\varepsilon_n,\, t)| \leq \delta\big(\partial_x F\big|_{[0,\,T]\times\mathbb{K}^X},3\gamma\big)\sqrt{[N,N]^{ucp}_{\varepsilon_n}(T)\,[X,X]^{ucp}_{\varepsilon_n}(T)},$$

and we get

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}|I^B_{13}(\varepsilon_n,t)| \leq \delta\big(\partial_x F\big|_{[0,\,T]\times\mathbb{K}^X},3\gamma\big)\sqrt{[N,N]_T\,[X,X]_T}. \tag{5.18}$$

Concerning $I_{13}^A(\varepsilon_n,\, t)$, we apply Lemma 2.11 to $Y = (Y^1, Y^2, Y^3) = (t, X, N)$ and

$$\phi(y_1, y_2) = (y_1^2 - y_2^2)\,(y_1^3 - y_2^3) \int_0^1 [\partial_x F(y_1^1,\, y_2^2 + a(y_1^2 - y_2^2)) - \partial_x F(y_1^1,\, y_2^2)]\, da.$$

Then $I_{13}^A(\varepsilon_n, t)$ converges uniformly in $t \in [0,\, T]$, as n goes to infinity, to

$$\sum_{i=1}^{M} \mathbb{1}_{]0,\,t]}(t_i)\,(X_{t_i} - X_{t_i-})(N_{t_i} - N_{t_i-}) \int_0^1 [\partial_x F(t_i,\, X_{t_i-} + a(X_{t_i} - X_{t_i-})) - \partial_x F(t_i,\, X_{t_i-})]\, da. \quad (5.19)$$

In particular, (5.19) equals zero since N is a continuous process. Then, recalling (5.18), we have

$$\limsup_{n\to\infty} \sup_{t\in[0,\,T]} |I_{13}(\varepsilon_n,\, t)| \le \delta(\partial_x F, 3\,\gamma)\sqrt{[N,N]_T\,[X,X]_T}.$$

Letting γ go to zero, we conclude that

$$\limsup_{n\to\infty} \sup_{t\in[0,\,T]} |I_{13}(\varepsilon_n,\, t)| = 0. \quad (5.20)$$

It remains to show the u.c.p. convergence to zero of $I_2(\varepsilon,\, t)$, as $\varepsilon \to 0$. To this end, let us write it as the sum of the two terms

$$I_{21}(\varepsilon,\, t) = \frac{1}{\varepsilon}\int_0^t (F(s+\varepsilon, X_s) - F(s, X_s))\,(N_{(s+\varepsilon)\wedge t} - N_s)\, ds,$$
$$I_{22}(\varepsilon,\, t) = \frac{1}{\varepsilon}\int_0^t (F((s+\varepsilon)\wedge t, X_s) - F(s+\varepsilon, X_s))\,(N_{(s+\varepsilon)\wedge t} - N_s)\, ds.$$

Concerning $I_{21}(\varepsilon,\, t)$, it can be written as

$$I_{21}(\varepsilon,\, t) = \int_{]0,\,t]} J_\varepsilon(r)\, dN_r \quad (5.21)$$

with

$$J_\varepsilon(r) = \int_{[(r-\varepsilon)_+,\, r[} \frac{F(s+\varepsilon, X_s) - F(s, X_s)}{\varepsilon}\, ds.$$

Since $J_\varepsilon(r) \to 0$ pointwise, it follows from the Lebesgue dominated convergence theorem that

$$\int_0^T J_\varepsilon^2(r)\, d\langle N,\, N\rangle_r \xrightarrow{\mathbb{P}} 0 \qquad \text{as } \varepsilon \to 0. \quad (5.22)$$

Therefore, according to [22], Problem 2.27 in Chapter 3,

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} |I_{21}(\varepsilon, t)| = 0. \quad (5.23)$$

As far as $I_{22}(\varepsilon, t)$ is concerned, we have

$$|I_{22}(\varepsilon,\, t)| \le \frac{1}{\varepsilon}\int_{t-\varepsilon}^t |F(t, X_s) - F(s+\varepsilon, X_s)|\,|N_t - N_s| \le 2\,\delta\big(F\big|_{[0,\,T]\times\mathbb{K}^X}, \varepsilon\big)\,||N||_\infty$$

and we get

$$\limsup_{\varepsilon\to 0} \sup_{t\in[0,T]} |I_{22}(\varepsilon, t)| = 0. \quad (5.24)$$

This concludes the proof of item (a).

(b) Let F^n be a sequence $C^{1,2}$ functions such that $F^n \to F$ and $\partial_x F^n \to \partial_x F$, uniformly on every compact subset. From item (ii), the process $\Gamma^{F^n}(t)$ in (5.5) equals

$$\int_0^t \partial_s F^n(s, X_s)\, ds + \int_0^t \partial_x F^n(s, X_s)\, d^- A_s + \int_0^t \partial^2_{xx} F^n(s, X_s)\, d[X, X]^c_s$$
$$+ \int_{[0,t]\times\mathbb{R}} (F^n(s, X_{s-} + x) - F^n(s, X_{s-}) - x\, \partial_x F^n(s, X_{s-}))\, \mathbb{1}_{\{|x|\le 1\}} \nu^X(ds\, dx),$$

which is predictable, see Remark 5.17. Since, by point (a), the map $\Gamma^F : C^{0,1} \to \mathbb{D}^{ucp}$ is continuous, Γ^{F^n} converges to Γ^F u.c.p. Then Γ^F is predictable because it is the u.c.p. limit of predictable processes. □

5.3 A class of particular weak Dirichlet processes

The notion of Dirichlet process is a natural extension of the one of semimartingale only in the continuous case. Indeed, if X is a càdlàg process, which is also Dirichlet, then $X = M + A'$ with $[A', A'] = 0$, and therefore A' is continuous because of Lemma 2.10. This class does not include all the càdlàg semimartingale $S = M + V$, perturbed by a zero quadratic variation process A'. Indeed, if V is not continuous, $S + A'$ is not necessarily a Dirichlet process, even though X is a weak Dirichlet process. Notice that, in general, it is even not a special weak Dirichlet process, since V is generally not predictable.

We propose then the following natural extension of the semimartingale notion in the weak Dirichlet framework.

Definition 5.19. *We say that X is an $(\mathcal{F}_t)$-**particular weak Dirichlet process** if it admits a decomposition $X = M + A$, where M is an $(\mathcal{F}_t)$-local martingale, $A = V + A'$ with V being a bounded variation adapted process and A' a continuous adapted process $(\mathcal{F}_t)$-orthogonal process such that $A'_0 = 0$.*

Remark 5.20.

1. A particular weak Dirichlet process is a weak Dirichlet process. Indeed by Proposition 2.14 we have $[V, N] = 0$, so
$$[A' + V, N] = [A', N] + [V, N] = 0.$$
2. By construction an $(\mathcal{F}_t)$-semimartingale is a particular $(\mathcal{F}_t)$-weak Dirichlet process, so also an $(\mathcal{F}_t)$-weak Dirichlet process.
3. There exist processes that are (even special) weak Dirichlet and not particular weak Dirichlet. As a matter of fact, let for instance consider the deterministic process $A_t = \mathbb{1}_{\mathbb{Q}\cap[0,T]}(t)$. Then A is predictable and $[A, N] = 0$ for any N continuous local martingale, since, the fact that $A \equiv 0$ Lebesgue a.e. implies that $[A, N]^{ucp}_\varepsilon \equiv 0$. Moreover, since A is totally discontinuous, it can not have bounded variation, so that A is special weak Dirichlet but not a particular weak Dirichlet process.

The result below is an extension of Proposition 5.7.

Proposition 5.21. *Let $X = M + V + A'$ be an-$(\mathcal{F}_t)$-particular weak Dirichlet process, with A' of zero quadratic variation. Then X is of finite quadratic variation, and*

$$[X, X] = [M, M] + \sum_{s\le\cdot} (\Delta V_s)^2 + 2 \sum_{s\le\cdot} \Delta V_s\, \Delta M_s.$$

Proof. By Proposition 2.14, $[V,V]_t = \sum_{s\le t}(\Delta V_s)^2$ and $[M,V]_t = \sum_{s\le t}\Delta V_s\,\Delta M_s$. By Proposition 2.15, $[M,A'] + [V,A'] \equiv 0$. The result follows by the bilinearity of the covariation. □

In Propositions 5.22 and 5.24 we extend some properties valid for semimartingales to the case of particular weak Dirichlet processes.

Proposition 5.22. *Let X be an $(\mathcal{F}_t)$-adapted càdlàg process satisfying assumption* (2.5)*. X is a particular weak Dirichlet process if and only if there exist a continuous local martingale M^c, a predictable process α of the type $\alpha^S + A'$, where α^S is predictable with bounded variation, A' is a $(\mathcal{F}_t)$-adapted continuous orthogonal process, $\alpha^S_0 = A'_0 = 0$, and*

$$X = M^c + \alpha + (x\,\mathbb{1}_{\{|x|\le 1\}}) * (\mu^X - \nu^X) + (x\,\mathbb{1}_{\{|x|>1\}}) * \mu^X. \tag{5.25}$$

In this case,

$$\Delta\alpha_t = \left(\int_{|x|\le 1} x\,\hat{\nu}^X_t(dx)\right), \quad t\in[0,T], \tag{5.26}$$

where $\hat{\nu}^X$ has been defined in (4.2)*.*

Remark 5.23. Assume that (5.25) holds. Then the process α is $(\mathcal{F}_t)$-orthogonal. Indeed, for every $(\mathcal{F}_t)$-local martingale N, $[A',N] = 0$ and $[\alpha^S, N] = 0$ by Proposition 2.14.

Proof. If we suppose that decomposition (5.25) holds, then X is a particular weak Dirichlet process satisfying

$$X = M + V + A', \quad M = M^c + (x\,\mathbb{1}_{\{|x|\le 1\}}) * (\mu^X - \nu^X), \quad V = \alpha^S + (x\,\mathbb{1}_{\{|x|>1\}}) * \mu^X.$$

Conversely, suppose that $X = M+V+A'$ is a particular weak Dirichlet process. Since $S = M+V$ is a semimartingale, by Theorem 11.25 in [19], it can be decomposed as

$$S = S^c + \alpha^S + (x\,\mathbb{1}_{\{|x|\le 1\}}) * (\mu^S - \nu^S) + (x\,\mathbb{1}_{\{|x|>1\}}) * \mu^S,$$

where μ^S is the jump measure of S and ν^S is the associated Lévy system, S^c a continuous local martingale, α^S a predictable process with finite variation such that $\alpha^S_0 = 0$ and

$$\Delta\alpha^S_s = \left(\int_{|x|\le 1} x\,\hat{\nu}^S_s(dx)\right).$$

Consequently, since A' is adapted and continuous, with $A'_0 = 0$, we have

$$X = S + A' = S^c + (\alpha^S + A') + (x\,\mathbb{1}_{\{|x|\le 1\}}) * (\mu^X - \nu^X) + (x\,\mathbb{1}_{\{|x|>1\}}) * \mu^X$$

and (5.25) holds with $\alpha = \alpha^S + A'$ and $M^c = S^c$. On the other hand, since $\Delta\alpha = \Delta\alpha^S$, (5.26) follows. □

The following condition on X will play a fundamental role in the sequel:

$$|x|\,\mathbb{1}_{\{|x|>1\}} * \mu^X \in \mathcal{A}^+_{\text{loc}}. \tag{5.27}$$

Proposition 5.24. *Let X be a particular $(\mathcal{F}_t)$-weak Dirichlet process verifying the jump assumption* (2.5)*. X is a special weak Dirichlet process if and only if* (5.27) *holds.*

Proof. Suppose the validity of (5.27). We can decompose

$$(x\,\mathbb{1}_{\{|x|>1\}}) * \mu^X = (x\,\mathbb{1}_{\{|x|>1\}}) * (\mu^X - \nu^X) + (x\,\mathbb{1}_{\{|x|>1\}}) * \nu^X.$$

By Proposition 5.22, using the notation of (5.25), we obtain

$$X = M + A, \quad M = M^c + M^d, \quad A = \alpha + (x\,\mathbb{1}_{\{|x|>1\}}) * \nu^X, \tag{5.28}$$

where $M^d = x * (\mu^X - \nu^X)$. First, the process $\alpha + (x\,\mathbb{1}_{\{|x|>1\}}) * \nu^X$ is predictable. Let N be a continuous local martingale. Then, $[A, N] = 0$ by Remark 5.23 and Proposition 2.14, because $(x\,\mathbb{1}_{\{|x|>1\}}) * \nu^X$ has bounded variation. Consequently X is a special Dirichlet process.

Conversely, let $X = M + V + A'$ be a particular weak Dirichlet process, with V bounded variation. We suppose that X is a special weak Dirichlet process. Since $[A', N] = 0$ for every continuous local martingale, then by additivity $X - A'$ is still a special weak Dirichlet process, A' being continuous adapted. But $S := X - A' = M + V$ is a semimartingale, and by Proposition 5.14 it is a special semimartingale. By Corollary 11.26 in [19], $|x|\,\mathbb{1}_{\{|x|>1\}} * \mu^S \in \mathcal{A}^+_{\text{loc}}$, where μ^S is the jump measure of S. On the other hand, since A' is continuous, μ^S coincides with μ^X and (5.27) holds. □

We give the following result on the stochastic integration theory.

Proposition 5.25. *Let $W \in \mathcal{G}^1_{\text{loc}}(\mu^X)$, and define $M^d_t = \int_{[0,t]\times\mathbb{R}} W_s(x)\,(\mu^X - \nu^X)(ds\,dx)$. Let moreover (Z_t) be a predictable process such that*

$$\sqrt{\sum_{s\le\cdot} Z_s^2 |\Delta M^d_s|^2} \in \mathcal{A}^+_{\text{loc}}. \tag{5.29}$$

Then $\int_0^\cdot Z_s\,dM^d_s$ is a local martingale and equals

$$\int_{]0,\cdot]\times\mathbb{R}} Z_s\,W_s(x)\,(\mu^X - \nu^X)(ds\,dx). \tag{5.30}$$

Proof. The conclusion follows by the definition of the stochastic integral (5.30), see Definition 3.63 in [20], provided we check the following three conditions.

(i) $\int_0^\cdot Z_s\,dM_s$ is a local martingale.

(ii) $\int_0^\cdot Z_s\,dM_s$ is a purely discontinuous local martingale; in agreement with Theorem 7.34, in [19], we will show $[\int_0^\cdot Z_s\,dM_s, N] = 0$ for every N continuous local martingale vanishing at zero.

(iii) $\Delta\left(\int_0^\cdot Z_s\,dM_s\right)_t = \int_{\mathbb{R}} Z_t\,W_t(e)\,(\mu(\{t\}, de) - \nu(\{t\}, de)), \quad t \in [0, T]$.

We prove now the validity of (i), (ii) and (iii). Condition (5.29) is equivalent to $\sqrt{\int_0^t Z_s^2\,d[M, M]_s} \in \mathcal{A}^+_{\text{loc}}$. According to Definition 2.46 in [20], $\int_0^t Z_s\,dM_s$ is the unique local martingale satisfying

$$\Delta\left(\int_0^\cdot Z_s\,dM_s\right)_t = Z_t\,\Delta M_t,\ t \in [0, T]. \tag{5.31}$$

This implies in particular item (i). By Theorem 29, Chapter II, in [23], it follows that

$$\left[\int_0^\cdot Z_s\,dM_s, N\right] = \int_0^\cdot Z_s\,d[M, N]_s,$$

and item (ii) follows because M is orthogonal to N, see Theorem 7.34, in [19] and the comments above, together with obvious localization arguments. Finally, by Definition 3.63 in [20], taking into account (5.31), $\Delta\left(\int_0^{\cdot} Z_s\, dM_s\right)_t$ equals

$$Z_t\, \Delta M_t = \int_{\mathbb{R}} Z_t\, W_t(e)\, (\mu(\{t\}, de) - \nu(\{t\}, de)),$$

for every $t \in [0, T]$, and this shows item (iii). □

Remark 5.26. Recalling that $\sqrt{[M, M]_t} \in \mathcal{A}^+_{\text{loc}}$ for any local martingale M (see, e.g. Theorem 2.34 and Proposition 2.38 in [20]), condition (5.29) is verified if for instance Z is locally bounded.

5.4 Stability of special weak Dirichlet processes under $C^{0,1}$ transformation

At this point, we investigate the stability properties of the class of special weak Dirichlet processes. We start with an important property.

Proposition 5.27. *Let X be a special weak Dirichlet process with its canonical decomposition $X = M^c + M^d + A$. We suppose that assumptions* (2.5), (5.27) *are verified. Then*

$$M^d_s = \int_{]0,s]\times\mathbb{R}} x\ (\mu^X - \nu^X)(dt\, dx). \tag{5.32}$$

Proof. Taking into account assumption (2.5), Corollary 4.6 together with condition (5.27) insures the fact that the right-hand side of (5.32) is well-defined. By definition, it is the unique purely discontinuous local martingale whose jumps are indistinguishable from

$$\int_{\mathbb{R}} x\, \mu^X(\{t\}, dx) - \int_{\mathbb{R}} x\, \nu^X(\{t\}, dx).$$

It remains to prove that

$$\Delta M^d_t = \int_{\mathbb{R}} x\, \mu^X(\{t\}, dx) - \int_{\mathbb{R}} x\, \nu^X(\{t\}, dx), \text{ up to indistinguishability.} \tag{5.33}$$

We have $\Delta M^d_t = \Delta X_t - \Delta A_t$, for all $t \geq 0$. We recall that $\Delta X_t = \int_{\mathbb{R}} x\, \mu^X(\{t\}, dx)$. Thus (5.33) can be established by additivity if we prove that, for any predictable time τ,

$$\Delta A_\tau\, \mathbb{1}_{\{\tau<\infty\}} = \int_{\mathbb{R}} x\, \mathbb{1}_{\{\tau<\infty\}}\, \nu^X(\{\tau\}, dx). \tag{5.34}$$

Indeed in this case the Predictable Section Theorem (see e.g. Proposition 2.18, Chapter I, in [21]) would insure that ΔA_t and $\int_{\mathbb{R}} x\, \nu^X(\{t\}, dx)$ are indistinguishable.

Let us prove (5.34). Let τ be a predictable time. Let (τ_l) be a sequence of localizing stopping times, $\tau_l \to +\infty$ as l goes to $+\infty$, such that $(M^d)^{\tau_l}$ is a martingale. In particular $(\Delta M^d)^{\tau_l}_\tau$ is integrable. Since τ is a stopping time and A is a predictable process, A_τ is $\mathcal{F}_{\tau-}$-measurable by Proposition 2.4-a), Chapter I, in [21]. We set $\Omega_n := \{A_\tau \leq n\}$: $\Omega_n \in \mathcal{F}_{\tau-}$, $\cup_n \Omega_n = \Omega$ a.s., so that $\Delta(M^d)^{\tau_l}_\tau\, \mathbb{1}_{\{\tau<\infty\}}\, \mathbb{1}_{\Omega_n}$, $\Delta A^{\tau_l}_\tau\, \mathbb{1}_{\{\tau<\infty\}}\, \mathbb{1}_{\Omega_n}$ are integrable. By additivity, also

$$\Delta X^{\tau_l}_\tau\, \mathbb{1}_{\{\tau<\infty\}}\, \mathbb{1}_{\Omega_n} \text{ is integrable.} \tag{5.35}$$

Being A predictable, A^{τ_l} (and therefore ΔA^{τ_l}) is also predictable, Proposition 2.4-b), Chapter I, in [21]. Since τ is a stopping time, again by item a) of the same proposition, $\Delta A^{\tau_l}_\tau\, \mathbb{1}_{\{\tau<\infty\}}\, \mathbb{1}_{\Omega_n}$ is $\mathcal{F}_{\tau-}$-measurable, so that

$$\Delta A^{\tau_l}_\tau\, \mathbb{1}_{\{\tau<\infty\}}\, \mathbb{1}_{\Omega_n} = \mathbb{E}\left[\Delta X^{\tau_l}_\tau\, \mathbb{1}_{\{\tau<\infty\}}\, \mathbb{1}_{\Omega_n} - \Delta(M^d_\tau)^{\tau_l}\, \mathbb{1}_{\{\tau<\infty\}}\, \mathbb{1}_{\Omega_n} | \mathcal{F}_{\tau-}\right]. \tag{5.36}$$

Now, by Corollary 2.31, Chapter I, in [21], recalling the definition of predictable projection, for any martingale L and any predictable time τ, we have $\mathbb{E}\left[\Delta L_\tau \, \mathbb{1}_{\{\tau<\infty\}} | \mathcal{F}_{\tau-}\right] = 0$. Therefore, by Remark 4.3, identity (5.36) gives

$$\begin{aligned}\Delta A^{\tau_l}_\tau \, \mathbb{1}_{\{\tau<\infty\}} \, \mathbb{1}_{\Omega_n} &= \mathbb{E}\left[\int_{\mathbb{R}} x \, \mathbb{1}_{\{\tau<\infty\}} \, \mathbb{1}_{\Omega_n} \, \mu^{X^{\tau_l}}(\{\tau\}, dx) | \mathcal{F}_{\tau-}\right] \\ &= \int_{\mathbb{R}} x \, \mathbb{1}_{\{\tau<\infty\}} \, \mathbb{1}_{\Omega_n} \, \nu^{X^{\tau_l}}(\{\tau\}, dx) \quad \text{a.s.,} \end{aligned} \tag{5.37}$$

where for the latter equality we have used Proposition 1.17, point b), Chapter II, in [21]. By the subsequent Remark 5.28, the right-hand side of (5.37) is well-defined.

Now we remark that, for any nonnegative random field $U \in \tilde{\mathcal{P}}$,

$$\begin{aligned}\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} U_s(x) \, \mathbb{1}_{]]0,\tau_l]]}(s) \, \nu^{X^{\tau_l}}(ds, dx)\right] &= \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} U_s(x) \, \mathbb{1}_{]]0,\tau_l]]}(s) \, \mu^{X^{\tau_l}}(ds, dx)\right] \\ &= \mathbb{E}\left[\sum_{s\le T} U_s(\Delta X^{\tau_l}_s) \, \mathbb{1}_{]]0,\tau_l]]}(s) \, \mathbb{1}_{\{\Delta X^{\tau_l}_s \neq 0\}}\right] \\ &= \mathbb{E}\left[\sum_{s\le T} U_s(\Delta X_s) \, \mathbb{1}_{]]0,\tau_l]]}(s) \, \mathbb{1}_{\{\Delta X_s \neq 0\}}\right] \\ &= \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} U_s(x) \, \mathbb{1}_{]]0,\tau_l]]}(s) \, \mu^{X}(ds, dx)\right] \\ &= \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} U_s(x) \, \mathbb{1}_{]]0,\tau_l]]}(s) \, \nu^{X}(ds, dx)\right].\end{aligned}$$

Therefore,

$$\mathbb{1}_{]]0,\tau_l]]}(s) \, \nu^{X^{\tau_l}}(ds, dx) = \mathbb{1}_{]]0,\tau_l]]}(s) \, \nu^{X}(ds, dx). \tag{5.38}$$

So, identity (5.37) becomes

$$\Delta A_{\tau\wedge\tau_l} \, \mathbb{1}_{]]0,\tau_l]]}(s) \, \mathbb{1}_{\{\tau<\infty\}} \, \mathbb{1}_{\Omega_n} = \int_{\mathbb{R}} x \, \nu^{X}(\{\tau\}, dx) \, \mathbb{1}_{]]0,\tau_l]]}(s) \, \mathbb{1}_{\{\tau<\infty\}} \, \mathbb{1}_{\Omega_n} \quad \text{a.s.} \tag{5.39}$$

We prove that $\int_{\mathbb{R}} |x| \, \nu^X(\{\tau\}, dx)$ is finite a.s. For every $n \in \mathbb{N}$, we set $\tilde{\Omega}^n := \{\tau_n > T\} \cap \Omega_n$. Obviously, $\Omega = \cup_n \tilde{\Omega}^n$ a.s. We have

$$\begin{aligned}\mathbb{E}\left[\int_{\mathbb{R}} |x| \, \nu^X(\{\tau\}, dx) \, \mathbb{1}_{\tilde{\Omega}^n}\right] &= \mathbb{E}\left[\int_{\mathbb{R}} |x| \, \nu^X(\{\tau\}, dx) \, \mathbb{1}_{]]0,\tau_n]]}(s) \, \mathbb{1}_{\tilde{\Omega}_n}\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} |x| \, \nu^{X^{\tau_n}}(\{\tau\}, dx) \, \mathbb{1}_{\tilde{\Omega}_n}\right],\end{aligned}$$

which is finite by Remark 5.28. The latter equality follows by (5.38). This implies that $\int_{\mathbb{R}} |x| \, \nu^X(\{\tau\}, dx)$ is finite a.s. on $\tilde{\Omega}^n$ and therefore on Ω. The conclusion follows letting first n, then l, go to infinity in (5.39).

□

Remark 5.28. For every $l, n \in \mathbb{N}$, we have

$$\mathbb{E}\left[\int_{\mathbb{R}} |x| \, \mathbb{1}_{\{\tau<\infty\}} \, \mathbb{1}_{\Omega_n} \, \nu^{X^{\tau_l}}(\{\tau\}, dx)\right] < \infty.$$

This follows from Proposition 1.17, point b), Chapter II, in [21], Remark 4.3, and (5.35). In particular this ensures that the right-hand side of (5.37) is well-defined.

Lemma 5.29. *Let X be a càdlàg process satisfying condition* (5.27)*. Let also $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a function of class $C^{0,1}$ such that*

$$\int_{]0,s]\times\mathbb{R}} |F(t, X_{t-} + x) - F(t, X_{t-}) - x\, \partial_x F(t, X_{t-})|\, \mathbb{1}_{\{|x|>1\}}\, \mu^X(dt\, dx) \in \mathcal{A}^+_{\text{loc}}. \tag{5.40}$$

Then

$$\int_{]0,s]\times\mathbb{R}} x\, \partial_x F(t, X_{t-}) \mathbb{1}_{\{|x|>1\}}\, \mu^X(dt\, dx) \ \in \mathcal{A}^+_{\text{loc}}, \tag{5.41}$$

$$\int_{]0,s]\times\mathbb{R}} |F(t, X_{t-} + x) - F(t, X_{t-})|\, \mathbb{1}_{\{|x|>1\}}\, \mu^X(dt\, dx) \in \mathcal{A}^+_{\text{loc}}. \tag{5.42}$$

Remark 5.30. Condition (5.40) is automatically verified if X is a càdlàg process satisfying (5.27) and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a function of $C^{0,1}$ class with $\partial_x F$ bounded.

Proof. Condition (5.27) together the fact that the process $(\partial_x F(t, X_{t-}))$ is locally bounded implies (5.41); then condition (5.42) follows from (5.40) and (5.41). □

Theorem 5.31. *Let X be a special weak Dirichlet process of finite quadratic variation with its canonical decomposition $X = M^c + M^d + A$. We denote by $\tilde{C}^{0,1}$ the space of functions $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying condition* (5.40)*. Then we have the following.*

(1) For every F of class $\tilde{C}^{0,1}$, $Y = F(\cdot, X)$ is a special weak Dirichlet process, with decomposition $Y = M^F + A^F$, where

$$M^F_t = F(0, X_0) + \int_{]0,t]} \partial_x F(s, X_s)\, d(M^c + M^d)_s$$
$$+ \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\, (\mu^X - \nu^X)(ds\, dx),$$

and $F \mapsto A^F$, $\tilde{C}^{0,1} \to \mathbb{D}^{ucp}$, is a linear map and, for every $F \in \tilde{C}^{0,1}$, A^F is a predictable $(\mathcal{F}_t)$-orthogonal process.

(2) If moreover condition (5.27) *holds, M^F reduces to*

$$M^F_t = F(0, X_0) + \int_0^t \partial_x F(s, X_s)\, dM^c_s + \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, (\mu^X - \nu^X)(ds\, dx).$$

Proof. (1) For every F of class $\tilde{C}^{0,1}$, we set

$$A^F = \Gamma^F + \bar{V}^F, \tag{5.43}$$

where Γ^F has been defined in Theorem 5.15, and

$$\bar{V}^F_t := \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}) - x\, \partial_x F(s, X_{s-}))\, \mathbb{1}_{\{|x|>1\}}\, \nu^X(ds\, dx),$$

which is well-defined by assumption (5.40).

The map $F \mapsto A^F$ is linear since $F \mapsto \Gamma^F$ and $F \mapsto \bar{V}^F$ are linear. Given $F \in \tilde{C}^{0,1}$, A^F is a $(\mathcal{F}_t)$-orthogonal process by Theorem 5.15-(a), taking into account that $[\bar{V}^F, N] = 0$ by Proposition 2.14. Using decomposition (5.43), Theorem 5.15-(b) and the fact that $\bar{V}$ is predictable, it follows that A^F is predictable.

(2) follows from (1), if we show that

$$\int_0^t \partial_x F(s, X_{s-})\, dM_s^d = \int_{]0,t]\times\mathbb{R}} x\, \partial_x F(s, X_{s-})\, (\mu^X - \nu^X)(ds\, dx),$$

This follows from Proposition 5.25 and Proposition 5.27, taking into account Remark 5.26.

□

Remark 5.32. In Theorem 5.31-(2), condition (5.27) is verified for instance if X is a particular weak Dirichlet process, see Proposition 5.24.

When X is a special weak Dirichlet process with finite quadratic and finite energy, and $F : \mathbb{R} \to \mathbb{R}$ is a function of class C_b^1, Corollary 3.2 in [8] states that $F(X)$ is again a special weak Dirichlet process. Below we extend that result.

Corollary 5.33. *Let X be a special weak Dirichlet process of finite quadratic variation with its canonical decomposition $X = M^c + M^d + A$, and such that $\mathbb{E}\left[\sum_{s\le T} |\Delta X_s|^2\right] < \infty$. Let $F : [0,T]\times\mathbb{R} \to \mathbb{R}$ of class $C^{0,1}$, with $\partial_x F$ bounded. Then $Y_t = F(t, X_t)$ is a special weak Dirichlet process, with decomposition $Y = M^F + A^F$, where A^F is a predictable $(\mathcal{F}_t)$-orthogonal process, and*

$$M_t^F = F(0, X_0) + \int_0^t \partial_x F(s, X_s)\, dM_s^c + \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, (\mu^X - \nu^X)(ds\, dx).$$

Proof. The result will be a consequence of Theorem 5.31, provided we verify conditions (5.27) and (5.40). Condition (5.27) follows from

$$\sum_{s\le T} |\Delta X_s|\, \mathbb{1}_{\{|\Delta X_s|>1\}} \le \sum_{s\le T} |\Delta X_s|^2\, \mathbb{1}_{\{|\Delta X_s|>1\}} \le \sum_{s\le T} |\Delta X_s|^2 \in L^1(\Omega).$$

Condition (5.40) follows by Remark 5.30. □

5.5 The case of special weak Dirichlet processes without continuous local martingale.

We end this section by considering the case of special weak Dirichlet processes with canonical decomposition $X = M + A$ where $M = M^d$ is a purely discontinuous local martingale. In particular there is no continuous martingale part. In this framework, under the assumptions of Theorem 5.31, if assumption (5.27) in verified, then item (2) says that

$$F(t, X_t) = F(0, X_0) + \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, (\mu^X - \nu^X)(ds\, dx) + A^F(t). \quad (5.44)$$

Since in the above formula no derivative appears, a natural question appears: is it possible to state a chain rule (5.44) when F is not of class $C^{0,1}$? Indeed we have the following result, which does not require any weak Dirichlet structure on X.

We first introduce some notations. Let E be a closed subset of $\mathbb{R}$ on which X takes values. Given a càdlàg function $\varphi : [0,\, T] \to \mathbb{R}$, we denote by $\mathcal{C}_\varphi$ the set of times $t \in [0,\, T]$ for which there is a left (resp. right) neighborhood $I_{t-} =]t-\varepsilon,\, t[$ (resp. $I_{t+} = [t,\, t+\varepsilon[$) such that φ is constant on I_{t-} and I_{t+}. We introduce the following assumption.

Hypothesis 5.34. *There exists $\mathcal{C} \in [0, T]$ such that for ω a.s. $\mathcal{C} \supset \mathcal{C}_{X(\omega)}$, and*

- *$\forall t \in \mathcal{C}$, $t \mapsto F(t, x)$ is continuous $\forall x \in E$;*
- *$\forall t \in \mathcal{C}^c$, $x \in E$, (t, x) is a continuity point of F.*

Remark 5.35. Hypothesis 5.34 is fulfilled in two typical situations.

1. $\mathcal{C} = [0, T]$. Almost surely X admits a finite number of jumps and $t \mapsto F(t, x)$ is continuous $\forall x \in E$.
2. $\mathcal{C} = \emptyset$ and $F|_{[0,T]\times E}$ is continuous.

From now on, we denote by $\Delta F(t, X_t)$ the jump of the time t the process $(F(t, X_t))$.

Remark 5.36. Assume that Hypothesis 5.34 holds. Then

(i) $F(t, X_t)$ is necessarily a càdlàg process.

(ii) $\forall t \in [0, T]$, $\Delta F(t, X_t) = F(t, X_t) - F(t, X_{t-})$.

Proposition 5.37. *Let X be an adapted càdlàg process. Let $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Hypothesis 5.34. Assume that the following holds.*

(i) $F(t, X_t)$ is an $(\mathcal{F}_t)$-orthogonal process such that $\sum_{s\le T} |\Delta F(s, X_s)| < \infty$, a.s.

(ii) $\int_{]0,\,\cdot]\times\mathbb{R}} |F(s, X_{s-} + x) - F(s, X_{s-})|\, \mu^X(ds\, dx) \in \mathcal{A}^+_{\mathrm{loc}}$.

Then

$$F(t, X_t) = F(0, X_0) + \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, (\mu^X - \nu^X)(ds\, dx) + A^F(t), \quad (5.45)$$

where A^F is a predictable $(\mathcal{F}_t)$-orthogonal process. In particular, $F(t, X_t)$ is a special weak Dirichlet process.

Proof. Since by condition (i), $\sum_{s\le t} |\Delta F(s, X_s)|$ is finite a.s., the process $B_t = \sum_{s\le t} \Delta F(s, X_s)$ is well-defined and has bounded variation. We set $A'_t := F(t, X_t) - B_t$. A' is a continuous process, and is $(\mathcal{F}_t)$-orthogonal by additivity, since by assumption $F(t, X_t)$ is $(\mathcal{F}_t)$-orthogonal and B is $(\mathcal{F}_t)$-orthogonal by Proposition 2.14. Recalling the definition of the jump measure μ^X, by Remark 5.36 and condition (ii), we get

$$\begin{aligned}
B_t &= \sum_{0<s\le t} (F(s, X_{s-} + \Delta X_s) - F(s, X_{s-})) \\
&= \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, \mu^X(ds\, dx) \\
&= \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, (\mu^X - \nu^X)(ds\, dx) \\
&\quad + \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, \nu^X(ds\, dx).
\end{aligned}$$

Finally, decomposition (5.45) holds with

$$A^F(t) := A'_t + \int_{]0,\,t]\times\mathbb{R}} (F(s, X_{s-} + x) - F(s, X_{s-}))\, \nu^X(ds\, dx). \quad (5.46)$$

The process A^F in (5.46) is clearly predictable. The $(\mathcal{F}_t)$-orthogonality property of A^F follows from the orthogonality of A' and by Proposition 2.14, noticing that the integral term in (5.46) is a bounded variation process. □

We provide below an example of $(\mathcal{F}_t)$-weak Dirichlet process, intervening in the framework of Piecewise Deterministic Processes, see e.g., Section 4.3 in [5].

Proposition 5.38. *Let $(T_n)_{n\geq 1}$ be a sequence of strictly increasing stopping times, such that $T_n \uparrow \infty$ a.s. Let X be a (càdlàg) process of the type*

$$X_t = \sum_{n=1}^{\infty} \mathbb{1}_{[T_{n-1},\, T_n[}(t)\, R_{n-1}(t),$$

where R_n is an $(\mathcal{F}_{T_n})$-measurable càdlàg process. Let $F : [0,\, T] \times \mathbb{R} \to \mathbb{R}$. We suppose one of the two following assumptions.

(i) $R_n(t)$ are constant $(\mathcal{F}_{T_n})$-random variables, $t \mapsto F(t,x)$, $\forall x \in E$, is continuous.

(ii) F is continuous.

Then $F(\cdot, X)$ is $(\mathcal{F}_t)$-orthogonal.

Proof. Let N be an $(\mathcal{F}_t)$-continuous local martingale. By Proposition 2.8-(ii),

$$\int_0^t F(s, X_{s-})\,(N_{(s+\varepsilon)\wedge t} - N_s)\,\frac{ds}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} \int_0^t F(s, X_{s-})\, dN_s \quad \text{u.c.p.} \tag{5.47}$$

By the definition of covariation it will be enough to prove

$$\int_0^t F((s+\varepsilon)\wedge t, X_{(s+\varepsilon)\wedge t})\,(N_{(s+\varepsilon)\wedge t} - N_s)\,\frac{ds}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} \int_0^t F(s, X_{s-})\, dN_s \quad \text{u.c.p.} \tag{5.48}$$

Indeed, by subtracting (5.47) from (5.48), we would obtain $[F(\cdot, X),\, N] = 0$.
(5.48) will be the consequence of

$$\int_0^t F(s+\varepsilon, \tilde{X}_{s+\varepsilon})\,(N_{(s+\varepsilon)\wedge t} - N_s)\,\frac{ds}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} \int_0^t F(s, X_{s-})\, dN_s \quad \text{u.c.p.,} \tag{5.49}$$

where $\tilde{X}$ is an extension of X to $]T, \infty[$ by X_T, and F is extended to $\mathbb{R}_+ \times \mathbb{R}$, setting $F(s,x) = F(T,x)$, if $s > T$, $x \in \mathbb{R}$. This happens because N is uniformly continuous on $[0,\, T]$ and F is locally bounded.
Let us now concentrate on proving (5.49). We set $\tau_n := T_n \wedge T$. By convention, we set $T_0 = 0$. The left-hand side of (5.49) gives $J_1(t,\varepsilon) + J_2(t,\varepsilon)$, where

$$J_1(t,\varepsilon) = \sum_{n=1}^{\infty} \int_0^t \mathbb{1}_{[\tau_{n-1},\, \tau_n - \varepsilon[}(s)\, F(s+\varepsilon, \tilde{X}_{s+\varepsilon})\,(N_{(s+\varepsilon)\wedge t} - N_s)\,\frac{ds}{\varepsilon}$$

$$J_2(t,\varepsilon) = \sum_{n=1}^{\infty} \int_0^t \mathbb{1}_{[\tau_n - \varepsilon,\, \tau_n[}(s)\, F(s+\varepsilon, \tilde{X}_{s+\varepsilon})\,(N_{(s+\varepsilon)\wedge t} - N_s)\,\frac{ds}{\varepsilon},$$

with the convention that $\mathbb{1}_{[a,\, b[}(s) \equiv 0$ if $b \leq a$. Now, since there is a.s. only a finite number of (τ_n) in $[0,\, T]$, we have a.s.

$$\sup_{t\in[0,\,T]} |J_2(t,\varepsilon)| \leq ||F||_\infty\, \delta(N,\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

where $\delta(N,\cdot)$ denotes the modulus of continuity of N. Concerning $J_1(\varepsilon,t)$, it will be enough to show that

$$\int_0^t \mathcal{Y}_s^\varepsilon \left(N_{(s+\varepsilon)\wedge t} - N_s\right) \frac{ds}{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} \int_0^t \sum_{n=1}^\infty \mathbb{1}_{[\tau_{n-1},\,\tau_n[}(s)\, F(s, X_{s-})\, dN_s \quad \text{u.c.p.,} \tag{5.50}$$

and

$$\mathcal{Y}_s^\varepsilon = \begin{cases} F(s+\varepsilon, R_{n-1}(s+\varepsilon)) & s\in[\tau_{n-1},\,\tau_n-\varepsilon[, \\ 0 & s\in[\tau_n-\varepsilon,\,\tau_n[. \end{cases}$$

Clearly, $\mathbb{1}_{[\tau_{n-1},\,\tau_n-\varepsilon[}(s)\, Y_s^\varepsilon$ is $\mathcal{F}_s$-measurable. So Y^ε is $(\mathcal{F}_s)$-adapted. Since, by assumption, Y^ε is càdlàg, it is then progressively measurable. The left-hand side of (5.50), using stochastic Fubini's theorem, gives

$$\int_0^t dN_u \int_{(u-\varepsilon)-}^u \mathcal{Y}_s^\varepsilon \frac{ds}{\varepsilon}. \tag{5.51}$$

To show that (5.51) goes u.c.p. to the right-hand side of (5.50), taking into account Problem 2.26, Chapter 3, in [22], it is enough to show that

$$\int_0^T d[N,N]_u \left(\int_{(u-\varepsilon)-}^u \mathcal{Y}_s^\varepsilon \frac{ds}{\varepsilon} - \sum_{n=1}^\infty \mathbb{1}_{[\tau_{n-1}-\varepsilon,\,\tau_n[}(u)\, F(u, X_u)\right)^2 \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in probability.}$$

That convergence will be proved to be also a.s. Indeed $|\mathcal{Y}_s^\varepsilon| \le ||F||_\infty$ and $|F(s,X_{s-})| \le ||F||_\infty$, where $||F||_\infty := \sup_{t\in[0,T]} |F(t,x)|$, with $x \in \mathrm{Im}(X_t,\, t\in[0,\,T])$. Moreover, for fixed ω, for $s\in[\tau_{n-1},\,\tau_n[$, for ε small enough,

$$\mathcal{Y}_s^\varepsilon = F(s+\varepsilon,\, R_{n-1}(s+\varepsilon)) \longrightarrow F(s,\, R_{n-1}(s)) = F(s, X_{s-}),$$

in both cases (i) and (ii) of the assumption. This shows the validity of (5.49) and completes the proof. □

Example 5.39. Previous proposition provides further examples of weak-Dirichlet processes. In particular, let us consider the two following cases, developed respectively in Sections 4.2 and 4.3 of [5], see also [4], [3], [1], [2].

1. X pure jump process.
2. X Piecewise Deterministic Markov Processes.

Then Proposition 5.38 applies to the case 1. with assumption (i), and to the case 2. with assumption (ii).

Appendix

A Additional results on calculus via regularization

In what follows, we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and an integer-valued random measure μ.

For every functions f, g defined on $\mathbb{R}$, let now set

$$\tilde{I}^-(\varepsilon, t, f, dg) = \int_{]0,\,t]} f(s)\, \frac{g(s+\varepsilon)-g(s)}{\varepsilon}\, ds, \tag{A.1}$$

$$C_\varepsilon(f,g)(t) = \frac{1}{\varepsilon}\int_{]0,\,t]} (f(s+\varepsilon)-f(s))(g(s+\varepsilon)-g(s))\, ds. \tag{A.2}$$

Definition A.1. *Assume that X, Y are two càdlàg processes. We say that* ***the forward integral of Y with respect to X exists in the pathwise sense****, if there exists some process $(I(t), t \geq 0)$ such that, for all subsequences (ε_n), there is a subsequence (ε_{n_k}) and a null set $\mathcal{N}$:*

$$\forall \omega \notin \mathcal{N}, \qquad \lim_{k\to\infty} |\tilde{I}^-(\varepsilon_{n_k}, t, Y, dX)(\omega) - I(t)(\omega)| = 0 \qquad \forall t \geq 0.$$

Definition A.2. *Let X, Y be two càdlàg processes.* ***the covariation between X and Y (the quadratic variation of X) exists in the pathwise sense****, if there exists a càdlàg process $(\Gamma(t), t \geq 0)$ such that, for all subsequences (ε_n) there is a subsequence (ε_{n_k}) and a null set $\mathcal{N}$:*

$$\forall \omega \notin \mathcal{N}, \qquad \lim_{k\to\infty} |C_{\varepsilon_{n_k}}(X, Y)(t)(\omega) - \Gamma(t)(\omega)| = 0 \qquad \forall t \geq 0.$$

Proposition A.3. *Let X, Y be two càdlàg processes. Then*

$$I^{-ucp}(\varepsilon, t, Y, dX) = \tilde{I}^-(\varepsilon, t, Y, dX) + R_1(\varepsilon, t) \tag{A.3}$$

$$[X, Y]_\varepsilon^{ucp}(t) = C_\varepsilon(X, Y)(t) + R_2(\varepsilon, t), \tag{A.4}$$

where

$$R_i(\varepsilon, t)(\omega) \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad i = 1, 2, \ \forall t \in [0, T], \ \forall \omega \in \Omega. \tag{A.5}$$

Moreover, if X is continuous, then the convergence in (A.5) *holds u.c.p.*

Proof. We fix $t \in [0, T]$. Let $\gamma > 0$. The definition of right continuity in t insures that there exists $\delta > 0$ small enough such that

$$|X(t) - X(a)| \leq \gamma \quad \text{if } a - t < \delta, \ a > t,$$
$$|Y(t) - Y(a)| \leq \gamma \quad \text{if } a - t < \delta, \ a > t.$$

We start proving (A.3). From decomposition (2.3) and the definition of $\tilde{I}^-(\varepsilon, t, Y, dX)$ we get

$$\begin{aligned} I^{-ucp}(\varepsilon, t, Y, dX) - \tilde{I}^-(\varepsilon, t, Y, dX) &= \frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^t Y(s)\,[X(t) - X(s)]\,ds \\ &\quad - \frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^t Y(s)\,[X(s+\varepsilon) - X(s)]\,ds \\ &= \frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^t Y(s)\,[X(t) - X(s+\varepsilon)]\,ds =: R_1(\varepsilon, t). \end{aligned}$$

Choosing $\varepsilon < \delta$ we get $|R_1(\varepsilon, t)| \leq \gamma\,||Y||_\infty$, and since γ is arbitrary, we conclude that $R_1(\varepsilon, t) \to 0$ as ε goes to zero, for every $t \in [0, T]$.

It remains to show (A.4). To this end we evaluate

$$\begin{aligned} [X, Y]_\varepsilon^{ucp}(t) - C_\varepsilon(X, Y)(t) &= \frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^t [X(t) - X(s)]\,[Y(t) - Y(s)]\,ds \\ &\quad - \frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^t [X(s+\varepsilon) - X(s)]\,[Y(s+\varepsilon) - Y(s)]\,ds \\ &=: R_2(\varepsilon, t). \end{aligned}$$

We have

$$R_2(\varepsilon, t) = \frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^t [X(t) - X(s)]\,[Y(t) - Y(s)]\,ds$$

$$
\begin{aligned}
&-\frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^{t}[X(s+\varepsilon)-X(s)]\,[Y(t)-Y(s)]\,ds\\
&+\frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^{t}[X(s+\varepsilon)-X(s)]\,[Y(t)-Y(s)]\,ds\\
&-\frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^{t}[X(s+\varepsilon)-X(s)]\,[Y(s+\varepsilon)-Y(s)]\,ds\\
&=\frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^{t}[X(t)-X(s+\varepsilon)]\,[Y(t)-Y(s)]\,ds\\
&+\frac{1}{\varepsilon}\int_{(t-\varepsilon)_+}^{t}[X(s+\varepsilon)-X(s)]\,[Y(t)-Y(s+\varepsilon)]\,ds.
\end{aligned}
$$

Choosing $\varepsilon < \delta$, the absolute value of previous expression is smaller than $2\,\gamma\,(||Y||_\infty + ||X||_\infty)$. Since γ is arbitrary, $R_2(\varepsilon,t) \to 0$ as ε goes to zero, for every $t \in [0,\,T]$. Suppose now that X is continuous. The expression of $R_2(\varepsilon,t)$ can be uniformly (in t) bounded by $2\delta(X,\varepsilon)\,\|Y\|_\infty$, where $\delta(X,\cdot)$ denotes the modulus of continuity of X; on the other hand $R_1(\varepsilon,t) \le 2\delta(X,\varepsilon)\,\|Y\|_\infty, \forall t \in [0,T]$. This concludes the proof of Proposition A.3. □

Corollary A.4. *Let X, Y be two càdlàg processes.*

1) *If the forward integral of Y with respect to X exists, then it exists in the pathwise sense. In particular, there is a null set $\mathcal{N}$ and, for any sequence $(\varepsilon_n) \downarrow 0$, a subsequence (ε_{n_k}) such that*

$$\tilde{I}^-(\varepsilon_{n_k},t,Y,dX)(\omega) \underset{k\to\infty}{\longrightarrow} \left(\int_{]0,\,t]} Y_s\,d^-X_s\right)(\omega) \quad \forall t\in[0,\,T],\ \forall\omega\notin\mathcal{N}. \tag{A.6}$$

2) *If the covariation between X and Y exists, then it exists in the pathwise sense. In particular, there is a null set $\mathcal{N}$ and, for any sequence $(\varepsilon_n) \downarrow 0$, a subsequence (ε_{n_k}) such that*

$$C_{\varepsilon_{n_k}}(X,Y)(t)(\omega) \underset{k\to\infty}{\longrightarrow} [X,Y]_t\,(\omega) \quad \forall t\in[0,\,T],\ \forall\omega\notin\mathcal{N}. \tag{A.7}$$

Proof. The result is a direct application of Proposition A.3. □

Lemma A.5. *Let $g : [0,\,T] \to \mathbb{R}$ be a càglàd process, X be a càdlàg process such that the quadratic variation of X exists in the pathwise sense, see Definition A.2. Setting (improperly) $[X,X] = \Gamma$, we have*

$$\int_0^s g_t\,(X_{(t+\varepsilon)\wedge s} - X_t)^2\,\frac{dt}{\varepsilon} \overset{\varepsilon\to 0}{\longrightarrow} \int_0^s g_t\,d[X,X]_t \quad \text{u.c.p.} \tag{A.8}$$

Proof. We have to prove that

$$\sup_{s\in[0,T]}\left|\int_0^s g_t\,(X_{(t+\varepsilon)\wedge s} - X_t)^2\,\frac{dt}{\varepsilon} - \int_0^s g_t\,d[X,X]_t\right| \overset{P}{\longrightarrow} 0 \quad \text{as } \varepsilon \text{ goes to zero.} \tag{A.9}$$

Let ε_n be a sequence converging to zero. Since $[X,X]$ exists in the pathwise sense, there is a subsequence ε_{n_k}, that we still symbolize by ε_n, such that

$$C_{\varepsilon_n}(X,X)(t) \overset{n\to\infty}{\longrightarrow} [X,X]_t, \quad \forall t\in[0,\,T] \text{ a.s.} \tag{A.10}$$

Let $\mathcal{N}$ be a null set such that

$$C_{\varepsilon_n}(X,X)(\omega,t) \overset{n\to\infty}{\longrightarrow} [X,X]_t(\omega) \quad \forall t\in[0,\,T],\ \forall\omega\notin\mathcal{N}. \tag{A.11}$$

From here on we fix $\omega\notin\mathcal{N}$. We have to prove that

$$\sup_{s\in[0,\,T]}\left|\int_0^s g_t\,(X_{(t+\varepsilon_n)\wedge s}-X_t)^2\,\frac{dt}{\varepsilon_n}-\int_0^s g_t\,d[X,X]_t\right| \overset{n\to\infty}{\longrightarrow} 0. \tag{A.12}$$

We will do it in two steps.

Step 1. We consider first the case of a càglàd process (g_t) with a finite number of jumps.

Let us fix $\gamma>0$, $\varepsilon>0$. We enumerate by $(t_i)_{i\geq 0}$ the set of jumps of $X(\omega)$ on $[0,\,T]$, union $\{T\}$. Without restriction of generality, we will assume that the jumps of (g_t) are included in $\{t_i\}_{i\geq 0}$. Let $N=N(\omega)$ such that

$$\sum_{i=N+1}^{\infty}|\Delta X_{t_i}|^2\leq\gamma^2,\qquad \sum_{i=N+1}^{\infty}|\Delta g_{t_i}|=0. \tag{A.13}$$

We define $A(\varepsilon,N)$ and $B(\varepsilon,N)$ as in (2.19)-(2.20). The term inside the supremum in (A.9) can be written as

$$\frac{1}{\varepsilon}\int_{]0,\,s]} g_t\,(X_{(t+\varepsilon)\wedge s}-X_t)^2\,dt-\int_{]0,\,s]} g_t\,d[X,X]_t=J_1(s,\,\varepsilon)+J_2(s,\,\varepsilon)+J_3(s,\,\varepsilon),$$

where

$$J_1(\varepsilon,\,N,\,s)=\frac{1}{\varepsilon}\int_{]0,\,s]\cap A(\varepsilon,N)} g_t\,(X_{(t+\varepsilon)\wedge s}-X_t)^2\,dt-\sum_{i=1}^{N}\mathbb{1}_{]0,\,s]}(t_i)\,(\Delta X_{t_i})^2\,g_{t_i},$$

$$J_2(\varepsilon,\,N,\,s)=\frac{1}{\varepsilon}\int_{]0,\,s]\cap B(\varepsilon,N)} g_t\,(X_{t+\varepsilon}-X_t)^2\,dt-\int_{]0,\,s]} g_t\,d[X,X]^c_t-\sum_{i=N+1}^{\infty}\mathbb{1}_{]0,\,s]}(t_i)\,(\Delta X_{t_i})^2\,g_{t_i},$$

$$J_3(\varepsilon,\,N,\,s)=\frac{1}{\varepsilon}\int_{]0,\,s]\cap B(\varepsilon,N)} g_t\,\left[(X_{(t+\varepsilon)\wedge s}-X_t)^2-(X_{t+\varepsilon}-X_t)^2\right]\,dt.$$

Applying Lemma 2.11 to $J_1(\varepsilon,\,N,\,s)$, with $Y=(Y^1,Y^2)=(t,X)$ and $\phi(y_1,y_2)=g_{y_2^1}(y_1^2-y_2^2)^2$, we get

$$\lim_{\varepsilon\to 0}\sup_{s\in[0,\,T]}|J_1(\varepsilon,\,N,\,s)|=0. \tag{A.14}$$

Concerning $J_3(\varepsilon,N,s)$, we have

$$|J_3(\varepsilon,N,s)|\leq\frac{||g||_\infty}{\varepsilon}\left(\int_{s-\varepsilon}^{s}\mathbb{1}_{B(\varepsilon,N)}(t)\,(|X_{t+\varepsilon}-X_t|^2+|X_s-X_t|^2)\,\frac{dt}{\varepsilon}\right).$$

We recall that $B(\varepsilon,N)=\bigcup_{i=1}^{N}]t_{i-1},\,t_i-\varepsilon]$. From Remark 2.13 it follows that, for every $t\in\,]t_{i-1},\,t_i-\varepsilon]$ and $s>t$, $(t+\varepsilon)\wedge s\in[t_{i-1},\,t_i]$. Therefore Lemma 2.12 applied successively to the intervals $[t_{i-1},\,t_i]$ implies that

$$\limsup_{\varepsilon\to 0}\sup_{s\in[0,\,T]}|J_3(\varepsilon,\,N,\,s)|\leq 18\gamma^2\,||g||_\infty. \tag{A.15}$$

It remains to evaluate the uniform limit of $J_2(\varepsilon_n,N,s)$. We show, in a first moment, that, for fixed $s\in[0,\,T]$, we have the pointwise convergence

$$J_2(\varepsilon_n,N,s)=\frac{1}{\varepsilon_n}\int_{]0,\,s]\cap B(\varepsilon_n,N)} g_t\,(X_{t+\varepsilon_n}-X_t)^2\,dt-\int_{]0,\,s]} g_t\,d[X,X]^c_t-\sum_{i=N+1}^{\infty}\mathbb{1}_{]0,\,s]}(t_i)\,(\Delta X_{t_i})^2\,g_{t_i}$$

$$\underset{n\to\infty}{\to} 0, \quad \forall s \in [0,\, T]. \tag{A.16}$$

For this, it will be useful to show that

$$\frac{dt}{\varepsilon_n}\mathbb{1}_{B(\varepsilon_n,N)}(t)\,(X_{t+\varepsilon_n}-X_t)^2 \Rightarrow d\Big(\sum_{\substack{t_i\le t\\ i=N+1}}^{\infty}(\Delta X_{t_i})^2+[X,X]_t^c\Big). \tag{A.17}$$

It will be enough to show that, $\forall s \in [0,\, T]$,

$$\int_0^s \frac{dt}{\varepsilon_n}\mathbb{1}_{B(\varepsilon_n,N)}(t)\,(X_{t+\varepsilon_n}-X_t)^2 \to_{n\to\infty} \sum_{\substack{t_i\le s\\ i=N+1}}^{\infty}(\Delta X_{t_i})^2+[X,X]_s^c. \tag{A.18}$$

By (A.7) in Corollary A.4-2) and Lemma 2.10, we have

$$\int_0^s (X_{t+\varepsilon_n}-X_t)^2\,\frac{dt}{\varepsilon_n} \overset{n\to\infty}{\longrightarrow} [X,X]_s^c+\sum_{t_i\le s}(\Delta X_{t_i})^2 \quad \forall s\in[0,\,T]. \tag{A.19}$$

On the other hand, we can show that

$$\int_0^s \frac{dt}{\varepsilon_n}\mathbb{1}_{A(\varepsilon_n,N)}(t)\,(X_{t+\varepsilon_n}-X_t)^2 \overset{n\to\infty}{\longrightarrow} \sum_{\substack{t_i\le s\\ i=1}}^{N}(\Delta X_{t_i})^2 \quad \forall s\in[0,\,T]. \tag{A.20}$$

Indeed

$$\begin{aligned}
&\Big|\int_0^s \frac{dt}{\varepsilon_n}\mathbb{1}_{A(\varepsilon_n,N)}(t)\,(X_{t+\varepsilon_n}-X_t)^2-\sum_{\substack{t_i\le s\\ i=1}}^{N}(\Delta X_{t_i})^2\Big|\\
&\le \Big|\int_0^s \frac{dt}{\varepsilon_n}\mathbb{1}_{A(\varepsilon_n,N)}(t)\,(X_{(t+\varepsilon_n)\wedge s}-X_t)^2-\sum_{\substack{t_i\le s\\ i=1}}^{N}(\Delta X_{t_i})^2\Big|\\
&+\Big|\int_0^s \frac{dt}{\varepsilon_n}\mathbb{1}_{A(\varepsilon_n,N)}(t)\,(X_{t+\varepsilon_n}-X_t)^2-\int_0^s \frac{dt}{\varepsilon_n}\mathbb{1}_{A(\varepsilon_n,N)}(t)\,(X_{(t+\varepsilon_n)\wedge s}-X_t)^2\Big| \quad \forall s\in[0,\,T].
\end{aligned}$$

The first addend converges to zero by Lemma 2.11 applied to $Y=X$ and $\phi(y)=(y_1-y_2)^2$. The second one converges to zero by similar arguments as those we have used to prove Proposition A.3. This establishes (A.20). Subtracting (A.19) and (A.20), we get (A.18), and so (A.17).

We remark that the left-hand side of (A.17) are positive measures. Moreover, we notice that, since the jumps of g are included in $\{t_1,...,t_N\}$, $t\mapsto g_t(\omega)$ is μ-continuous, where μ is the measure on the right-hand side of (A.17). At this point, Portmanteau theorem and (A.17) insure that $J_2(\varepsilon_n,N,s)$ converges to zero as n goes to infinity, for every $s\in[0,\,T]$.

Finally, we control the convergence of $J_2(\varepsilon_n,N,s)$, uniformly in s. We make use of Lemma 2.16. We set

$$\begin{aligned}
G_n(s) &= \frac{1}{\varepsilon_n}\int_{]0,\,s]}\mathbb{1}_{B(\varepsilon_n,N)}(t)\,(X_{t+\varepsilon_n}-X_t)^2\,g_t\,dt,\\
F(s) &= \int_{]0,\,s]} g_t\,d[X,X]_t^c,
\end{aligned}$$

$$G(s) \quad = \quad - \sum_{i=N+1}^{\infty} \mathbb{1}_{]0,\,s]}(t_i)\,(\Delta X_{t_i})^2\, g_{t_i}.$$

By (A.16), $F_n := G_n + G$ converges pointwise to F as n goes to infinity. Since G_n is continuous and increasing, F is continuous and G is càdlàg, Lemma 2.16 implies that

$$\limsup_{n\to\infty} \sup_{s\in[0,\,T]} |J_2(\varepsilon_n,\, N,\, s)| \le 2\gamma^2\, ||g||_\infty. \tag{A.21}$$

Collecting (A.14), (A.15) and (A.21), it follows that

$$\limsup_{n\to\infty} \sup_{s\in[0,\,T]} \left| \int_0^s g_t\,(X_{(t+\varepsilon_n)\wedge s} - X_t)^2\, \frac{dt}{\varepsilon_n} - \int_0^s g_t\, d[X,X]_t \right| \le 20\gamma^2\, ||g||_\infty.$$

Since γ is arbitrarily small, (A.12) follows.

Step 2. We treat now the case of a general càglàd process (g_t).

Let us fix $\gamma > 0$, $\varepsilon > 0$. Without restriction of generality, we can write $g_t = g_t^{\gamma,BV} + g_t^{\gamma}$, where $g_t^{\gamma,BV}$ is a process with a finite number of jumps and g_t^{γ} is such that $|\Delta g_t^{\gamma}| \le \gamma$ for every $t \in [0,\, T]$. From Step 1, we have

$$I_s^{1,n} := \int_0^s g_t^{\gamma,BV}\,(X_{(t+\varepsilon_n)\wedge s} - X_t)^2\, \frac{dt}{\varepsilon_n} - \int_0^s g_t^{\gamma,BV}\, d[X,X]_t \tag{A.22}$$

converges to zero, uniformly in s, as n goes to infinity.

Concerning (g_t^{γ}), by Lemma 2.12 we see that there exists $\bar\varepsilon_0 = \bar\varepsilon_0(\gamma)$ such that

$$\sup_{\substack{a,\,t\in I\\ |a-t|\le\bar\varepsilon_0}} |g_a^{\gamma} - g_t^{\gamma}| \le 3\gamma. \tag{A.23}$$

At this point, we introduce the càglàd process

$$g_t^{k,\gamma} = \sum_{i=0}^{2^k-1} g^{\gamma}_{i\,2^{-k}T}\, \mathbb{1}_{]i2^{-k}T,(i+1)2^{-k}T]}(t), \tag{A.24}$$

where k is such that $2^{-k} < \bar\varepsilon_0$. From (A.24), taking into account (A.23), for every $t \in [0,\, T]$ there is $i \in \{0, ..., 2k-1\}$, such that

$$|g_t^{\gamma} - g_t^{k,\gamma}| = |g_t^{\gamma}\, \mathbb{1}_{]i2^{-k}\,T,(i+1)2^{-k}\,T]}(t) - g^{\gamma}_{i\,2^{-k}}| \le 3\gamma. \tag{A.25}$$

We set

$$I_s^{2,n} := \int_0^s (g_t^{\gamma} - g_t^{k,\gamma})\,(X_{(t+\varepsilon_n)\wedge s} - X_t)^2\, \frac{dt}{\varepsilon_n} - \int_0^s (g_t^{\gamma} - g_t^{k,\gamma})\, d[X,X]_t.$$

From (A.25) we have $\sup_{s\in[0,\,T]} |I_s^{2,n}| \le 3\gamma\,\Gamma$, with

$$\Gamma = \sup_{n\in\mathbb{N},s\in[0,\,T]} \left| \int_0^s (X_{(t+\varepsilon_n)\wedge s} - X_t)^2\, \frac{dt}{\varepsilon_n} \right| + [X,X]_T. \tag{A.26}$$

Notice that Γ is finite, since the term inside the absolute value in (A.26) converges uniformly by Step 1 with $g = 1$. On the other hand, by definition, $(g_t^{k,\gamma})$ has a finite number of jumps, therefore from Step 1 we get that

$$I_s^{3,n} = \int_0^s g_t^{k,\gamma}\,(X_{(t+\varepsilon_n)\wedge s} - X_t)^2\, \frac{dt}{\varepsilon_n} - \int_0^s g_t^{k,\gamma}\, d[X,X]_t \tag{A.27}$$

converges to zero, uniformly in s, as n goes to infinity. Finally, collecting all the terms, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{s\in[0,T]} \left| \int_0^s g_t \left(X_{(t+\varepsilon_n)\wedge s} - X_t\right)^2 \frac{dt}{\varepsilon_n} - \int_0^s g_t \, d[X,X]_t \right| \\
&\leq \limsup_{n\to\infty} \sup_{s\in[0,T]} |I_s^{1,n}| + \limsup_{n\to\infty} \sup_{s\in[0,T]} |I_s^{2,n}| + \limsup_{n\to\infty} \sup_{s\in[0,T]} |I_s^{3,n}| \\
&\leq 3\,\gamma\Gamma. \qquad \text{(A.28)}
\end{aligned}$$

Since γ is arbitrarily small, (A.12) follows. □

Proposition A.6. *Let X, Y be two càdlàg processes. The following properties are equivalent.*

(1) $[X,X]$, $[X,Y]$, $[Y,Y]$ exist in the pathwise sense.

(2) Suppose the existence of processes Γ_1, Γ_2, Γ_3, fulfilling the following properties. For every càglàd process (g_t),

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_0^s g_t \, \frac{\left(X((t+\varepsilon)\wedge s) - X(t)\right)\left(Y((t+\varepsilon)\wedge s) - Y(t)\right)}{\varepsilon} \, dt &= \int_0^s g_t \, d\Gamma_1(t) \quad \text{u.c.p.,} \\
\lim_{\varepsilon\to 0} \int_0^s g_t \, \frac{\left(X((t+\varepsilon)\wedge s) - X(t)\right)^2}{\varepsilon} \, dt &= \int_0^s g_t \, d\Gamma_2(t) \quad \text{u.c.p.,} \\
\lim_{\varepsilon\to 0} \int_0^s g_t \, \frac{\left(Y((t+\varepsilon)\wedge s) - Y(t)\right)^2}{\varepsilon} \, dt &= \int_0^s g_t \, d\Gamma_3(t) \quad \text{u.c.p.}
\end{aligned}$$

In particular, setting $g = 1$, we have $\Gamma_1 = [X,Y]$, $\Gamma_2 = [X,X]$, $\Gamma_3 = [Y,Y]$.

Proof. Without loss of generality, we first reduce to the case $g \geq 0$. Using polarity arguments of the type

$$\begin{aligned}
[X+Y, X+Y]_t &= [X,X]_t + [Y,Y]_t + 2\,[X,Y]_t \\
[X+Y, X+Y]_\varepsilon^{ucp}(t) &= [X,X]_\varepsilon^{ucp}(t) + [Y,Y]_\varepsilon^{ucp}(t) + 2\,[X,Y]_\varepsilon^{ucp}(t), \\
C_\varepsilon(X+Y, X+Y)(t) &= C_\varepsilon(X,X)(t) + C_\varepsilon(Y,Y)(t) + 2\,C_\varepsilon(X,Y)(t),
\end{aligned}$$

we can reduce to the case $X = Y$.

(1) implies (2) by Lemma A.5.

(1) follows from (2) choosing $g = 1$ and Corollary A.4-2).

□

Remark A.7. Let X, Y be two càdlàg processes. Implication (1) ⇒ (2) in Proposition A.6 with $g = 1$, together with Corollary A.4-2), implies that the following are equivalent:

- (X,Y) admits all its mutual brackets;
- $[X,X]$, $[X,Y]$, $[Y,Y]$ exist in the pathwise sense.

In that case, the covariation processes above equal the corresponding processes Γ related to Definition A.2. In particular, the following properties are equivalent:

- X is a finite quadratic variation process;
- $[X,X]$ exists in the pathwise sense.

Proposition A.8. *Let X be a finite quadratic variation process. The following are equivalent.*

(i) X is a weak Dirichlet process;

(ii) $X = M + A, \quad [A, N] = 0$ in the pathwise sense for all N continuous local martingale.

Proof. (i) $\Rightarrow$ (ii) obviously. Assume now that (ii) holds. Taking into account Corollary A.4 2), it is enough to prove that $[A, N]$ exists. Now, we recall that, whenever M and N are local martingale, $[M, N]$ exists by Proposition 2.8. Let N be a continuous local martingale. By Remark A.7, $[X, X]$ and $[N, N]$ exist in the pathwise sense. By additivity and item (ii), $[X, N] = [M, N]$ exists in the pathwise sense. Still by Remark A.7, (X, N) admits all its mutual brackets. Finally, by bilinearity

$$[A, N] = [X, N] - [M, N] = 0.$$

□

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